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On the Plane Cubics over $\mathbb{Q}_p$ and the Associated Igusa Zeta Function

by
DENIS IBADULA*

Abstract

In this paper we determine the explicit form of the Igusa local zeta function associated to the plane cubics defined over the field of p-adic numbers $\mathbb{Q}_p$ and classify them. We will use this in a next paper [Iba05b] in order to classify the plane cubics over $\mathbb{Q}_p$ from the point of view of their Igusa local zeta function.

Key Words: Igusa local zeta function, Plane cubics.
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1 Introduction

Let k be a field with $char(k) \notin \{2,3\}$, k^* its multiplicative group and $\tilde{k}$ an algebraic closure of k.

Let $\mathcal{F}^k_{2,3}$ be the category of non-zero homogeneous polynomials of degree 3 in x_1, x_2 over k. An object $F(x_1,x_2)$ from $\mathcal{F}^k_{2,3}$ is called **a binary cubic form** (over k).

The discriminant of a binary cubic form $F(x_1,x_2) = ax_1^3+bx_1^2x_2+cx_1x_2^2+dx_2^3$ is defined as $\Delta(F) = -27a^2d^2+18abcd+b^2c^2-4b^3d-4ac^3$, and F is called **non-degenerated** if $\Delta(F) \neq 0$.

Let $\mathcal{F}^k \subset \mathcal{F}^k_{2,3}$ be the category of non-degenerated binary cubic forms. For example, the Fermat form $F_0(x_1,x_2) = x_1^3+x_2^3$ is non-degenerated, since $\Delta(F_0) = -27 \neq 0$.

It is known that *any non-degenerated plane cubic is a Fermat form.* What is the meaning of this? In geometric terms, this means that the projective variety $V(F)$ defined by the equation $F = 0$, embedded in the projective line $\mathbb{P}^1(\tilde{k})$,

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is smooth (without singular points). Obviously, $V(F)$ is finite with at most 3 points corresponding to the roots in $\tilde{k}$ of the polynomial of degree 3 in $\frac{x_2}{x_1}$, $x_1^{-3}F$. The smooth condition means that this three roots are simple, or, equivalent, the system $F = \partial F/\partial x_1 = \partial F/\partial x_2 = 0$ has a unique solution $x_1 = x_2 = 0$ in the affine plane over $\tilde{k}$.

The important fact here is the following: if F is non-degenerated then F **is a Fermat form**, i.e. there exists an automorphism f of the projective line $\mathbb{P}^1(\tilde{k})$ that maps $V(F)$ in a projective variety W of dimension 0, defined by the Fermat cubic $x_1^3 + x_2^3$, consisting of the three points: $(-1, 1)$, $(-1, \zeta)$, $(-1, \zeta^2)$, where ζ is a primitive root of unity of degree 3 or, equivalent, there exists a matrix $g \in GL_2(\tilde{k})$ such that $F((x_1, x_2)g^t) = x_1^3 + x_2^3$. This is actually a consequence of the following fact: the action of the automorphisms group of the projective line $PGL_2(\tilde{k}) = GL_2(\tilde{k})/\tilde{k}^*$ on the projective line is 3-transitive.

Using this facts, we can determine the isomorphism types of non-degenerated plane cubics with coefficients in k. We identify the plane cubics F_1 and F_2 (i.e. F_1 and F_2 are in the same isomorphism classes) if F_1 can be obtained from F_2 applying an automorphism defined over k (using a matrix with coefficients in k), i.e. if there exists $g \in GL_2(k)$ such that $F_2((x_1, x_2)g^t) = F_1(x_1, x_2)$, where g^t denotes the transposed of the matrix g.

The isomorphism types of plane cubics can be obtained as a particular case of the general theory of Fermat's forms. In this paper we apply effectively the algorithm given in the Theorem 3.5 for the case we are interested in: $k = \mathbb{Q}_p$. The next step consists in computing of the Igusa local zeta function of the representative binary cubics.

2 Preliminaries

For p prime, denote the field of p-adic numbers by $\mathbb{Q}_p$, the ring of p-adic integers by $\mathbb{Z}_p$ and the finite field with p elements by $\mathbb{F}_p$.

Definition 2.1. *Let $F(x) = F(x_1, x_2) \in \mathbb{Z}_p[x_1, x_2]$. The **Igusa local zeta function** associated to F is*

$$\boxed{Z_F(s) = \int_{(x_1,x_2)\in\mathbb{Z}_p^2} |F(x_1, x_2)|^s |dx_1||dx_2|},$$

where $s \in \mathbb{C}$, $Re(s) > 0$ and $|dx| = |dx_1||dx_2|$ denotes the Haar measure on $\mathbb{Q}_p^2$ so normalized that $\mathbb{Z}_p^2$ has measure 1.

It was proved by Igusa, using resolution of singularities, that $Z_F(s)$ is a rational function of $t = p^{-s}$. An entirely different proof was obtained ten years later by Denef [Den84] using p-adic cell decomposition instead of resolution of singularities.

We also note that if $P(t)$ is **the Poincaré series** defined by

$$\boxed{P(t) = \sum_{n=0}^{\infty} N_e p^{-ne} t^e},$$

with $N_e = Card\{x \mod (p\mathbb{Z}_p)^2 | F(x) \equiv 0 \mod (p\mathbb{Z}_p)^2\}$, there is the relation

$$\boxed{P(t) = \frac{1 - tZ_F}{1 - t}}.$$

Hence, one can obtain explicit formulas for N_e from the explicit form of Z_F.

We can also remark the followings: the fact that $Z_F(s)$ is a rational function in p^{-s} shows that infinitely many informations, namely the numbers N_e, for all $e \in \mathbb{N}$, are contained in a finite numbers of coefficients of the nominator and denominator of the Igusa zeta function.

Igusa introduced what he called a stationary phase formula (abbreviated SPF), which can be an effective method to compute the local zeta function.

Theorem 2.2. ***(Stationary phase formula)*** *We take a subset $\bar{E}$ of $\mathbb{F}_p^2$ and denote by $\bar{S}$ its subset consisting of all $\bar{a}$ in $\bar{E}$ such that $\bar{F}(\bar{a}) = (\partial \bar{F}/\partial x_i)(\bar{a}) = 0$, for $1 \leq i \leq 2$; we further denote by E, S the preimages of $\bar{E}$, $\bar{S}$ under the canonical projection $\mathbb{Z}_p^2 \to \mathbb{Z}_p^2/(p\mathbb{Z}_p)^2 \cong \mathbb{F}_p^2$ and by N the number of zeros of $\bar{F}(x)$ in $\bar{E}$. Then we have:*

$$\int_{x \in E} |F(x)|^s |dx| = p^{-2}\left(card(\bar{E}) - N\right) \quad + \quad p^{-2}\frac{\left(N - card(\bar{S})\right)\left(1 - p^{-1}\right)t}{1 - p^{-1}t} + \\ + \int_{x \in S} |F(x)|^s |dx|.$$

Proof: See [Igu00]. □

3 The Isomorphism Classes Over $\mathbb{Q}_p$ of the Plane Cubics with Coefficients in $\mathbb{Q}_p$

3.1 The Isomorphism Theorem

Definition 3.1. *For $\delta \in k^*/k^{*2}$, we define:*

$$L_\delta := \begin{cases} k \times k, & \text{if} \quad \delta \in k^{*2} \\ k(\sqrt{\delta}), & \text{else.} \end{cases}$$

Proposition 3.2. *The map $\delta \mapsto L_\delta$ defines a bijection between k^*/k^{*2} and the isomorphism classes of separable k-algebras of degree 2 over k.*

Proof: Clear. □

Proposition 3.3. *For any $\delta \in k^*/k^{*2}$, we have $Aut_k L_\delta \cong \mathbb{Z}/2\mathbb{Z}$, where $Aut_k\,(L_\delta)$ is the group of k-automorphisms of L_δ.*

Proof: Obviously, the group $Aut_k L_\delta$ is cyclic of order 2 generated by the automorphism σ, where

$$\begin{array}{rcl} \sigma : k^* \times k^* & \to & k^* \times k^* \\ \sigma(x,y) & = & (y,x), \text{ for all} (x,y) \in k^* \times k^*, \end{array}$$

if $\delta \in k^{*2}$, or

$$\begin{array}{rcl} \sigma : k(\sqrt{\delta}) & \to & k(\sqrt{\delta}) \\ \sigma(x + y\sqrt{\delta}) & = & x - y\sqrt{\delta}, \text{ for all } x + y\sqrt{\delta} \in k(\sqrt{\delta}), \end{array}$$

else. □

For $\delta \in k^*/k^{*2}$, let L_δ be the commutative, separable k-algebra of degree 2 from the Definition 3.1, $\{e_1, e_2\}$ be a k-basis of L_δ and $\sigma \in Aut_k L_\delta$ be the generator of the group $Aut_k L_\delta$.

Since the subgroup $k^* L_\delta^{*3}$ of L_δ^* is invariated by σ, the action of σ on L_δ^* induces an action of $Aut_k L_\delta$ on the factor group $L_\delta^*/k^* L_\delta^{*3}$

$$\begin{array}{rcl} Aut_k\,(L_\delta) \times L_\delta^*/k^* L_\delta^{*3} & \to & L_\delta^*/k^* L_\delta^{*3} \\ \left(\sigma, u \mod k^* L_\delta^{*3}\right) & \mapsto & \sigma(u) \mod k^* L_\delta^{*3}, \text{ for all } u \in L_\delta^*. \end{array}$$

Definition 3.4. *For $\beta \in L_\delta^*$ a representative element of the set*

$$Aut_k L_\delta \backslash \left(L_\delta^*/k^* L_\delta^{*3}\right),$$

we denote by $P\left\{(L_\delta, \beta)\right\}$ the following trace form:

$$P\left\{(L_\delta, \beta)\right\} := Tr_{L_\delta \backslash k}\left[\beta \cdot (e_1 x_1 + e_2 x_2)^3\right] \in k[x_1, x_2].$$

Theorem 3.5 (The isomorphism theorem). *The map*

$$\begin{array}{rcl} \bigsqcup_{\delta \in k^*/k^{*2}} \left[Aut_k L_\delta \backslash \left(L_\delta^*/k^* L_\delta^{*3}\right)\right] & \to & \left\{\text{The isomorphism classes over } k \text{ of } \mathcal{F}^k\right\} \\ (\delta, \beta) & \mapsto & P\left\{(L_\delta, \beta)\right\}, \end{array}$$

is a bijection.

Proof: See [Bru02], page 90. □

3.2 The Computation of the Isomorphism Classes over $\mathbb{Q}_p$ of Plane Cubics with Coefficients in $\mathbb{Q}_p$

In order to determine effectively the isomorphism classes over $\mathbb{Q}_p$ of non-degenerated plane cubics with coefficients in $\mathbb{Q}_p$, we will apply the Theorem 3.5 for $k = \mathbb{Q}_p$, the field of p-adic numbers.

Let $\boxed{k = \mathbb{Q}_p}$, let $\mathcal{F}_{2,3} := \mathcal{F}_{2,3}^{\mathbb{Q}_p}$ be the category of non-zero binary cubic forms over $\mathbb{Q}_p$ and $\mathcal{F} := \mathcal{F}^{\mathbb{Q}_p} \subset \mathcal{F}_{2,3}$ be the category of non-degenerated, non-zero binary cubic forms over $\mathbb{Q}_p$.

From the Theorem 3.5, for a fixed $\delta \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$, the isomorphism classes over $\mathbb{Q}_p$ of plane cubics with coefficients in $\mathbb{Q}_p$, corresponds bijectively to the set $Aut_k L_\delta \backslash \left(L_\delta^*/k^* L_\delta^{*3}\right)$, where L_δ is as in Definition 3.1.

Since

$$\mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

(see [Neu99]) we can choose as a representative system $\{1, p, \zeta, p\zeta\}$, where ζ is a primitive $(p-1)$-root of unity.

Definition 3.6. *For $\delta \in \{1, p, \zeta, p\zeta\}$, we define:*

$$L_\delta := \begin{cases} \mathbb{Q}_p \times \mathbb{Q}_p, & \text{if} \quad \delta = 1 \\ \mathbb{Q}_p(\sqrt{\delta}), & \text{else.} \end{cases}$$

Let now p be a prime number, $p \geq 5$. We will follow one of the following four paths, depending on the choice of δ.

3.2.1 The extension $\mathbb{Q}_p \subset \mathbb{Q}_p \times \mathbb{Q}_p$

Case I: Let $\delta = 1$ and let $L := L_1 := \mathbb{Q}_p \times \mathbb{Q}_p$ be the separable $\mathbb{Q}_p$-algebra of degree 2 over $\mathbb{Q}_p$.

Theorem 3.7. *For $L := \mathbb{Q}_p \times \mathbb{Q}_p$, we get:*

1) If $p \equiv 1 \mod 3$, then a representative system of isomorphism classes over k of plane cubics with coefficients in k is given by

$$\left\{x_1^3 + x_2^3,\ px_1^3 + x_2^3,\ \zeta x_1^3 + x_2^3,\ p\zeta x_1^3 + x_2^3,\ p\zeta^2 x_1^3 + x_2^3\right\};$$

2) If $p \equiv 2 \mod 3$, then a representative system of isomorphism classes over k of plane cubics with coefficients in k is given by

$$\left\{x_1^3 + x_2^3,\ px_1^3 + x_2^3\right\}.$$

Proof: Let $L^*/L^{*3} \xrightarrow{\varphi} L^*/k^*L^{*3}$ given by $\varphi(x \mod L^{*3}) = x \mod k^*L^{*3}$, $\forall x \in L^*$ be the surjection having $\ker(\varphi) = k^*L^{*3}/L^{*3} \cong k^*/k^* \cap L^{*3}$. Since $k^* \cap L^{*3} = k^{*3}$, we get the exact sequence:

$$1 \to k^*/k^{*3} \to L^*/L^{*3} \to L^*/k^*L^{*3} \to 1,$$

and, consequently, $L^*/k^*L^{*3} \cong k^*/k^{*3}$.

We use now the structure theorem of $\mathbb{Q}_p^*$ $(p \neq 2)$ (see for example [Neu99]) and we get:

$$L^*/k^*L^{*3} \cong \mathbb{Q}_p^*/\mathbb{Q}_p^{*3} \cong (\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p) \,/\, (3\mathbb{Z} \times 3\mathbb{Z}/(p-1)\mathbb{Z} \times 3\mathbb{Z}_p)\,.$$

Case I.1: If $p \equiv 1 \mod 3$, for

$$L^*/k^*L^{*3} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

we can choose $\{p^i\zeta^j e_1 + e_2 | i,j = 0,1,2\}$ as a representative system modulo k^*L^{*3}, where ζ is a primitive $p-1$ root of unity and e_1, e_2 is the canonical base of L.

In order to determine the isomorphism classes over $\mathbb{Q}_p$ of plane cubics with coefficients in $\mathbb{Q}_p$, we have to determine the orbits corresponding to the action on cyclic group of order 2 $Aut_k L$ generated by σ (see Proposition 3.3) on L^*/k^*L^{*3} and choose a representative for each orbit.

Because $p^2e_1+e_2$ and pe_1+e_2, $p^2\zeta e_1+e_2$ and $p\zeta^2e_1+e_2$, $\zeta^2e_1+e_2$ and ζe_1+e_2, respectively $p^2\zeta^2e_1+e_2$ and $p\zeta e_1+e_2$ defines, in pairs, the same equivalence classes in $Aut_kL\backslash\left(L^*/k^*L^{*3}\right)$ (because, for example, $\sigma\left(p^2e_1+e_2\right) = e_1+p^2e_2 \equiv pe_1+e_2 \mod k^*L^{*3}$), a representative system for this factor set contains five elements: ae_1+e_2, with $a \in \{1, p, \zeta, p\zeta, p\zeta^2\}$.

For such a representative, the corresponding plane cubic can be obtained calculating the trace (see Definition 3.4) of the elements
$(ae_1+e_2)(x_1e_1+x_2e_2)^3 = ax_1^3e_1 + x_2^3e_2$:

$$P\left\{\left(\mathbb{Q}_p^2, ae_1+e_2\right)\right\} := Tr_{L/k}\left(ax_1^3e_1 + x_2^3e_2\right) = ax_1^3 + x_2^3,$$

cu $a \in \{1, p, \zeta, p\zeta, p\zeta^2\}$. In this way we obtain 5 non isomorphic cubics over $\mathbb{Q}_p$:

$$x_1^3+x_2^3,\ px_1^3+x_2^3,\ \zeta x_1^3+x_2^3,\ p\zeta x_1^3+x_2^3,\ p\zeta^2x_1^3+x_2^3.$$

Case I.2: If $p \equiv 2 \mod 3$, for

$$L^*/k^*L^{*3} \cong \mathbb{Z}/3\mathbb{Z}$$

we can choose $\{p^ie_1+e_2 | i = 0,1,2\}$ as a representative system modulo k^*L^{*3}. Because $p^2e_1+e_2$ and pe_1+e_2 belongs to the same orbit, the action of the cyclic group of order 2 Aut_kL on L^*/k^*L^{*3} determines two orbits. It results that a representative system for $Aut_kL\backslash\left(L^*/k^*L^{*3}\right)$ is given by: $p^ie_1+e_2$, with $i \in \{0,1\}$. In the same way as in the previous case we obtain the corresponding cubics $x_1^3+x_2^3$ and $px_1^3+x_2^3$, as required. □

3.2.2 The extension $\mathbb{Q}_p \subset \mathbb{Q}_p\left(\sqrt{p}\right)$

Case II: Let $\delta = \sqrt{p}$ and let $L := L_{\sqrt{p}} := \mathbb{Q}_p\left(\sqrt{p}\right)$ be the extension of degree 2 over $\mathbb{Q}_p$ with $\{1,\ \sqrt{p}\}$ a basis of the k-vector space L. We describe next the valuation ring $v(L)$ of L, the maximal ideal $\underline{m}_L$, the group of units $\mathcal{O}_L^*$ and the residue class field $\bar{L}$.

Lemma 3.8. *Let $L := \mathbb{Q}_p(\sqrt{p})$ be the extension of degree 2 of $\mathbb{Q}_p$.*

Then the unique extension v to L of the p-adic valuation v_p is given by:

$$v(x_1 + \sqrt{p}x_2) = min\left\{v_p(x_1), \frac{1}{2} + v_p(x_2)\right\},$$

for all $x_1 + \sqrt{p}x_2 \in \mathbb{Q}_p(\sqrt{p})$.

The valuation group of L is $v(L^*) = (1/2)\mathbb{Z}$. In particular, $1/2$ is the smallest positive element from the abelian total ordered group $v(L) \cong \mathbb{Z}$ and $\sqrt{p}$ is a local uniformizer of L.

Proof: Since $\mathbb{Q}_p$ is henselian, the p-adic valuation v_p can be uniquely extended to a valuation v of the algebraic extension of degree 2, L.

From $v(p) = 1$ we get $2v(\sqrt{p}) = v(p) = 1$, i.e. $v(\sqrt{p}) = \frac{1}{2}$. Now, for $x_1,\ x_2 \in \mathbb{Q}_p$, since $v(x_1) \neq v(x_2) + 1/2$ it follows that

$$v(x_1 + \sqrt{p}x_2) = min\left\{v_p(x_1), \frac{1}{2} + v_p(x_2)\right\},$$

that ends lemma's proof. □

In order to simplify the notations, we denote by v both p-adic valuation v_p and its extension v to L.

Using the previous proposition we can describe the valuation ring $\mathcal{O}_L$, the maximal ideal $\underline{m}_L$ and the group of units $\mathcal{O}_L^*$:

Corollary 3.9. *For the 2 degree extension of $\mathbb{Q}_p$, $L := \mathbb{Q}_p(\sqrt{p})$ we get:*

1) The valuation ring $\mathcal{O}_L = \mathbb{Z}_p[\sqrt{p}] = \{x_1 + x_2\sqrt{p}\ |x_1,\ x_2 \in \mathbb{Z}_p\}$;

2) The maximal ideal $\underline{m}_L$ is equal to $\underline{m}_L = \{x_1 p + x_2\sqrt{p}\ |x_1,\ x_2 \in \mathbb{Z}_p\}$;

3) The group of units $\mathcal{O}_L^*$ has the form $\mathcal{O}_L^* = \{x_1 + x_2\sqrt{p}\ |x_1 \in \mathbb{Z}_p^*,\ x_2 \in \mathbb{Z}_p\}$.

Proof: **1)** For an element $\alpha = x_1 + x_2\sqrt{p}$ from $\mathbb{Z}_p[\sqrt{p}]$, because $v(\alpha) = v(x_1 + x_2\sqrt{p}) = min\left\{v(x_1),\ v(x_2) + \frac{1}{2}\right\}$, it follows that $\alpha \in \mathcal{O}_L$, since $v(x_1), v(x_2) \geq 0$. Conversely, let $\alpha = x_1 + x_2\sqrt{p}$ be an element such that $min\left\{v(x_1), \frac{1}{2} + v(x_2)\right\} \geq 0$.

Case i: If $min\left\{v(x_1), \frac{1}{2} + v(x_2)\right\} = v(x_1)$, then $v(x_1) \geq 0$ and $\alpha \in \mathbb{Z}_p[\sqrt{p}]$.

Case ii: If $min\left\{v(x_1), \frac{1}{2} + v(x_2)\right\} = \frac{1}{2} + v(x_2)$, then $v(x_2) \geq 0$ and $\alpha \in \mathbb{Z}_p[\sqrt{p}]$, as contended.

2) Since the valuation v is discrete ($v(L) \cong \mathbb{Z}$), the maximal ideal is principal generated by $\sqrt{p}$:

$$\underline{m}_L = \sqrt{p}\mathcal{O}_L = \sqrt{p}\mathbb{Z}_p[\sqrt{p}] = \{x_1 p + x_2\sqrt{p}\ |x_1,\ x_2 \in \mathbb{Z}_p\}.$$

3) It follows from 1) and 2) and from $\mathcal{O}_L^* = \mathcal{O}_L - \underline{m}_L$. □

From the previous corollary it follows that the ramification index is $e = [v(L^*) : v(k^*)] = 2$, that is $\mathbb{Q}_p\left(\sqrt{p}\right)$ is a totally ramified extension of $\mathbb{Q}_p$.

Lemma 3.10. *For* $L := \mathbb{Q}_p\left(\sqrt{p}\right)$ *we get* $L^* \cong \mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p^2$.

Proof: From the structure theorem of the multiplicative local compact group L^* of the local field L (see [Neu99]), since the extension is totally ramified of degree 2, it follows that: $L^* = \mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p^a\mathbb{Z} \times \mathbb{Z}_p^2$, where a the maximal order of a p-power root of unity in L. Since $\mathbb{Q}_p\left(\sqrt{p}\right)$ could not contain a p^a-power root of unity, with $a \in \mathbb{N}^*$, we obtain that $a = 0$ and $L^* \cong \mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p^2$, as contained. □

Corollary 3.11. *With the above notations,*

$$\mathcal{O}_L^* \cong \mathbb{Z}_p^* \times \mathbb{Z}_p.$$

Proof: Since the multiplicative group L^* is isomorphic with the direct product of cyclic group generated by the local uniformizer of $\sqrt{p}$ and the group of units $\mathcal{O}_L^*$, using Lemma 3.10 we get $\mathcal{O}_L^* \cong \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p^2 \cong \mathbb{Z}_p^* \times \mathbb{Z}_p$, as contained. □

Theorem 3.12. *Let* $L := \mathbb{Q}_p\left(\sqrt{p}\right)$ *be the separable k-algebra of degree* 2.

Then a representative system of isomorphism classes over $\mathbb{Q}_p$ *of plane cubics with coefficients in* $\mathbb{Q}_p$ *is given by*

$$x_1^3 + 3px_1x_2^2 \text{ and } 3x_1^2x_2 + px_2^3.$$

Proof: As in Theorem 3.7, we get the exact sequence

$$1 \to k^*/\left(k^* \cap L^{*3}\right) \to L^*/L^{*3} \to L^*/k^*L^{*3} \to 1.$$

But $k^* \cap L^{*3} = k^{*3}$: for an element $x \in k^* = \mathbb{Q}_p^*$, from $x = \left(a + b\sqrt{p}\right)^3$, with $a + b\sqrt{p} \in L^*$, identifying the coefficients we get $b = 0$ as a unique possibility, that is $x \in k^{*3}$. Consequently,

$$Aut_k L \backslash \left(L^*/k^*L^{*3}\right) = \{1\}.$$

Calculating the trace of the corresponding element $1 \cdot \left(x_1 + x_2\sqrt{p}\right)^3$, as in Theorem 3.7, we obtain the corresponding cubic: $2\left(x_1^3 + 3px_1x_2^2\right)$, which is equivalent over k with $x_1^3 + 3px_1x_2^2$, as contained. □

3.2.3 The extension $\mathbb{Q}_p \subset \mathbb{Q}_p\left(\sqrt{\zeta}\right)$

Case III: Let $\delta = \sqrt{\zeta}$ and let $L := L_\zeta := \mathbb{Q}_p\left(\sqrt{\zeta}\right)$ be the extension of degree 2 over $\mathbb{Q}_p$, where $\zeta \in \mathbb{Q}_p$ is a primitive $(p-1)$-root of unity.

Lemma 3.13. *Let $L := \mathbb{Q}_p(\sqrt{\zeta})$ be the extension of degree 2 of $\mathbb{Q}_p$.*
Then the unique extension v of p-adic valuation v_p to L is given by

$$v\left(x_1 + \sqrt{\zeta}x_2\right) = min\left\{v_p(x_1), v_p(x_2)\right\},$$

for all $x_1 + \sqrt{\zeta}x_2 \in \mathbb{Q}_p[\sqrt{\zeta}]$.
The valuation group of L is $v(L^*) = \mathbb{Z}$. In particular, p is the smallest positive element from total ordered abelian group $v(L^*) = \mathbb{Z}$ and p is a local uniformizer of L.

Proof: Since $\mathbb{Q}_p$ is henselian, the p-adic valuation v_p can be uniquely extended to a valuation v of the algebraic extension of degree 2, L. The map $|.| : L \to \mathbb{R}_+$ defined by $|\alpha| = \sqrt{|N_{L/\mathbb{Q}_p}(\alpha)|_p}$, for all $\alpha \in L := \mathbb{Q}_p[\sqrt{\zeta}]$, is a nonarchimedean absolute value on L which extends the p-adic absolute value.

The element $x_1 + \sqrt{\zeta}x_2 \in \mathbb{Q}_p[\sqrt{\zeta}]$ has the norm $x_1^2 - \zeta x_2^2$, and so the valuation on L is defined by $v(x_1 + x_2\sqrt{\zeta}) = \frac{1}{2}v_p\left(x_1^2 - \zeta x_2^2\right)$. For $\alpha = x_1 + \sqrt{\zeta}x_2 \in \mathbb{Q}_p[\sqrt{\zeta}]$, with $v_p(x_1) \neq v_p(x_2)$ we get

$$v\left(x_1 + \sqrt{\zeta}x_2\right) = \frac{1}{2}min\left\{2v_p(x_1), 2v_p(x_2)\right\} = min\left\{v_p(x_1), v_p(x_2)\right\}.$$

Let now $\alpha = x_1 + \sqrt{\zeta}x_2 \in \mathbb{Q}_p[\sqrt{\zeta}]$, with $x_1 = p^n u_1$, $x_2 = p^n u_2$, $u_1,\ u_2 \in \mathbb{Z}_p^*$ and $n \in \mathbb{Z}$. Then, since $v_p\left(u^2 - \zeta v^2\right) = 0$, it follows that

$$v_p\left(x_1^2 - \zeta x_2^2\right) = 2n + v_p\left(u^2 - \zeta v^2\right) = 2n,$$

that ends lemma's proof. □

In order to simplify the notations, we denote by v both p-adic valuation v_p and its extension v to L.

From the previous lemma we can obtain easily the following corollary, which proof is analogous with the proof of Corrolary 3.9:

Corollary 3.14. *For the 2 degree extension of $\mathbb{Q}_p$, $L := \mathbb{Q}_p[\sqrt{\zeta}]$ we get:*
1) The valuation ring $\mathcal{O}_L = \mathbb{Z}_p[\sqrt{\zeta}] = \left\{x_1 + x_2\sqrt{\zeta}\ |x_1,\ x_2 \in \mathbb{Z}_p\right\}$;
2) The maximal ideal $\underline{m}_L$ is equal to $\underline{m}_L = \left\{x_1 p + x_2 p\sqrt{\zeta}\ |x_1,\ x_2 \in \mathbb{Z}_p\right\}$;
3) The group of units $\mathcal{O}_L^*$ has the form

$$\begin{aligned}\mathcal{O}_L^* &= \left\{x_1 + px_2\sqrt{\zeta}\ |x_1 \in \mathbb{Z}_p^*, x_2 \in \mathbb{Z}_p\right\} \cup \left\{px_1 + x_2\sqrt{\zeta}\ |x_1 \in \mathbb{Z}_p, x_2 \in \mathbb{Z}_p^*\right\} \cup \\ &\cup \left\{x_1 + x_2\sqrt{\zeta}\ |x_1, x_2 \in \mathbb{Z}_p^*\right\} \square.\end{aligned}$$

In this case, the extension L/k has the ramification index $e=[v(L^*):v(k^*)]=1$, that is L is unramified over k, and the residual degree is $[\bar{L}:\bar{k}]=2$.

Lemma 3.15. *For $L:=\mathbb{Q}_p\left(\sqrt{\zeta}\right)$, it follows that $L^*\cong\mathbb{Z}\times\mathbb{Z}/(p^2-1)\mathbb{Z}\times\mathbb{Z}_p^2$.*

Proof: Analogous with the proof of Lemma 3.10. □

Corollary 3.16. *With the above notations, the group of units $\mathcal{O}_L^*$ is isomorphic with*

$$\mathcal{O}_L^*\cong\mathbb{Z}/(p^2-1)\mathbb{Z}\times\mathbb{Z}_p^2.$$

Proof: It follows from the Lemma 3.15. □

Theorem 3.17. *Let $L:=\mathbb{Q}_p\left(\sqrt{\zeta}\right)$ be the separable k-algebra of degree 2.*

1) If $p\equiv 1 \mod 3$, then there exists a unique isomorphism classes over k of plane cubics with coefficients in k with the representative $x_1^3+3\zeta x_1x_2^2$.

2) If $p\equiv 2 \mod 3$, then a representative system of isomorphism classes over k of plane cubics with coefficients in k is given by:
$\{x_1^3+3\zeta x_1x_2^2,\ Tr_{L/k}(\theta)x_1^3+3Tr_{L/k}(\theta)\zeta x_1x_2^2+3Tr_{L/k}(\theta\sqrt{\zeta})x_1^2x_2+$ $+Tr_{L/k}(\theta\sqrt{\zeta})\zeta x_2^3\}$, *where θ is a primitive root of unity of order p^2-1 such that $\theta^{p+1}=\zeta$.*

Proof: As in the proof of Theorems 3.12 and 3.7, using the exact sequence

$$1\to k^*/k^{*3}\to L^*/L^{*3}\to L^*/k^*L^{*3}\to 1,$$

and the Lemma 3.15, we obtain that the quotient group L^*/k^*L^{*3} is trivial, if $p\equiv 1 \mod 3$, respectively cyclic of order three, if $p\equiv 2 \mod 3$.

Case III.1: If $p\equiv 1 \mod 3$, then $L^*/k^*L^{*3}=\{1\}$ and, consequently,

$$Aut_kL\backslash\left(L^*/k^*L^{*3}\right)=\{1\}.$$

As in the previous cases we obtain that the associated plane cubic is

$$\begin{aligned}P(\mathbb{Q}_p(\sqrt{\zeta}),1): &= Tr_{L/k}\left(1\cdot(x_1+x_2\sqrt{\zeta})^3\right)=\left(x_1+x_2\sqrt{\zeta}\right)^3+\\ &\quad +\sigma\left(\left(x_1+x_2\sqrt{\zeta}\right)^3\right)=\\ &= \left(x_1+x_2\sqrt{\zeta}\right)^3+\left(x_1-x_2\sqrt{\zeta}\right)^3=\\ &= 2\left(x_1^3+3\zeta x_1x_2^2\right),\end{aligned}$$

equivalent over k with the cubic $x_1^3+3\zeta x_1x_2^2$.

Case III.2: Daca $p\equiv 2 \mod 3$, i.e. for

$$L^*/k^*L^{*3}\cong\mathbb{Z}/3\mathbb{Z}$$

we can choose as a representative system modulo k^*L^{*3} the elements

$$\left\{\theta^{i} | i=0,1,2\right\},$$

where θ is a primitive root of unity of order p^2-1 such that $\theta^{p+1}=\zeta$, and θ generates L, i.e. $L:=\mathbb{Q}_p\left(\sqrt{\zeta}\right)=\mathbb{Q}_p\left(\theta\right)$. The irreducible polynomial of θ over $\mathbb{Q}_p$ is

$$x^2-\underbrace{\left(\theta+\theta^p\right)}_{Tr_{L/k}\in\mathbb{Q}_p}x+\underbrace{\theta^{p+1}}_{\zeta\in\mathbb{Q}_p}=0,$$

and the conjugate of θ over $\mathbb{Q}_p$ is θ^p, since $\sigma\left(\theta\right)=\theta^p$, where $<\sigma>=Aut_k L=\cong \mathbb{Z}/2\mathbb{Z})$.

The $Aut(L/k)$-orbit of $\theta \mod k^*L^{*3}\in L^*/k^*L^{*3}$ contains $\theta^2 \mod k^*L^{*3}$: $\sigma\left(\theta \mod k^*L^{*3}\right)=\theta^p \mod k^*L^{*3}$, and $\theta^p \mod k^*L^{*3}=\theta^2 \mod k^*L^{*3}$ because $\theta^{p-2}=\zeta(\theta^{-1})^3\in k^*L^{*3}$. Hence, corresponding to the action of Aut_kL on L^*/k^*L^{*3} we get 2 orbits, and a representative system of $Aut_kL\backslash\left(L^*/k^*L^{*3}\right)$ contains, for example, the following elements:

$$Aut_kL\backslash\left(L^*/k^*L^{*3}\right)=\left\{1,\theta\right\}.$$

As in the previous cases, we obtain the associated plane cubis forms:

$$P\left\{\left(\mathbb{Q}_p\left(\sqrt{\zeta}\right),1\right)\right\}=2\left(x_1^3+3\zeta x_1x_2^2\right),$$

equivalent over k with the cubic $x_1^3+3\zeta x_1x_2^2$. Analogously, for θ we obtain the cubic (having coefficients in $\mathbb{Q}_p$):

$$\begin{aligned}P\left\{\left(\mathbb{Q}_p\left(\sqrt{\zeta}\right),\theta\right)\right\} &= \theta\left(x_1^3+3x_1^2x_2\sqrt{\zeta}+3\zeta x_1x_2^2+x_2^3\zeta\sqrt{\zeta}\right)+\\ &\quad +\theta^p\left(x_1^3-3x_1^2x_2\sqrt{\zeta}+3\zeta x_1x_2^2-x_2^3\zeta\sqrt{\zeta}\right)=\\ &= \underbrace{\left(\theta+\theta^p\right)}_{Tr_{L/k}(\theta)}x_1^3+3\underbrace{\left(\theta+\theta^p\right)}_{Tr_{L/k}(\theta)}\zeta x_1x_2^2+3\underbrace{\left(\theta-\theta^p\right)\sqrt{\zeta}}_{Tr_{L/k}(\theta\sqrt{\zeta})}x_1^2x_2+\\ &\quad +\underbrace{\left(\theta-\theta^p\right)\sqrt{\zeta}}_{Tr_{L/k}(\theta\sqrt{\zeta})}\zeta x_2^3.\end{aligned}$$

□

3.2.4 The extension $\mathbb{Q}_p\subset\mathbb{Q}_p\left(\sqrt{\zeta p}\right)$

Case IV: Let $\delta=\sqrt{\zeta p}$ and let $L:=L_{\sqrt{\zeta p}}:=\mathbb{Q}_p\left(\sqrt{\zeta p}\right)$ be the extension of degree 2 over $\mathbb{Q}_p$, where $\zeta\in\mathbb{Q}_p$ is a primitive $(p-1)$-root of unity.

Lemma 3.18. *Let $L:=\mathbb{Q}_p(\sqrt{\zeta p})$ be the extension of degree 2 of $k:=\mathbb{Q}_p$.*

The unique extension v of p-adic valuation v_p to L is given by:

$$v\left(x_1+\sqrt{\zeta p}x_2\right)=min\left\{v_p(x_1),\frac{1}{2}+v_p(x_2)\right\},$$

for all $x_1+x_2\sqrt{\zeta p}\in\mathbb{Q}_p(\sqrt{\zeta p})$.

The valuation group of L is $v(L^*)=(1/2)\mathbb{Z}$. In particular, $1/2$ is the smallest positive element from the total ordered abelian group $v(L)\cong\mathbb{Z}$ and $\sqrt{\zeta p}$ is a local uniformizer of L.

Proof: Analogously with the Lemma 3.8, the p-adic valuation v_p can be uniquely extended to L. Norming $v(p)=1$, we get $2v(\sqrt{p})=v(p)=1$, i.e. $v(\sqrt{p})=\frac{1}{2}$. For $x_1,\ x_2\in\mathbb{Q}_p$, from $v(x_1)\neq v(x_2)+1/2$ it follows that

$$v\left(x_1+\sqrt{\zeta p}x_2\right)=min\left\{v_p(x_1),\frac{1}{2}+v_p(x_2)\right\},$$

as contained. □

In order to simplify the notations, we denote by v both p-adic valuation v_p and its extension v to L.

As in the previous cases, using the lemma 3.18 we can describe the valuation ring $\mathcal{O}_L$, the maximal ideal $\underline{m}_L$ and the group of units $\mathcal{O}_L^*$ as follows:

Corollary 3.19. *For the 2 degree extension of $\mathbb{Q}_p$, $L:=\mathbb{Q}_p(\sqrt{\zeta p})$ we get:*

1) The valuation ring $\mathcal{O}_L=\mathbb{Z}_p[\sqrt{\zeta p}]=\left\{x_1+x_2\sqrt{\zeta p}\,|x_1,\ x_2\in\mathbb{Z}_p\right\}$;

2) The maximal ideal $\underline{m}_L$ is equal to $\underline{m}_L=\left\{x_1p\zeta+x_2\sqrt{\zeta p}\,|x_1,\ x_2\in\mathbb{Z}_p\right\}$;

3) The group of units $\mathcal{O}_L^*$ has the form $\mathcal{O}_L^*=\left\{x_1+x_2\sqrt{\zeta p}\,|x_1\in\mathbb{Z}_p^*,\ x_2\in\mathbb{Z}_p\right\}$.

Proof: **1)** Let $\alpha=x_1+x_2\sqrt{\zeta p}$ be an element from $\mathcal{O}_L$, that is

$$min\left\{v(x_1),\tfrac{1}{2}+v(x_2)\right\}\geq 0.$$

If $min\left\{v(x_1),\frac{1}{2}+v(x_2)\right\}=v(x_1)$, than $v(x_1)\geq 0$ and $\alpha\in\mathbb{Z}_p[\sqrt{\zeta p}]$; if

$$min\left\{v(x_1),\tfrac{1}{2}+v(x_2)\right\}=\tfrac{1}{2}+v(x_2),$$

then $v(x_2)\geq 0$ and $\alpha\in\mathbb{Z}_p[\sqrt{\zeta p}]$. The converse inclusion is easy to verify.

2) Is easy to verify (as in Corollary 3.9).

3) It follows from 1) and 2). □

Using the previous corollary, we obtain that the ramification index is $e=[v(L^*):v(k^*)]=2$, i.e. L is a totally ramified extension of k.

Lemma 3.20. *For* $L := \mathbb{Q}_p\left(\sqrt{\zeta p}\right)$ *we have* $L^* \cong \mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p^2$.

Proof: Analogously with the Lemma 3.10. □

Analogously with the Corollary 3.11, we can obtain the next result:

Corollary 3.21. *With the previous notations, the group of units* $\mathcal{O}_L^*$ *can be described as:*

$$\mathcal{O}_L^* \cong \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p^2 \cong \mathbb{Z}_p^* \times \mathbb{Z}_p.$$

Theorem 3.22. *Let* p *be a prime number,* $p \geq 5$ *and* $L := \mathbb{Q}_p\left(\sqrt{\zeta p}\right)$ *the separable* k*-algebra of degree* 2.

Then the representative system of isomorphism classes over k *of plane cubics with coefficients in* k *is given by*

$$x_1^3 + 3\zeta p x_1 x_2^2 \text{ and } 3x_1^2 x_2 + \zeta p x_2^3.$$

Proof: As in the proof of Theorem 3.12, we obtain:

$$1 \to k^*/k^{*3} \to L^*/L^{*3} \to L^*/k^* L^{*3} \to 1,$$

which, together with the Lema 3.20 proves that L^*/k^*L^{*3} is trivial. Consequently,

$$Aut_k L \backslash \left(L^*/k^*L^{*3}\right) = \{1\}.$$

The associated cubic is obtained as in the previous cases, calculating the trace of the element $1 \cdot \left(x_1 + x_2\sqrt{\zeta p}\right)^3$:

$$\begin{aligned} P\left\{\left(\mathbb{Q}_p(\sqrt{\zeta p}), 1\right)\right\} &:= Tr_{L/k}\left(1 \cdot \left(x_1 + x_2\sqrt{\zeta p}\right)^3\right) = \\ &= 2\left(x_1^3 + 3\zeta p x_1 x_2^2\right), \end{aligned}$$

which is equivalent over k with the cubic $x_1^3 + 3\zeta p x_1 x_2^2$. □

4 The Igusa-equivalence relation on the $GL_2(\mathbb{Q}_p)$-orbits of the representatives of the isomorphism classes

4.1 The Igusa equivalence relation

4.1.1 The definition of the Igusa-equivalence relation and its properties

Let n and d be two natural numbers, different from zero. Let

$$\mathbb{Q}_p[x] = \mathbb{Q}_p[x_1, ..., x_n]$$

be the set of polynomials in n undeterminates with coefficients in $\mathbb{Q}_p$.

Definition 4.1. *To a polynomial $F \in \mathbb{Q}_p[x]$ we associate a natural invariant $v(F) \in \mathbb{Z}$ as follows:*

$$v : \mathbb{Q}_p[x] \to \mathbb{Z} \cup \infty$$

$$v(F) = \begin{cases} \text{the minimum of } p\text{-adic valuation of the coefficients of } F, & \text{if } F \neq 0 \\ \infty, & \text{if } F = 0 \end{cases}$$

For $F \in \mathbb{Q}_p[x]$, $v(F)$ is called ***" the valuation" of the polynomial*** *F.*

Definition 4.2. *A polynomial $F \in \mathbb{Q}_p[x]$ is called* ***primitive*** *if $v(F) = 0$. If*

$$F = p^{v(F)} F_1,$$

then F_1 is called ***the primitive polynomial associated to*** *F.*

Examples 4.1. *1) The plane cubic $F_1(x_1, x_2) := p^3 x_1^3 + x_2^3 \in \mathbb{Q}_p[x_1, x_2]$ is obviously a primitive polynomial.*

2) For $F_2(x_1, x_2) := x_1^3 + p^{-3} x_2^3 \in \mathbb{Q}_p[x_1, x_2]$, we get $v(F_2) = -3$, and the associated primitive cubic is F_1.

Let $F \in \mathbb{Q}_p[x]$; the canonical map $F \mapsto Z_F$, where Z_F is the Igusa local zeta function associated to F (we see Z_F as a rational function in one undeterminate $t := p^{-s}$ with coefficients in $\mathbb{Q}$), induce an equivalence relation called *Igusa equivalence:*

Definition 4.3. *Two polynomials F_1 and F_2 from $\mathbb{Q}_p[x]$ are called* ***Igusa equivalent (I.E.)*** *if the associated zeta functions Z_{F_1} and Z_{F_2} are equal.*

Obviously, the previous relation is an equivalence relation.

Our goal is to describe the equivalence classes modulo the Igusa equivalence relation. In order to do this, we will make the first reduction of our problem: from the definition of the Igusa zeta function we observe that, if we multiply with a $p-$adic unit, the function does not modify. It is natural then to give the following definition:

Definition 4.4. *Let F_1 and F_2 be two polynomial from $\mathbb{Q}_p[x]$. F_1 and F_2 are called* ***homothetical*** *if*

$$F_1 = uF_2, \text{ with } u \in \mathbb{Z}_p^*.$$

Proposition 4.5. *Any two homothetical polynomials F_1, $F_2 \in \mathbb{Q}_p[x]$, with $F_1 = uF_2$, $u \in \mathbb{Z}_p^*$, are Igusa-equivalent.*

Proof: The proof is easy and it uses the definition of Igusa zeta function on the Definition 4.4. □

Remark 4.6. *The reciprocal of the previous proposition, in general, is not true:*

For example , if $p \equiv 1 \mod 3$ and ζ is a $(p-1)$-root of unity, the plane cubics:

$$F_1(x_1,x_2) := x_1^3 + px_2^3, \quad F_2(x_1,x_2) := x_1^3 + p\zeta x_2^3, \quad F_3(x_1,x_2) := x_1^3 + p\zeta^2 x_2^3$$

are not homothetical, but they have the same Igusa zeta function, since for any p-adic unit u,

$$v_p(x_1^3 + pux_2^3) = v_p(x_1^3 + px_2^3) = \begin{cases} 3v_p(x_1), & \text{if} \quad v_p(x_1) \le v_p(x_2), \\ 3v_p(x_2) + 1, & \text{if} \quad v_p(x_1) > v_p(x_2). \square \end{cases}$$

Next, we consider $\mathcal{F}_{n,d}$ the category of homogeneous polynomials (forms) of degree d, with $d \in \mathbb{N}^*$, in $x_1, x_2, ..., x_n$, over $\mathbb{Q}_p$.

For $F \in \mathcal{F}_{n,d}$ is immediate, from the definition of Z_F, that $v(F)$ coincides with the order of formal series Z_F, if $p > d$ (i.e. the prime number p is big enough), hypothesis accepted for now on.

Remark 4.7. *If F_1, $F_2 \in \mathbb{Q}_p[x]$ are two Igusa equivalent forms, then $v(F_1) = v(F_2)$. The reciprocal affirmation, in general, is not true.*

For example, for $p \equiv 1 \mod 3$, the primitive cubics

$$F_1(x_1,x_2) := x_1^3 + px_2^3, \quad F_2(x_1,x_2) := x_1^3 + x_2^3,$$

with $v(F_1) = v(F_2) = 0$, does not have the same zeta functions, because

$$Z_{F_1} := \frac{(p-1)(p+t)}{p^2 - t^3}, \; Z_{F_2} := \frac{(p-1)\left(p^2 - 2p + 2tp - t\right)}{(p-t)(p^2 - t^3)},$$

as we can verify using the Stationary Phase Formula 2.2$\square$.

From the above remark, we observe that considered forms $F \in \mathcal{F}_{n,d}$ can be chosen primitive, i.e. $v(F) = 0$.

4.1.2 The relation of $\mathbb{Z}_p$-, $\mathbb{Q}_p$- and functional equivalence

The set $\mathcal{F}_{n,d}/I.E.$ of classes of Igusa equivalent forms is obviously countable. This fact suggest that it could be possible to describe the classes of Igusa equivalent forms using some numeric invariants. For example, for $F_1, F_2 \in \mathcal{F}_{n,d}$, $v(F_1) = v(F_2)$ is a necessary condition (which is not sufficient) for the forms F_1 and F_2 to be Igusa equivalent. It is natural then to look for some sufficient conditions for Igusa equivalence of forms.

Definition 4.8. *Let $F_1, F_2 \in \mathcal{F}_{n,d}$. F_1 and F_2 are called **$\mathbb{Z}_p$-equivalent** if there exists a matrix $g \in GL_n(\mathbb{Z}_p)$ and a unit $u \in \mathbb{Z}_p^*$ such that*

$$F_1(xg^t) = uF_2(x),$$

where x is the vector $x = (x_1, .., x_n)$, and g^t is the transpose of the matrix g.

Analogously, we can define $\mathbb{Q}_p$-equivalence, with $g \in GL_n(\mathbb{Q}_p)$ and $u \in \mathbb{Q}_p^*$.

Remark 4.9. *We observe that two $\mathbb{Z}_p$-equivalent form are also $\mathbb{Q}_p$-equivalent. The reciprocal of the affirmation is not true: for example, the plane cubics $F_1(x_1, x_2) = px_1^3 + x_2^3$ and $F_2(x_1, x_2) = p^{-2}x_1^3 + x_2^3$ are $\mathbb{Q}_p$-equivalent: for $g = \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \in GL_2(\mathbb{Q}_p)$, $F_1(xg^t) = F_2(x)$, where $x = (x_1, x_2)$.*

But this two forms are not $\mathbb{Z}_p$-equivalent: if there exists a matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ from $GL_2(\mathbb{Z}_p)$ such that $F_1(xg^t) = F_2(x)$, where $x = (x_1, x_2)$ and $u \in \mathbb{Z}_p^*$, it follows that $a^3p + c^3 = p^{-2}u$. Forward, we get $v_p(a^3p + c^3) = -2$, i.e. $min\{1 + 3v_p(a), 3v_p(c)\} = -2$, impossible since $v_p(a)$, $v_p(c) \geq 0\Box$.

Instead the following statement is true:

Proposition 4.10. *If two forms F_1, $F_2 \in \mathcal{F}_{n,d}$ are $\mathbb{Z}_p$-equivalent, they are also Igusa-equivalent.*

Proof: Let F_1, $F_2 \in \mathcal{F}_{n,d}$, $g \in GL_n(\mathbb{Z}_p)$ and $u \in \mathbb{Z}_p^*$ such that $F_1(xg^t) = uF_2(x)$, where x denote the vector $(x_1, ..., x_n)$, and g^t is the transpose of the matrix g.

The Igusa zeta function corresponding to F_2 is then:

$$\begin{aligned} Z_{F_2} \quad &:= \quad Z_{F_2}(s) = \int_{x\in\mathbb{Z}_p^n} |F_2(x)|^s |dx| = \\ &= \int_{x\in\mathbb{Z}_p^n} |u^{-1}F_1(xg^t)|^s |dx| = \int_{x\in\mathbb{Z}_p^n} |F_1(xg^t)|^s |dx| = \\ &= \int_{yg^{-t}\in\mathbb{Z}_p^n} |F_1(y)|^s |\det g||dy| = \\ &= \int_{yg^{-t}\in\mathbb{Z}_p^n} |F_1(y)|^s |dy|, \end{aligned}$$

where we used the following property of the Haar measure (see [Igu00]):

$$meas(gA) = |det(g)|_p \cdot meas(A) = meas(A),$$

for all $g \in GL_n(\mathbb{Z}_p)$ and any open, compact subset A from $\mathbb{Z}_p^n$.

In order to finish the prove, it suffices to observe that, for $g \in GL_2(\mathbb{Z}_p)$, the map $\mathbb{Z}_p^n \to \mathbb{Z}_p^n$, $x \mapsto xg^t$ is a bijection. □

In order to describe the Igusa-equivalent forms, we made the first reductions of the problem: we have reduced our study to the primitive forms and we have identified of the homotetical forms. But this reductions are not strong enough because they did not preserve the specific nature of Igusa local zeta function, as an integral of a function with values determined exclusively of valuation (absolute value). This is why we have to make another reduction of the problem:

Definition 4.11. *Two forms $F_1, F_2 \in \mathcal{F}_{n,d}$ are called **functional equivalent** if $v_p(F_1(x)) = v_p(F_2(x))$, for all $x \in \mathbb{Z}_p^n$, where*

$$\begin{array}{rcl} \mathbb{Z}_p^n & \to & \mathbb{Z} \cup \infty, \\ x & \mapsto & v_p(F_i(x)), \forall x \in \mathbb{Z}_p^n \end{array}$$

for $i = 1, 2$.

The Igusa-equivalence relation identifies the functional equivalent forms:

Proposition 4.12. *Two forms $F_1, F_2 \in \mathcal{F}_{n,d}$ are functional equivalent if they are Igusa-equivalent.*

Proof: Let F_1 and F_2 be two functional equivalent forms. Then obviously,

$$Z_{F_1} = \int_{x \in \mathbb{Z}_p^n} |F_1(x)|^s |dx| = \int_{x \in \mathbb{Z}_p^n} |F_2(x)|^s |dx| = Z_{F_2}.$$

□

In the previous section, in the Theorems 3.7, 3.12, 3.17 and 3.22, we have determined the following representatives of isomorphism classes over $\mathbb{Q}_p$ of plane cubics with coefficients in $\mathbb{Q}_p$:

Corresponding to the extension $\mathbb{Q}_p \subset \mathbb{Q}_p \times \mathbb{Q}_p$, for $p \equiv 1 \mod 3$, we have obtained the plane cubics $x_1^3 + x_2^3$, $px_1^3 + x_2^3$, $\zeta x_1^3 + x_2^3$, $p\zeta x_1^3 + x_2^3$, $p\zeta^2 x_1^3 + x_2^3$, where ζ is a $(p-1)$-root of unity, and for $p \equiv 2 \mod 3$ the cubics $x_1^3 + x_2^3$ and $px_1^3 + x_2^3$.

Since for any p-adic unit u, $v(pux_1^3 + x_2^3) = v(px_1^3 + x_2^3) = 3v(x_1) + 1$, if $v(x_2) > v(x_1)$, respectively $v(x_2)$, if $v(x_2) \leq v(x_1)$, the cubics $p\zeta x_1^3 + x_2^3$, $p\zeta x_1^3 + x_2^3$ and $p\zeta^2 x_1^3 + x_2^3$ are functional equivalent, and so Igusa equivalent. Analogously, the Igusa zeta function associated to the forms $x_1^3 + x_2^3$ and $\zeta x_1^3 + x_2^3$ coincide. Denote $\boxed{F_0(x_1, x_2) := x_1^3 + x_2^3}$, $\boxed{F_1(x_1, x_2) := px_1^3 + x_2^3}$ and, for $p \equiv 1 \mod 3$, we add $F_2(x_1, x_2) := \zeta x_1^3 + x_2^3$.

Corresponding to the extension $\mathbb{Q}_p \subset \mathbb{Q}_p(\sqrt{p})$, we obtained the form

$$\boxed{F_3(x_1, x_2) := x_1^3 + 3px_1x_2^2}.$$

Corresponding to the extension $\mathbb{Q}_p \subset \mathbb{Q}_p(\sqrt{\zeta})$, where ζ is a $(p-1)$-root of unity, we get the cubic $\boxed{F_4(x_1, x_2) := x_1^3 + 3\zeta x_1 x_2^2}$ to which we add, for $p \equiv 2 \mod 3$,

$\boxed{F_5(x_1, x_2) := Tr_{L/k}(\theta)x_1^3 + 3Tr_{L/k}(\theta\sqrt{\zeta})x_1^2x_2 + 3Tr_{L/k}(\theta)\zeta x_1 x_2^2 + Tr_{L/k}(\theta\sqrt{\zeta})\zeta x_2^3} \in \mathbb{Q}_p[x_1, x_2]$, where θ is a (p^2-1) root of unity such that $\theta^{p+1} = \zeta$.

Corresponding to the extension $\mathbb{Q}_p \subset \mathbb{Q}_p(\sqrt{\zeta p})$, we obtained the form $x_1^3 + 3p\zeta x_1 x_2^2$ which is functional equivalent with F_3.

4.2 The Igusa local zeta function of plane cubic F_0

Let us remember that we denoted by F_0 the Fermat's plane cubic

$$\boxed{F_0(x) := F_0(x_1, x_2) := x_1^3 + x_2^3 \in \mathcal{F}_{2,3}}.$$

Proposition 4.13. *The Igusa zeta function Z_{F_0} associated to Fermat's plane cubic $x_1^3 + x_2^3$ is:*

$$\boxed{Z_{F_0} = \begin{cases} \frac{(p-1)(p^2-2p+2tp-t)}{(p-t)(p^2-t^3)}, & \text{if} \quad p \equiv 1 \mod 3 \\ \frac{(p-1)(p^2-t)}{(p-t)(p^2-t^3)}, & \text{if} \quad p \equiv 2 \mod 3, \end{cases}}$$

where $t := p^{-s}$.

Proof: With the notations from Stationary Phase Formula, the only solution of the system $\bar{F}_0(x_1, x_2) = \frac{\partial \bar{F}_0}{\partial x_1}(x_1, x_2) = \frac{\partial \bar{F}_0}{\partial x_2}(x_1, x_2) = 0$ is $(0, 0)$. It follows that

$$Z_{F_0} := Z_{F_0}(s) = \frac{(1 - p^{-1})(1 - p^{-2})t + (1 - p^{-2}N)(1 - t)}{(1 - p^{-1}t)(1 - p^{-2}t^3)},$$

where N is the number of elements of the set $\{(x_1, x_2) \in \mathbb{F}_p^2 | \bar{F}_0(x_1, x_2) = 0\}$, i.e. $N = 3(p - 1) + 1 = 3p - 2$, if $p \equiv 1 \mod 3$, respectively $N = p$, if $p \equiv 2 \mod 3$. The proof is now complete. □

4.3 The Igusa local zeta function of plane cubic F_1

Denoted by F_1 the plane cubic $\boxed{F_1(x) := F_1(x_1, x_2) := px_1^3 + x_2^3 \in \mathcal{F}_{2,3}}$, and by Z_{F_1} the associated Igusa zeta function.

Proposition 4.14. *The Igusa zeta function Z_{F_1} associated to the plane cubic $px_1^3 + x_2^3$ is:*

$$\boxed{Z_{F_1} = \frac{(p - 1)(p + t)}{p^2 - t^3}}.$$

Proof: As in Proposition 4.13, since $N = p$ and $S = \mathbb{Z}_p \times p\mathbb{Z}_p$, we get:

$$Z_{F_1} := Z_{F_1}(s) = 1 - p^{-1} + p^{-1}t \int_{\mathbb{Z}_p^2} |x_1^3 + p^2x_2^3|^s |dx_1||dx_2|.$$

Applying again SPF in order to calculate the integral $\int_{\mathbb{Z}_p^2} |x_1^3 + p^2x_2^3|^s |dx_1||dx_2|$, we obtain

$$Z_{F_1} = \frac{(p - 1)(p + t)}{p^2 - t^3},$$

as contained. □

4.4 The Igusa local zeta function of plane cubic F_2

For $p \equiv 1 \mod 3$ and ζ a $(p-1)$-root of unity, we denote by F_2 the plane cubic $\boxed{F_2(x) := F_2(x_1, x_2) := \zeta x_1^3 + x_2^3 \in \mathcal{F}_{2,3}}$, and by Z_{F_2} the associated Igusa zeta function.

Proposition 4.15. *The Igusa zeta function Z_{F_2} associated to the plane cubic $\zeta x_1^3 + x_2^3$ is:*

$$\boxed{Z_{F_2} = \frac{p^2 - 1}{p^2 - t^3}}.$$

Proof: Using the notation from the Stationary Phase Formula, $\bar{S} = \{(0,0)\}$ and $N = 1$; forward we obtain:

$$\begin{aligned} Z_{F_2} &= \frac{p^{-1}(p-1)^2}{p-t} + p^{-1}t[(1-p^{-1}) + p^{-1}t^2 \int_{\mathbb{Z}_p^2} |3x_1^2 x_2 + p x_2^3|^s |dx_1||dx_2|] = \\ &= \frac{p^{-1}(p-1)^2}{p-t} + p^{-2}t(p-1) + p^{-2}t^3 Z_{F_2}, \end{aligned}$$

from where we obtain Z_{F_2} as required. □

4.5 The Igusa local zeta function of plane cubic F_3

Proposition 4.16. *The Igusa zeta function Z_{F_3} associated to the plane cubic $x_1^3 + 3p x_1 x_2^2$ is:*

$$\boxed{Z_{F_3} = \frac{(p-1)(p^2 + t^2 p - pt - t^2)}{(p-t)(p^2 - t^3)}}.$$

Proof: As in previous propositions, since $\bar{S} = \{(0, x_2) | x_2 \in \mathbb{F}_p\}$, using the stationary phase formula, one can checks that:

$$\begin{aligned} Z_{F_3} &= (1-p^{-1}) + p^{-1}t^2[\frac{p^{-1}(p-1)^2}{p-t} + p^{-1}t \int_{\mathbb{Z}_p^2} |x_1^3 + 3p x_1 x_2^2|^s |dx_1||dx_2|] = \\ &= (1-p^{-1}) + \frac{p^{-2}t^2(p-1)^2}{p-t} + p^{-2}t^3 Z_{F_3}, \end{aligned}$$

and we obtain Z_{F_3} as required. □

4.6 The Igusa local zeta function of plane cubic F_4

Denote by F_4 the plane cubic $F_4(x) := F_4(x_1, x_2) := x_1^3 + 3\zeta x_1 x_2^2 \in \mathcal{F}_{2,3}$, where ζ is a $(p-1)$-root of unity, and by Z_{F_4} the associated Igusa zeta function.

Lemma 4.17. *The Legendre symbol* $\left(\frac{-3\zeta}{p}\right)$ *is:*

$$\left(\frac{-3\zeta}{p}\right) = \begin{cases} -1, & \textit{daca} \quad p \equiv 1 \mod 3, \\ 1, & \textit{daca} \quad p \equiv 2 \mod 3 \end{cases}$$

Proof: Let us remark first that ζ can not be a square modulo p: $\mathbb{F}_p^*$ is a cyclic group of even order $p-1$ and the subgroup of squares $\mathbb{F}_p^{*2}$ is the only subgroup of order 2; consequently, the generator $\bar{\zeta}$ of the group $\mathbb{Z}_p^*/1 + p\mathbb{Z}_p \cong \mathbb{F}_p^*$ can not be a square. That is, $\left(\frac{\zeta}{p}\right) = -1$.

From the properties of Legendre symbol, we get $\left(\frac{-3\zeta}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right)\left(\frac{\zeta}{p}\right)$. Since $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$ and $\left(\frac{\zeta}{p}\right) = -1$, we have: $\left(\frac{-3\zeta}{p}\right) = (-1)^{\frac{p+1}{2}}\left(\frac{3}{p}\right)$.

On the other hand, since p and 3 are prime odd, different numbers we can apply the qadratic reciprocity law: from $\left(\frac{3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{\frac{3-1}{2}\frac{p-1}{2}}$, we obtain $\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right)$. But, for $p \equiv 1 \mod 3$ the Legendre symbol is $\left(\frac{p}{3}\right) = 1$, and for $p \equiv 2 \mod 3$ the Legendre symbol is $\left(\frac{p}{3}\right) = -1$. In this way, we obtained

$$\left(\frac{-3\zeta}{p}\right) = \begin{cases} -1, & \textit{if} \quad p \equiv 1 \mod 3, \\ 1, & \textit{if} \quad p \equiv 2 \mod 3 \end{cases}$$

□

Proposition 4.18. *The Igusa zeta function* Z_{F_4} *associated to the plane cubic* $x_1^3 + 3\zeta x_1 x_2^2$ *is:*

$$Z_{F_4} = \begin{cases} \frac{(p-1)(p^2-t)}{(p^2-t^3)(p-t)}, & \textit{if} \quad p \equiv 1 \mod 3, \\ \\ \frac{(p-1)(p^2+2pt-2p-t)}{(p^2-t^3)(p-t)}, & \textit{if} \quad p \equiv 2 \mod 3, \end{cases}$$

Proof: The reduction modulo p of the given polynomial is $\bar{F}_4(x_1, x_2) = x_1^3 + 3\zeta x_1 x_2^2$, $\frac{\partial \bar{F}_4}{\partial x_1}(x_1, x_2) = 3x_1^2 + 3\zeta x_2^2$, and $\frac{\partial \bar{F}_4}{\partial x_2}(x_1, x_2) = 6\zeta x_1 x_2$. So, the unique solution of the system $\bar{F}_4(x_1, x_2) = 0$, $\frac{\partial \bar{F}_4}{\partial x_1}(x_1, x_2) = 0$, $\frac{\partial \bar{F}_4}{\partial x_2}(x_1, x_2) = 0$ is $(0, 0)$.

We have to count next the number of elements of the set

$$\{(x_1, x_2) \in \mathbb{F}_p^2 | \bar{F}_4(x_1, x_2) = 0\}.$$

Because $\bar{F}_4(x_1, x_2) = x_1^3 + 3\zeta x_1 x_2^2 = x_1(x_1^2 + 3\zeta x_2^2)$, we get p solutions $\mod p$ of the form $\{(0, x_2) | x_2 \in \mathbb{F}_p\}$, to which we add the points $(x_1, x_2) \in \mathbb{F}_p^2$ having both coordinates different from zero, with $\left(\frac{x_1}{x_2}\right)^2 = -3\zeta \mod p$, that is another $2(p-1)$ points of the form $\{(\pm a x_2, x_2) | x_2 \in \mathbb{F}_p\}$, where $a^2 = -3\zeta \mod p$. From the Lemma 4.17, for $p \equiv 2 \mod 3$, there exists such an element a in $\mathbb{F}_p^*$. Thus, we obtain:

$$N = \begin{cases} p, & daca \quad p \equiv 1 \mod 3, \\ 3p-2, & daca \quad p \equiv 2 \mod 3 \end{cases}$$

In order to calculate Z_{F_4} we apply SPF:

$$Z_{F_4} := Z_{F_4}(s) = p^{-2}(p^2 - N) + \frac{p^{-2}(N-1)(1-p^{-1})t}{1-p^{-1}t} + \\ + \int_{p\mathbb{Z}_p \times p\mathbb{Z}_p} |(x_1 + 3\zeta x_1 x_2^2)|^s |dx_1||dx_2|.$$

Since

$$\int_{p\mathbb{Z}_p \times p\mathbb{Z}_p} |(x_1^3 + 3\zeta x_1 x_2^2)|^s |dx_1||dx_2| = p^{-2}t^3 \int_{\mathbb{Z}_p \times \mathbb{Z}_p} |(x_1^3 + 3\zeta x_1 x_2^2)|^s |dx_1||dx_2|,$$

replacing N from the previous relation, we obtain Z_{F_4} as required. □

4.7 The Igusa local zeta function of plane cubic F_5

Let p be a prime number different from 2 and 3, with $p \equiv 2 \mod 3$.

Denote by F_5 the plane cubic

$$F_5(x) := F_5(x_1, x_2) := Tr_{L/k}(\theta)x_1^3 + 3Tr_{L/k}(\theta\sqrt{\zeta})x_1^2 x_2 + \\ + 3Tr_{L/k}(\theta)\zeta x_1 x_2^2 + Tr_{L/k}(\theta\sqrt{\zeta})\zeta x_2^3 \in \mathcal{F}_{2,3}^{pr},$$

where ζ is a $(p-1)$-root of unity and θ is a (p^2-1)-rot of unity such that $\theta^{p+1} = \zeta$, and with Z_{F_5} the associated Igusa zeta function.

Proposition 4.19. *Let $F_5(x)$ be the polynomial*

$$F_5(x_1, x_2) := Tr_{L/k}(\theta)x_1^3 + 3Tr_{L/k}(\theta\sqrt{\zeta})x_1^2 x_2 + \\ + 3Tr_{L/k}(\theta)\zeta x_1 x_2^2 + Tr_{L/k}(\theta\sqrt{\zeta})\zeta x_2^3 \in \mathcal{F}_{2,3}^{pr},$$

and let $\bar{F}_5(x) \in \mathbb{F}_p[x]$ be the polynomial obtained from F_5 reducing modulo p the coefficients.

Then, the equation $\bar{F}_5(x) = 0$ has a unique solution in $\mathbb{F}_p^2$: $(0,0)$.

In the proof of this proposition, we need the following result

Proposition 4.20. *Let $x^3+ax^2+bx+c=0$ be the general form of the equation of degree three, with $a,b,c \in k$ and let $y^3+qy+r=0$, $q,r \in k$ be the equation of degree three obtained from the previous one by the substitution $y = x+\frac{a}{3}$.*

Then the solutions of the second equation are $\alpha+\beta$, $\epsilon\alpha+\epsilon^2\beta$, $\epsilon^2\alpha+\epsilon\beta$, with ϵ a root of unity of degree 3 and

$$\alpha=\sqrt[3]{\frac{1}{2}\left(-r+\sqrt{r^2+\frac{4q^3}{27}}\right)}, \qquad \beta=\sqrt[3]{\frac{1}{2}\left(-r-\sqrt{r^2+\frac{4q^3}{27}}\right)}.$$

Proof: [**Proof of the Proposition 4.20:**] See [Ion05]. □

Proof: [**Proof of the Proposition 4.19:**] Obviously, $(0,0)$ is o solution for $\bar{F}_5(x)=0$. It remains to prove that the equation

$$\overline{Tr_{L/k}(\theta)}x_1^3+3\overline{Tr_{L/k}(\theta\sqrt{\zeta})}x_1^2x_2+3\overline{Tr_{L/k}(\theta)}\bar{\zeta}x_1x_2^2+\overline{Tr_{L/k}(\theta\sqrt{\zeta})}\bar{\zeta}x_2^3=0$$

has no solutions in $\mathbb{F}_p$.

Let us first remark that $\overline{Tr_{L/k}(\theta)} = \overline{\theta}\left(\overline{\theta}^{p-1}+1\right)$ can not be 0 modulo p: since θ is a root of unity, $\overline{\theta} \not\equiv \bar{0} \mod p$. On the other hand, $\bar{\theta}^{p-1}$ can nor be $-\bar{1}$ because, otherwise, $\bar{\theta}^{2p-2}=\bar{1}$, which contradicts the fact that θ is a primitive (p^2-1)-root of unity.

Dividing now the previous equation by $\overline{Tr_{L/k}(\theta)}$, we obtain:

$$t^3+3\frac{\overline{Tr_{L/k}(\theta\sqrt{\zeta})}}{\overline{Tr_{L/k}(\theta)}}t^2+3\bar{\zeta}t+\frac{\overline{Tr_{L/k}(\theta\sqrt{\zeta})}}{\overline{Tr_{L/k}(\theta)}}\bar{\zeta}=0.$$

By the substitution $y=t+\frac{\overline{Tr_{L/k}(\theta\sqrt{\zeta})}}{\overline{Tr_{L/k}(\theta)}}$, we get:

$$y^3-3\left(\frac{\overline{Tr^2_{L/k}(\theta\sqrt{\zeta})}}{\overline{Tr^2_{L/k}(\theta)}}-\bar{\zeta}\right)y+2\frac{\overline{Tr^3_{L/k}(\theta\sqrt{\zeta})}}{\overline{Tr^3_{L/k}(\theta)}}-2\bar{\zeta}\frac{\overline{Tr_{L/k}(\theta\sqrt{\zeta})}}{\overline{Tr_{L/k}(\theta)}}=0.$$

In order to simplify the notations, until the end of this proof, we will not use the "bar" anymore. In the case of the given equation,

$$\begin{aligned} q &= -3\left(\frac{Tr^2_{L/k}(\theta\sqrt{\zeta})}{Tr^2_{L/k}(\theta)}-\zeta\right) = -3\left(\frac{\left(\theta\sqrt{\zeta}-\theta^p\sqrt{\zeta}\right)^2}{(\theta+\theta^p)^2}-\zeta\right) = \\ &= 3\left(\frac{2\zeta}{\theta+\theta^p}\right)^2, \end{aligned}$$

$$\begin{aligned} r &= 2\frac{Tr_{L/k}(\theta\sqrt{\zeta})}{Tr_{L/k}(\theta)}\left(\frac{Tr^2_{L/k}(\theta\sqrt{\zeta})}{Tr^2_{L/k}(\theta)}-\zeta\right) = \frac{\theta\sqrt{\zeta}-\theta^p\sqrt{\zeta}}{\theta+\theta^p}\left(\frac{\left(\theta\sqrt{\zeta}-\theta^p\sqrt{\zeta}\right)^2}{(\theta+\theta^p)^2}-\zeta\right) = \\ &= \frac{-8\zeta^2Tr_{L/k}(\theta\sqrt{\zeta})}{(\theta+\theta^p)^3}, \end{aligned}$$

and the discriminant of the equation is

$$\Delta := r^2 + \frac{4q^3}{27} = \frac{2^6\zeta^5}{Tr^4_{L/k}(\theta)} \neq 0,$$

which means that the roots of the previous equation are simple.

Let us calculate now α^3 and β^3. Replacing r, q and Δ determined above in the formula from the Proposition 4.20 we obtain the followings:

$$\alpha^3 = \frac{1}{2}\left(-r + \sqrt{\Delta}\right) = \left(\frac{2}{Tr_{L/k}(\theta)}\right)^3 \theta^{\frac{5p+7}{2}},$$

and

$$\beta^3 = \frac{1}{2}\left(-r - \sqrt{\Delta}\right) = \left(\frac{-2}{Tr_{L/k}(\theta)}\right)^3 \theta^{\frac{7p+5}{2}}.$$

But p is a prime, odd number. If $p \equiv 1 \mod 3$, i.e. $p = 6u+1$, with $u \in \mathbb{N}^*$, then $\frac{5p+7}{2} = 15u + 6 \equiv 0 \mod 3$ and $\frac{7p+5}{2} = 21u + 6 \equiv 0 \mod 3$. If $p \equiv 2 \mod 3$, i.e. $p = 6u+5$, with $u \in \mathbb{N}^*$, it follows that $\frac{5p+7}{2} = 15u + 16 \equiv 1 \mod 3$ and $\frac{7p+5}{2} = 21u + 20 \equiv 2 \mod 3$. Hence, α^3, $\beta^3 \in \mathbb{F}_{p^2} - \mathbb{F}_p$ iff $p \equiv 2 \mod 3$.

Consequently, for $p \equiv 2 \mod 3$, the given equation in t has tree distinct roots in $\mathbb{F}_{p^2} - \mathbb{F}_p$ and hence is irreducible over $\mathbb{F}_p$. Then the initial equation has a unique solution: $(0,0)$. □

Proposition 4.21. *The zeta function Z_{F_5} associated to the cubic $Tr_{L/k}(\theta)x_1^3 + 3Tr_{L/k}(\theta\sqrt{\zeta})x_1^2x_2 + 3Tr_{L/k}(\theta)\zeta x_1x_2^2 + Tr_{L/k}(\theta\sqrt{\zeta})\zeta x_2^3$, where θ is a (p^2-1)-root of unity, is:*

$$\boxed{Z_{F_5^g} = \frac{p^2-1}{p^2-t^3}},$$

where $t := p^{-s}$.

Proof: The reduction modulo p of the given polynomial is

$$\bar{F}_5(x_1,x_2) = \overline{Tr_{L/k}(\theta)}x_1^3 + 3\overline{Tr_{L/k}(\theta\sqrt{\zeta})}x_1^2x_2 + 3\overline{Tr_{L/k}(\theta)}\bar{\zeta}x_1x_2^2 + \overline{Tr_{L/k}(\theta\sqrt{\zeta})}\bar{\zeta}x_2^3 = 0,$$

$$\frac{\partial \bar{F}_5}{\partial x_1}(x_1,x_2) = 3(\overline{Tr_{L/k}(\theta)}x_1^2 + 2\overline{Tr_{L/k}(\theta\sqrt{\zeta})}x_1x_2 + \overline{Tr_{L/k}(\theta)}\bar{\zeta}x_2^2),$$

and $\frac{\partial \bar{F}_5}{\partial x_2}(x_1,x_2) = 3(\overline{Tr_{L/k}(\theta\sqrt{\zeta})}x_1^2 + 2\overline{Tr_{L/k}(\theta)}\bar{\zeta}x_1x_2 + \overline{Tr_{L/k}(\theta\sqrt{\zeta})}\bar{\zeta}x_2^2)$.

From the Proposition 4.19, it results that the only solution of the system $\bar{F}_5(x_1,x_2) = 0$, $\frac{\partial \bar{F}_5}{\partial x_1}(x_1,x_2) = 0$, $\frac{\partial \bar{F}_5}{\partial x_2}(x_1,x_2) = 0$ is $(0,0)$.

In order to calculate Z_{F_5} we apply the Stationary Phase Formula 2.2. With the notations from there, we have: $\bar{S} = \{(0,0)\}$ and $N = 1$. Then:

$$Z_{F_5} = Z_{F_5}(s) = 1 - p^{-1} + p^{-2}t^3 \int_{\mathbb{Z}_p^2} |F_5(x_1,x_2)|^s |dx_1||dx_2|,$$

from where we obtain Z_{F_5} as contained. □

Remark 4.22. *From the proof of the Proposition 4.19 it results that, for* $p \equiv 1 \mod 3$*, the number* N *of solutions of the equation* $\bar{F}_5(x) = 0$ *is* $N = 3(p-1)+1 = 3p-2$.

Proposition 4.23. *For* $p \equiv 1 \mod 3$*, the zeta function* Z_{F_5} *associated to the cubic* $Tr_{L/k}(\theta)x_1^3 + 3Tr_{L/k}(\theta\sqrt{\zeta})x_1^2x_2 + 3Tr_{L/k}(\theta)\zeta x_1 x_2^2 + Tr_{L/k}(\theta\sqrt{\zeta})\zeta x_2^3$*, where* θ *is a* (p^2-1)*-root of unity is*

$$Z_{F_5^g} = Z_{F_0}.$$

Proof: Analogously to the proof of the Proposition 4.21, using the Remark 4.22, we get $\bar{S} = \{(0,0)\}$ and $N = 3p-2$. Then

$$\begin{aligned} Z_{F_5} &= Z_{F_5}(s) = p^{-2}(p^2-3p+2) + p^{-2}(3p-3)\frac{(1-p^{-1})t}{1-p^{-1}t} + \\ &\quad + \int_{(p\mathbb{Z}_p)^2} |F_5(x_1,x_2)|^s |dx_1||dx_2| = \\ &= p^{-2}(p^2-3p+2) + 3p^{-2}(p-1)\frac{(p-1)t}{p-t} + p^{-2}t^3 Z_{F_5}, \end{aligned}$$

and, consequently, $Z_{F_5} = \frac{(p-1)(p^2+2pt-t-2p)}{(p-t)(p^2-t^3)} = Z_{F_0}$, as required. □

References

[Bru02] L.Brünjes. *Über die Zetafunktion von Formen von fermatgleichungen. Dissertattion zur erlangung des doktorgrades* Ph.D. Thesis, Regensburg University, 2002; http://www.bibliothek.uni-regensburg.de/opus/volltexte/2002/98/

[Den84] J. Denef. *The rationality of Poincaré series associated to the p-adic points on a variety.* Invent.Math., 77: 1-23, 1984;

[Iba05a] D. Ibadula. *The arboreal structure of the metric space* $X := GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^* GL_2(\mathbb{Z}_p)$, Preprint 2005;

[Iba05b] D. Ibadula. *On the Igusa Local Zeta Function of Non-degenerated Binary Cubic Forms*, Communication on the XIV-th edition of the National School of Algebra, Constanta 2005;

[Igu00] J.-I. Igusa. *An Introduction to the Theory of Local Zeta Functions.* volume 14 of Studies in Advanced Mathematics. Amer.Math.Soc.International Press, U.S.A., 2000;

[Ion05] Cristodor Ionescu. *Ecuatii algebrice*. Ovidius University Press, Constanta, 2005;

[Neu99] J.Neukirch. *Algebraic Number Theory*. Translated from the German by Norbert Schappacher, Springer Verlag, Berlin-Heidelberg-New York, 1999.

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Faculty of Mathematics and Informatics,
"Ovidius" University, Constanţa
Mamaia Bd. 124
RO – 900527 Constanţa, ROMANIA
E-mail: denis@univ-ovidius.ro