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Some properties of the class of $n-$starlike functions

by
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Abstract

In this paper we study some properties of the classes $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ of univalent functions with negative coefficients.

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1 Introduction

Let U denote the open unit disc: $U = \{z \ ; \ z \in \mathbb{C} \ , \ |z| < 1\}$, let $\mathcal{A}$ denote the class of functions

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1}$$

which are analytic in U, and let S denote the class of functions of the form (1) wich are analytic and univalent in U.

A function $f(z) \in \mathcal{A}$ is said to be starlike of order $\alpha \, (0 \leq \alpha < 1)$ in the unit disk U if

$$Re \frac{zf'(z)}{f(z)} > \alpha$$

for all $z \in U$.

For $f \in S$ we define the differential operator D^n (Sălăgean [3])

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= Df(z) = zf'(z) \end{aligned}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad ; \quad n \in \mathbb{N}^* = \{1, 2, 3, ...\} .$$

We note that if

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j,$$

then

$$D^nf(z)=z+\sum_{j=2}^{\infty}j^na_jz^j\ ;\ z\in U.$$

Let T denotes the subclass of S containing the functions which can be expressed in the form:

$$f(z)=z-\sum_{k=2}^{\infty}a_kz^k\ ;\quad a_k\geq 0\ ,\quad \forall\ k\geq 2.$$

We say that a function $f\in T$ is in the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$, $0\leq\alpha<1$, $0<\beta\leq 1$, $-1\leq A<B\leq 1$, $0<B\leq 1$ and

$$\frac{B}{B-A}<\gamma\leq\begin{cases}\dfrac{B}{(B-A)\alpha} & ;\quad \alpha\neq 0\\ 1 & ;\quad \alpha=0\end{cases}$$

if

$$\left|\frac{\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)}-1}{(B-A)\gamma\left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)}-\alpha\right]-B\left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)}-1\right]}\right|<\beta\quad ,\quad z\in U$$

where

$$F_{n,\lambda}(z)=(1-\lambda)D^nf(z)+\lambda D^{n+1}f(z)\ \ ;\ \lambda\geq 0\ \ ;\ f\in T$$

For this class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$, in Holhoş [1] it is showed the following lemma:

Lemma 1. *Let* $f\in T$, $f(z)=z-\sum\limits_{k=2}^{\infty}a_kz^k$; $a_k\geq 0$, $\forall\ k\geq 2$. *Then* $f(z)$ *is in the class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ *if and only if*

$$\sum_{k=2}^{\infty}a_kk^n\left[1+\lambda(k-1)\right]\left\{(k-1)+\beta\left[(B-A)\gamma(k-\alpha)-B(k-1)\right]\right\}\leq\beta\gamma(B-A)(1-\alpha)\tag{2}$$

and the result is sharp.

If we denote

$$D_n\left(k,A,B,\alpha,\beta,\gamma,\lambda\right)=k^n\left[1+\lambda\left(k-1\right)\right]\left\{(k-1)+\beta\left[(B-A)\gamma\left(k-\alpha\right)-B\left(k-1\right)\right]\right\}$$

then (2) can be rewritten

$$\sum_{k=2}^{\infty}a_kD_n\left(k,A,B,\alpha,\beta,\gamma,\lambda\right)\leq\beta\gamma(B-A)(1-\alpha).$$

2 Some properties of the classes $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$.

Remark 2. *From the definition of the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ it is easy to see that if $0<\beta_1\leq\beta_2\leq 1$, then $T_{n,\lambda}(A,B,\alpha,\beta_1,\gamma)\subset T_{n,\lambda}(A,B,\alpha,\beta_2,\gamma)$.*

Theorem 3. *Let $0\leq\alpha_2\leq\alpha_1<1$; $0<\beta_1\leq\beta_2\leq 1$ and*

$$\frac{B}{B-A}<\gamma_1\leq\gamma_2\leq\begin{cases}\frac{B}{(B-A)\alpha};\ \alpha\neq 0\\ 1;\ \alpha=0\end{cases}.$$

Then we have $T_{n,\lambda}(A,B,\alpha_1,\beta_1,\gamma_1)\subset T_{n,\lambda}(A,B,\alpha_2,\beta_2,\gamma_2)$.

Proof: Let $f\in T_{n,\lambda}(A,B,\alpha_1,\beta_1,\gamma_1)$ and $\alpha_1=\alpha_2+\delta$; $\beta_1=\beta_2-\varepsilon$; $\gamma_1=\gamma_2-\theta$ where $\delta,\varepsilon,\theta\geq 0$. Then, by using Lemma 1 we have

$$\sum_{k=2}^{\infty}a_kD_n\left(k,A,B,\alpha_1,\beta_1,\gamma_1,\lambda\right)\leq\beta_1\gamma_1(B-A)(1-\alpha_1)\Leftrightarrow$$

$$\sum_{k=2}^{\infty}a_kD_n\left(k,A,B,\alpha_2+\delta,\beta_2-\varepsilon,\gamma_2-\theta,\lambda\right)\leq\beta_1\gamma_1(B-A)(1-\alpha_1)$$

because

$$D_n\left(k,A,B,\alpha_2+\delta,\beta_2-\varepsilon,\gamma_2-\theta,\lambda\right)=D_n\left(k,A,B,\alpha_2,\beta_2,\gamma_2,\lambda\right)-$$

$$-k^n\left[1+\lambda\left(k-1\right)\right]\beta_2\left(B-A\right)\left[\theta\left(k-\alpha_2-\delta\right)+\gamma_2\delta\right]$$

$$-k^n\left[1+\lambda\left(k-1\right)\right]\varepsilon\left[\left(B-A\right)\left(\gamma_2-\theta\right)\left(k-\alpha_2-\delta\right)-B\left(k-1\right)\right]$$

we have

$$\sum_{k=2}^{\infty}a_kD_n\left(k,A,B,\alpha_2,\beta_2,\gamma_2,\lambda\right)\leq\beta_1\gamma_1(B-A)(1-\alpha_1)+$$

$$+\sum_{k=2}^{\infty}a_kk^n\left[1+\lambda\left(k-1\right)\right]\beta_2\left(B-A\right)\left[\theta\left(k-\alpha_2-\delta\right)+\gamma_2\delta\right]+$$

$$+\sum_{k=2}^{\infty}a_kk^n\left[1+\lambda\left(k-1\right)\right]\varepsilon\left[\left(B-A\right)\left(\gamma_2-\theta\right)\left(k-\alpha_2-\delta\right)-B\left(k-1\right)\right]=$$

$$=\beta_1\gamma_1(B-A)(1-\alpha_1)+\beta_2\left(B-A\right)\theta\sum_{k=2}^{\infty}a_kk^n\left[1+\lambda\left(k-1\right)\right]\left(k-\alpha_2-\delta\right)+$$

$$+\beta_2\left(B-A\right)\gamma_2\delta\sum_{k=2}^{\infty}a_kk^n\left[1+\lambda\left(k-1\right)\right]+$$

$$+\varepsilon\sum_{k=2}^{\infty}a_k k^n\left[1+\lambda\left(k-1\right)\right]\left[\left(B-A\right)\left(\gamma_2-\theta\right)\left(k-\alpha_2-\delta\right)-B\left(k-1\right)\right]\leq$$
$$\leq\beta_1\gamma_1(B-A)(1-\alpha_1)+\beta_2\left(B-A\right)\theta\left(1-\alpha_1\right)+\beta_2\left(B-A\right)\gamma_2\delta+$$
$$+\varepsilon\gamma_1\left(B-A\right)\left(1-\alpha_1\right)=\beta_2\gamma_2(B-A)(1-\alpha_2)$$

According to Lemma 1 we obtain $f\in T_{n,\lambda}(A,B,\alpha_2,\beta_2,\gamma_2)$ and $T_{n,\lambda}(A,B,\alpha_1,\beta_1,\gamma_1)\subset T_{n,\lambda}(A,B,\alpha_2,\beta_2,\gamma_2)$. □

Corollary 4. *Let $0\leq\alpha_1\leq\alpha_2<1$. Then we have $T_{n,\lambda}(A,B,\alpha_1,\beta,\gamma)\supset T_{n,\lambda}(A,B,\alpha_2,\beta,\gamma)$.*

Corollary 5. *Let*

$$\frac{B}{B-A}<\gamma_1\leq\gamma_2\leq\begin{cases}\frac{B}{(B-A)\alpha};\ \alpha\neq0\\ 1;\ \alpha=0\end{cases}.$$

Then $T_{n,\lambda}(A,B,\alpha,\beta,\gamma_1)\subset T_{n,\lambda}(A,B,\alpha,\beta,\gamma_2)$.

Definition 6. *Let $f,g\in T$,*

$$f(z)=z-\sum_{k=2}^{\infty}a_k z^k;\ a_k\geq0,\ \forall\ k\geq2 \tag{3}$$

and

$$g(z)=z-\sum_{k=2}^{\infty}b_k z^k;\ b_k\geq0,\ \forall\ k\geq2, \tag{4}$$

then we define the Hadamard product of f and g by

$$f*g(z)=z-\sum_{k=2}^{\infty}a_k b_k z^k.$$

The folowing teorem is proved in [2].

Theorem 7. *If the functions f and g defined by (3) and (4) belong to the same class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$, then the Hadamard product $f*g$ belongs to $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$.*

Theorem 8. *Let $0\leq\alpha_2\leq\alpha_1<1$; $0<\beta_1\leq\beta_2\leq1$ and*

$$\frac{B}{B-A}<\gamma_1\leq\gamma_2\leq\begin{cases}\frac{B}{(B-A)\alpha};\ \alpha\neq0\\ 1;\ \alpha=0\end{cases}.$$

If the functions f defined by (3) be in the class $T_{n,\lambda}(A,B,\alpha_1,\beta_1,\gamma_1)$ and g defined by (4) be in the class $T_{n,\lambda}(A,B,\alpha_2,\beta_2,\gamma_2)$, then the Hadamard product $f*g$ belongs to the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$, where $\alpha=\min\left(\alpha_1,\alpha_2\right)$, $\beta=\max\left(\beta_1,\beta_2\right)$ and $\gamma=\max\left(\gamma_1,\gamma_2\right)$.

Proof: Since

$$\alpha = \min(\alpha_1, \alpha_2) \Rightarrow \alpha \le \alpha_1 \text{ and } \alpha \le \alpha_2$$

$$\beta = \max(\beta_1, \beta_2) \Rightarrow \beta \ge \beta_1 \text{ and } \beta \ge \beta_2$$

$$\gamma = \max(\gamma_1, \gamma_2) \Rightarrow \gamma \ge \gamma_1 \text{ and } \gamma \ge \gamma_2$$

from Theorem 3 we have $f \in T_{n,\lambda}(A, B, \alpha_1, \beta_1, \gamma_1) \Rightarrow f \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ and $g \in T_{n,\lambda}(A, B, \alpha_2, \beta_2, \gamma_2) \Rightarrow g \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$. From Theorem 7 we have $f * g \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$. □

Theorem 9. *Let* $-1 \le A_2 \le A_1 < B_1 \le B_2 \le 1$, $0 < B_1$. *Then we have* $T_{n,\lambda}(A_1, B_1, \alpha, \beta, \gamma) \subset T_{n,\lambda}(A_2, B_2, \alpha, \beta, \gamma)$.

Proof: Let $f \in T_{n,\lambda}(A_1, B_1, \alpha, \beta, \gamma)$, $A_1 \ge A_2$ and $B_2 = B_1 + \delta$ where $\delta \ge 0$. Then, by using Lemma 1 we have

$$\sum_{k=2}^{\infty} a_k D_n(k, A_1, B_1, \alpha, \beta, \gamma, \lambda) \le \beta\gamma(B_1 - A_1)(1 - \alpha)$$

and

$$\sum_{k=2}^{\infty} a_k k^n [1 + \lambda(k-1)] \beta [\gamma(k-\alpha) - (k-1)] \le \sum_{k=2}^{\infty} a_k D_n(k, A_1, B_1, \alpha, \beta, \gamma, \lambda).$$

From this

$$\sum_{k=2}^{\infty} a_k D_n(k, A_2, B_2, \alpha, \beta, \gamma, \lambda) \le \sum_{k=2}^{\infty} a_k D_n(k, A_1, B_2, \alpha, \beta, \gamma, \lambda) =$$

$$= \sum_{k=2}^{\infty} a_k D_n(k, A_1, B_1 + \delta, \alpha, \beta, \gamma, \lambda) = \sum_{k=2}^{\infty} a_k D_n(k, A_1, B_1, \alpha, \beta, \gamma, \lambda) +$$

$$+ \delta \sum_{k=2}^{\infty} a_k k^n [1 + \lambda(k-1)] \beta [\gamma(k-\alpha) - (k-1)] \le \beta\gamma(B_1 - A_1)(1-\alpha) +$$

$$+ \delta\beta\gamma(1-\alpha) \le \beta\gamma(B_1 - A_2)(1-\alpha) + \delta\beta\gamma(1-\alpha) = \beta\gamma(B_2 - A_2)(1-\alpha)$$

and according to Theorem 2 we obtain $f \in T_{n,\lambda}(A_2, B_2, \alpha, \beta, \gamma)$ which implies that $T_{n,\lambda}(A_1, B_1, \alpha, \beta, \gamma) \subset T_{n,\lambda}(A_2, B_2, \alpha, \beta, \gamma)$. □

Theorem 10. $T_{n,\lambda}(A, B, \alpha, \beta, \gamma) \supset T_{n+1,\lambda}(A, B, \alpha, \beta, \gamma)$.

Proof: Since $f(z) \in T_{n+1,\lambda}(A, B, \alpha, \beta, \gamma)$, by using Lemma 1 we have

$$\sum_{k=2}^{\infty} a_k D_{n+1}(k, A, B, \alpha, \beta, \gamma, \lambda) \leq \beta\gamma(B - A)(1 - \alpha);$$

if

$$k^n < k^{n+1};\ \forall\, k \geq 2 \text{ and } \forall\, n \geq 0,$$

then

$$D_n(k, A, B, \alpha, \beta, \gamma, \lambda) \leq D_{n+1}(k, A, B, \alpha, \beta, \gamma, \lambda) \ ; \quad \forall\, n \geq 0$$

and

$$\sum_{k=2}^{\infty} a_k D_n(k, A, B, \alpha, \beta, \gamma, \lambda) \leq \sum_{k=2}^{\infty} a_k D_{n+1}(k, A, B, \alpha, \beta, \gamma, \lambda) \leq \beta\gamma(B - A)(1 - \alpha)$$

According to Lemma 1 we obtain $f \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma) \Rightarrow T_{n,\lambda}(A, B, \alpha, \beta, \gamma) \supset T_{n+1,\lambda}(A, B, \alpha, \beta, \gamma)$. □

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