Bull. Math. Soc. Sc. Math. Roumanie
Tome 49(97) No. 2, 2006, 157–169

Continuous selections of solution sets of differential-difference inclusions

by
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Abstract

Extending some results in the literature, certain continuous versions of Filippov theorem for Lipschitz differential inclusions and Filippov-Ważewski Relaxation theorem are obtained for parameterized differential-difference inclusions.

Key Words: Differential-difference inclusion, Filippov's theorem, relaxation.
2000 Mathematics Subject Classification: Primary: 34A60, Secondary 34K99.

1 Introduction

In this note we study the parameterized differential-difference inclusion

$$\dot{x}(t) \in F(t, x(t), x(t-\Delta), s), \quad x(t_0) = a(s) \tag{1}$$

and its relaxation

$$\dot{x}(t) \in \overline{co}F(t, x(t), x(t-\Delta), s), \quad x(t_0) = a(s) \tag{2}$$

where $t \in I := [t_o, T], \Delta \in (0, T-t_o),\ s \in S,\ F(.,.,.,.) : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times S \to \mathcal{P}(\mathbb{R}^n)$ is a set-valued map and S is a Banach space.

A fundamental result for Lipschitz differential inclusions is the Filippov-Ważewski Relaxation theorem, which provides approximations of trajectories of a relaxed inclusion in terms of the solutions of the initial inclusion. In this paper, we obtain continuous versions of Filippov theorem and Filippov-Ważewski Relaxation theorem for differential-difference inclusions. Such results can be found in literature also for other classes of differential inclusions (e.g., [2], [5], [6], [8] etc.).

The results obtained in this note may be interpreted, on one hand, as extensions of the results in [5], [6], [8] to the differential-difference inclusions and, on the other hand, as extensions to the parametric (continuous) case of the results in [4]. We mention that our proof is based on an argument different from the one used in [4]; namely, it relies on a selection theorem of Bressan and Colombo, concerning selections of l.s.c. set-valued maps with decomposable values ([1]), instead of the selection theorem due to Kuratowski and Ryll-Nardzewski, used in [4].

The motivation for this work is given by its possible applications to the study of optimal control problems defined by differential-difference inclusions. In more precise terms, the obtained results play an important part in study both of necessary optimality conditions and in the study of local controllability sufficient conditions for control problems defined by differential-difference inclusions ([3]).

Definitions, notations and basic results are given in the next section and the main results are presented in Section 3.

2 Preliminaries

In what follows, X is a separable Banach space whose norm is denoted by $\|.\|$ and $\mathcal{P}(X)$ will stand for the set of all subsets of X. If $A \subset X$, by $\mathrm{cl}(A)$ and $\overline{co}A$ we mean the closure and the closed convex hull of A, respectively. For any interval $I := [a, b] \subset \mathbb{R}$, denote by $\mathcal{L}(I)$ the σ-field of Lebesgue measurable subsets of I and by $\mathcal{B}(S)$ the family of Borel subsets of S. Let $L^1(I, X)$ be the space of integrable functions $x(.) : I \to X$ equipped with norm $||x(.)||_1 = \int_I \|x(s)\|ds$ and $AC(I, X)$ the Banach space of absolutely continuous functions $x(.) : I \to X$, endowed with the norm $\|x(.)\|_{AC} = \|x(a)\| + \|\dot{x}(.)\|_1$.

A subset $K \subseteq L^1(I, X)$ is said to be decomposable if, for any $u, v \in K$ and any $A \in \mathcal{L}(I)$,

$$u \cdot \chi_A + v \cdot \chi_{I \setminus A} \in K,$$

where χ_A stands for the characteristic function of A. The family of all nonempty closed and decomposable subsets of $L^1(I, X)$ is denoted by $\mathcal{D}(I, X)$.

Given a set $A \in \mathcal{P}(X)$ and $x \in X$, by $d(x, A)$ we mean the distance from the point x to the set A. For any $A, B \in \mathcal{P}(X)$, the Haussdorff distance between A and B is defined as

$$d_H(A, B) := \max\big\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\big\}.$$

Denote by $\mathcal{F}([t_0 - \Delta, T], \mathbb{R}^n)$ the family of all functions $x(.) : [t_0 - \Delta, T] \to \mathbb{R}^n$ which are absolutely continuous on I and essentially bounded on $[t_0 - \Delta, t_0)$. Given $s \in S$, a solution to the differential-difference inclusion (1) is a function $x(.) \in \mathcal{F}([t_0 - \Delta, T], \mathbb{R}^n)$ which satisfies (1) for almost all t in I. Let S_F be the solution set of (1) and let $R_F(T)$ be the "reachable set" of (1), defined by:

$$R_F(T) := \{x(T);\ x(\cdot) \in S_F\}.$$

The following lemmas are useful in proving the results in Section 3.

Lemma 2.1. **([9])** *Let $u(.) : I \to X$ be a measurable function and $G(.) : I \to \mathcal{P}(X)$ a set-valued map with nonempty closed images.*

Then, for any measurable function $r(.) : I \to (0, \infty)$, there exists an integrable selection $g(.) : I \to X$ of $G(.)$ (i.e. $g(t) \in G(t)$ a.e.(I),) such that

$$||u(t) - g(t)|| < d(u(t), G(t)) + r(t) \quad a.p.t.(I).$$

Lemma 2.2. **([5])** *Let $F^*(.,.) : I \times S \to \mathcal{P}(X)$ be a set-valued map with nonempty closed images, $\mathcal{L}(I) \otimes \mathcal{B}(S)$-measurable and such that $F^*(t,.)$ is lower semi-continuous (l.s.c.), for all $t \in I$.*

Then, $G(.) : S \to \mathcal{D}(I, X)$ defined by

$$G(s) = \{v(.) \in L^1(I, X); \quad v(t) \in F^*(t, s) \quad a.p.t.(I)\}$$

is lower semi-continuous from S to $\mathcal{D}(I, X)$ iff there exists a continuous function $p(.) : S \to L^1(I, X)$ such that

$$d(0, F^*(t, s)) \leq p(s)(t) \quad a.p.t.(I), \quad \forall s \in S.$$

Lemma 2.3. **([1])** *Let $G(.) : S \to \mathcal{D}(I, X)$ be a l.s.c set-valued map and the continuous functions $\phi(.) : S \to L^1(I, X)$, $\psi(.) : S \to L^1(I, \mathbb{R})$ such that $H(.) : S \to \mathcal{D}(I, X)$ given by*

$$H(s) = cl\{v(.) \in G(s); \; ||v(t) - \phi(s)(t)|| < \psi(s)(t) \; a.p.t.(I)\}.$$

has nonempty values. Then $H(.)$ admits a continuous selection.

Lemma 2.4. **([7])** *Let $H(.) : I \to \mathcal{P}(X)$ be a measurable set-valued map, integrable bounded, with nonempty closed values. Then:*

$$cl \int_{t_0}^{T} \overline{co} H(t) dt = cl \int_{t_0}^{T} H(t) dt.$$

Lemma 2.5. **([6])** *Let (S, d) be a separable metric space, $J \subset \mathbb{R}$, and let $G(.) : S \to \mathcal{D}(J, X)$ be a l.s.c set-valued map. Then, for each continuous selection $f(s) \in F(s) := cl(\int_J g(s)(t)dt)$ and for any $\varepsilon > 0$, there exists a continuous selection $h(s) \in G(s), \forall s \in S$ such that*

$$||f(s) - \int_J h(s)(t) dt|| < \varepsilon, \quad \forall s \in S.$$

We shall assume the following Hypothesis on $F(.,.,.,.)$ and $a(.)$.

Hypothesis 2.6. (H1) *$F(.,.,.,.) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathcal{P}(\mathbb{R}^n)$ is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n \times S)$-measurable, with nonempty closed values and $a(.) : S \to \mathbb{R}^n$ is a continuous function.*

(H2) *There exists a continuous map $r(.) : S \to L^1(I, \mathbb{R}_+^*)$ such that, for any $s \in S$ and almost all $t \in I$, $F(t,.,.,s)$ is $r(s)(t)$-Lipschitz, i.e. for all $x_i, y_i \in \mathbb{R}^n, i = \overline{1,2}$,*

$$d_H(F(t,x_1,y_1,s), F(t,x_2,y_2,s)) \le r(s)(t)\big[|x_1 - x_2| + |y_1 - y_2|\big].$$

(H3) *For any $(t,x,y) \in I \times \mathbb{R}^n \times \mathbb{R}^n$, the set-valued map $F(t,x,y,.)$ is l.s.c. on S.*

(H4) *There exists a continuous map $p_0(.) : S \to L^1(I, \mathbb{R})$ such that, for any $s_0 \in S$ there is a neighborhood $V(s_0) \in \mathcal{V}(s_0)$, satisfying:*

$$d(0_n, F(t, 0_n, 0_n, s)) \le p_0(s)(t) \quad a.e.(I), \forall s \in V(s_0).$$

Remark 2.7. In view of assumption (H2), the assumption (H4) may be replaced by:

(H4′) *For any map $y(.) : S \to \mathcal{F}([t_o - \Delta, T], \mathbb{R}^n)$ such that $s \to y(s)(.)|_I$ is continuous, there exists a continuous map $p_y(.) \to L^1(I, \mathbb{R}_+)$ such that, for every $s \in S$*

$$d(\dot{y}(s)(t), F(t, y(s)(t), y(s)(t-\Delta, s)) \le p_y(s)(t) \quad a.e.(I). \tag{3}$$

Indeed, it easily follows from the inequality

$$d(\dot{y}(s)(t), F(t, y(s)(t), y(s)(t-\Delta, s)) \le$$
$$\le \|\dot{y}(s)(t)\| + d(0_n, F(t, 0_n, 0_n, s)) + r(s)(t)\big[\|y(s)(t)\| + y(s)(t-\Delta)\|\big], \ a.e.(I).$$

3 Main results

We prove first a continuous version of Filippov's theorem for Lipschitz differential-difference inclusions. For this purpose, let $m(s,t) = \int_{t_0}^t r(s)(u)du$ and $b(s) = ||a(s) - y(s)(t_0)||$; for $y(.) : S \to \mathcal{F}([t_o - \Delta, T], \mathbb{R}^n)$ and $p_y(.) \to L^1(I, \mathbb{R}_+)$ verifying (3) and for any $s \in S$, we define

$$\xi_y(s,t) = \begin{cases} e^{2m(s,t)}[\varepsilon(t-t_0) + b(s)] + \int_{t_0}^t p_y(s)(u) e^{2(m(s,t)-m(s,u))} du, & t \ge t_0 \\ 0, \quad t \in [t_0 - \Delta, t_0). \end{cases}$$

Theorem 3.1. *Assume that Hypothesis 2.6 is satisfied.*

Then, for any $\varepsilon > 0$, any function $y(.) : S \to \mathcal{F}([t_0 - \Delta, T], \mathbb{R}^n)$ such that $s \to y(s)(.)|_I$ is continuous and for any map $p_y(.) : S \to L^1(I, \mathbb{R})$ satisfying (3), there exists a function $x(.) : S \to \mathcal{F}([t_0 - \Delta, T], \mathbb{R}^n)$ such that:

i) for every $s \in S$, $s \to x(s)(.)|_I$ is continuous, $x(s)(.)$ is a solution of (1) and $x(s)(t) = y(s)(t), \forall t \in [t_0 - \Delta, t_0), s \in S$;

ii) for every $s \in S$ and $t \in I$,

$$\|x(s)(t) - y(s)(t)\| \le \xi_y(s,t); \tag{4}$$

iii) for every $s \in S$ and almost every $t \in I$,

$$\|\dot{x}(s)(t) - \dot{y}(s)(t)\| \le r(s)(t)[\xi_y(s,t) + \xi_y(s,t-\Delta)] + p_y(s)(t) + \varepsilon. \tag{5}$$

Proof: Let $\varepsilon > 0$ and $\varepsilon_n = \varepsilon\frac{n+1}{n+2}, n \geq 0$. For any $s \in S$, we define the following sequence

$$p_n(s)(t) = \int_{t_0}^{t} p_y(s)(u)\frac{[2(m(s,t)-m(s,u))]^{n-1}}{(n-1)!}du+ \\ + \frac{[2m(s)(t)]^{n-1}}{(n-1)!}(\varepsilon_n(t-t_0)+b(s)), \forall t \in (t_0, T]$$

and $p_n(s)(t) = 0, \forall t \in [t_0 - \Delta, t_0]$ and every $n \geq 1$.
Set $x_0(s)(t) = y(s)(t)$ and consider the set-valued maps $G_0(.), H_0(.)$ given by

$$G_0(s) = \{v \in L^1(I, \mathbb{R}^n), v(t) \in F(t, y(s)(t), y(s)(t-\Delta), s) \ a.e.(I)\},$$

$$H_0(s) = cl\{v \in G_0(s) : \|v(t) - \dot{y}(s)(t)\| < p_y(s)(t) + \varepsilon_0\}.$$

Since $d(\dot{y}(s)(t), F(t, y(s)(t), y(s)(t-\Delta), s)) \leq p_y(s)(t)$, from Lemma 2.1, it follows that $H_0(s) \neq \varnothing, \forall s \in S$. Moreover,

$$d(0, F(t, y(s)(t), y(s)(t-\Delta), s)) \leq \|\dot{y}(s)(t)\| + p_y(s)(t)$$

and $s \to \|\dot{y}(s)(t)\| + p_y(s)(t)$ is continuous from S into $L^1(I, \mathbb{R})$. Due to Lemma 2.2, $G_0(.) : S \to \mathcal{P}(L^1(I, \mathbb{R}^n))$ is l.s.c., with decomposable values and, applying Lemma 2.3, there exists a continuous selection $f_0(.)$ of $H_0(.)$ such that $f_0(s)(t) \in F(t, y(s)(t), y(s)(t-\Delta), s)$, for almost every $t \in I, \forall s \in S$ and

$$\|f_0(s)(t) - \dot{y}(s)(t)\| \leq p_y(s)(t) + \varepsilon_0, \quad s \in S.$$

Define

$$x_1(s)(t) = \begin{cases} a(s) + \int_{t_0}^{t} f_0(s)(u)du, & t \in [t_0, T] \\ y(s)(t), & t \in [t_0 - \Delta, t_0). \end{cases}$$

One has $x_1(s)(t_0) = a(s)$ and $\dot{x}_1(s)(t) = f_0(s)(t), \forall t \in (t_0, T]$. Notice that

$$\|\dot{x}_1(s)(t) - x_0(s)(t)\| = \|f_0(s)(t) - \dot{y}(s)(t)\| \leq p_y(s)(t) + \varepsilon_0;$$

$$\|x_1(s)(t) - x_0(s)(t)\| = \|a(s) - y(s)(t) + \int_{t_0}^{t} f_0(s)(u)du\| \leq \\ \leq \|a(s) - y(s)(t_0)\| + \|\int_{t_0}^{t}[f_0(s)(u) - \dot{y}(s)(u)]du\| \leq \\ \leq b(s) + \int_{t_0}^{t}\|f_0(s)(u) - \dot{y}(s)(u)\|du \leq b(s) + \int_{t_0}^{t}[p_y(s)(u) + \varepsilon_0]du = \\ = b(s) + \varepsilon_0(t - t_0) + \int_{t_0}^{t} p_y(s)(u)du \leq p_1(s)(t).$$

We shall construct a sequence $x_n(.) : S \to \mathcal{F}([t_0 - \Delta, T], \mathbb{R}^n), n \geq 1$ such that

$$x_n(s)(t) = y(s)(t), t \in [t_0 - \Delta, t_0) \text{ and } x_n(s)(t_0) = a(s); \tag{6}$$

$$\dot{x}_n(s)(t) \in F(t, x_{n-1}(s)(t), x_{n-1}(s)(t - \Delta), s); \tag{7}$$

$$\|\dot{x}_n(s)(t) - \dot{x}_{n-1}(s)(t)\| \leq r(s)(t)[p_{n-1}(s)(t) + p_{n-1}(s)(t - \Delta)]. \tag{8}$$

Now, assuming that for $k = \overline{1, n}$ we have defined $x_k(.)$ satisfying (6)-(8), we obtain $x_{n+1}(.)$ as follows. From (7) and (8), for any $n \geq 2$ one has

$$\begin{aligned}
\|x_n(s)(t) - x_{n-1}(s)(t)\| &\leq \int_{t_0}^{t} r(s)(u)[p_{n-1}(s)(u) + p_{n-1}(s)(u - \Delta)]du \leq \\
&\leq 2\int_{t_0}^{t} r(s)(u) p_{n-1}(s)(u) du = \\
2\int_{t_0}^{t} \Big[r(s)(u)\int_{t_0}^{u} p_y(s)(\tau)\frac{[2(m(s)(u) - m(s)(\tau))]^{n-2}}{(n-2)!}d\tau \\
&+ \frac{[2m(s)(u)]^{n-2}}{(n-2)!}(\varepsilon_{n-1}(u - t_0) + b(s))\Big]du \leq \\
\leq \int_{t_0}^{t}\int_{t_0}^{u} \frac{\partial}{\partial u}\Big[p_y(s)(\tau)\frac{[2(m(s)(u) - m(s)(\tau)]^{n-1}}{(n-1)!}\Big]d\tau du + \\
&+ [\varepsilon_{n-1}(t - t_0) + b(s)]\int_{t_0}^{t} \frac{\partial}{\partial u}\Big[\frac{(2m(s)(u))^{n-1}}{(n-1)!}\Big]du = \\
= \Bigg[\int_{t_0}^{u} p_y(s)(\tau)\frac{[2(m(s)(u) - m(s)(\tau)]^{n-1}}{(n-1)!}d\tau + \\
[\varepsilon_n(t - t_0) + b(s)]\frac{(2m(s,u))^{n-1}}{(n-1)!}\Bigg]\Bigg|_{u=t_0}^{t} = \\
= \int_{t_0}^{t} p_y(s)(\tau)\frac{[2(m(s)(t) - m(s)(\tau)]^{n-1}}{(n-1)!}d\tau + [\varepsilon_n(t - t_0) + b(s)]\frac{(2m(s,t))^{n-1}}{(n-1)!} = \\
&= p_n(s)(t) - \frac{[2m(s)(t)]^{n-1}}{(n-1)!}[(\varepsilon_n - \varepsilon_{n-1})(t - t_0) + b(s)].
\end{aligned}$$

We conclude that

$$\|x_n(s)(t) - x_{n-1}(s)(t)\| \leq p_n(s)(t) - \frac{\varepsilon(t - t_0) \cdot [2m(s)(t)]^{n-1}}{(n+2)(n+1)(n-1)!} \leq p_n(s)(t). \tag{9}$$

Set d $c_n(s)(t) = \frac{\varepsilon(t-t_0)[2m(s)(t)]^{n-1}}{(n+2)(n+1)(n-1)!} r(s)(t)$. From (7) and (9) one may infer

$$\begin{aligned} d\big(\dot{x}_n(s)(t), F(t, x_n(s)(t), x_n(s)(t-\Delta), s)\big) \le \\ \le d\big(\dot{x}_n(s)(t), F(t, x_{n-1}(s)(t), x_{n-1}(s)(t-\Delta), s)\big) + \\ + d_H\big(F(t, x_{n-1}(s)(t), x_{n-1}(s)(t-\Delta), s), F(t, x_n(s)(t), x_n(s)(t-\Delta), s)\big) \le \\ \le r(s)(t)\Big(\|x_n(s)(t) - x_{n-1}(s)(t)\| + \|x_n(s)(t-\Delta) - x_{n-1}(s)(t-\Delta)\|\Big) \le \\ \le r(s)(t)\big[p_n(s)(t) + p_n(s)(t-\Delta)\big] - c_n(s)(t). \end{aligned}$$

For every $s \in S$, consider the set-valued maps

$$G_n(s) = \{v \in L^1(I, \mathbb{R}^n), v(t) \in F(t, x_n(s)(t), x_n(s)(t-\Delta), s), a.e.(I)\}$$

$$H_n(s) = cl\{v \in G_n(s), \|v(t) - \dot{x}_n(s)(t)\| \le r(s)(t)(p_n(s)(t) + p_n(s)(t-\Delta)), a.e.(I)\}$$

As the map $t \to c_n(s)(t)$ is Lebesgue measurable, $\forall s \in S$, due to Lemma 2.1, there exists $v(.) \in L^1(I, \mathbb{R}^n)$ such that $v(t) \in F(t, x_n(s)(t), x_n(s)(t-\Delta), s)$, a.e.(I) and

$$\|v(t) - \dot{x}_n(s)(t)\| \le d\big(\dot{x}_n(s)(t), F(t, x_n(s)(t), x_n(s)(t-\Delta), s)\big) + c_n(s)(t) \le$$

$$\le r(s)(t)[p_n(s)(t) + p_n(s)(t-\Delta)],$$

hence $H_n(s) \neq \varnothing, \forall s \in S$. Moreover,

$$d(0, F(t, x_n(s)(t), x_n(s)(t-\Delta), s) \le \|\dot{x}_n(s)(t)\| + r(s)(t)[p_n(s)(t) + p_n(s)(t-\Delta)]$$

and $\beta_n(.) : S \to L^1(I, \mathbb{R}^n), \beta_n(s)(t) := \|\dot{x}_n(s)(t)\| + r(s)(t)[p_n(s)(t) + p_n(s)(t-\Delta)]$ is continuous. Lemmas 2.2, 2.3, provide the existence of a continuous selection $f_n(.)$ of $H_n(s)$, $f_n(.) : S \to L^1(I, \mathbb{R}^n)$, satisfying the following

$$f_n(s)(t) \in F(t, x_n(s)(t), x_n(s)(t-\Delta), s), a.e.(I);$$

$$\|f_n(s)(t) - \dot{x}_n(s)(t)\| \le r(s)(t)[p_n(s)(t) + p_n(s)(t-\Delta)], a.e.(I).$$

Define

$$x_{n+1} = \begin{cases} a(s) + \int_{t_0}^{t} f_n(s)(u)du, & t \in I \\ y(s)(t), & t \in [t_0 - \Delta, t_0) \end{cases}.$$

Clearly, $x_{n+1}(.)$ satisfies conditions (6)-(8) and we have

$$\begin{aligned} \|x_{n+1}(s)(.) - x_n(s)(.)\|_{AC} = \int_{t_0}^{T} \|\dot{x}_{n+1}(s)(u) - \dot{x}_n(s)(u)\| du \le \\ \le \int_{t_0}^{T} r(s)(u)[p_n(s)(u) + p_n(s)(u-\Delta)]du \le p_{n+1}(s)(T). \quad (10) \end{aligned}$$

Further,

$$p_{n+1}(s)(T) \leq \frac{[2m(s)(T)]^n}{n!}\Big[\int_{t_0}^{T}|p_y(s)(u)|du + \varepsilon_{n+1}(T-t_0)+b(s)\Big] \leq \\ \leq \frac{[2m(s)(T)]^n}{n!}[\|p_y(s)(.)\|_1 + \varepsilon(T-t_0)+b(s)]. \tag{11}$$

Thus, from (10) and (11), we have

$$\|x_{n+1}(s)(.) - x_n(s)(.)\|_{AC} \leq \frac{[2m(s)(T)]^n}{n!}\big[\|p_y(s)(.)\|_1 + \varepsilon(T-t_0)+b(s)\big]. \tag{12}$$

The latter inequality combined with the continuity of the map $s \mapsto \|p_y(s)(.)\|_1$ implies that the sequence $\{x_n(s')(.)\}_n$ satisfies the Cauchy condition uniformly in s' on some neighborhood of s, in the Banach space $AC(I,\mathbb{R}^n)$. Let $x(.) : S \to AC(I,\mathbb{R}^n), x(s)(t) = \lim_{n\to\infty} x_n(s)(t)$. Hence $x(.)$ is continuous from S into $AC(I,\mathbb{R}^n)$. Now, we shall prove that $x(s)(.)$ is a solution of (1). For any $s \in S$ and a.e.(I), we have

$$d(\dot{x}_{n+1}(s)(t), F(t, x(s)(t), x(s)(t-\Delta), s) \leq \\ \leq d_H(F(t, x_n(s)(t), x_n(s)(t-\Delta), s), F(t, x(s)(t), x(s)(t-\Delta), s)) \leq \\ \leq r(s)(t)[\|x_n(s)(t) - x(s)(t)\| + \|x_n(s)(t-\Delta) - x(s)(t-\Delta)\|].$$

Passing to the limit along a subsequence of the sequence $\{x_n(.)\}_n$ which point wise converges to $x(.)$, we find that

$$\dot{x}(s)(t) \in F(t, x(s)(t), x(s)(t-\Delta), s), \quad a.e(I), \forall s \in S.$$

Taking into account (8), for every $s \in S$, the following estimates hold

$$\|f_{n+1}(s)(t) - \dot{y}(s)(t)\| = \|\sum_{k=1}^{n+1}[f_k(s)(t) - f_{k-1}(s)(t)] + f_0(s)(t) - \dot{y}(s)(t)\| \leq \\ \leq \sum_{k=1}^{n+1} r(s)(t)[p_{k-1}(s)(t) + p_{k-1}(s)(t-\Delta)] + p_y(s)(t) + \varepsilon_0 \leq \\ \leq p_y(s)(t)+\varepsilon+r(s)(t)\Big[\int_{t_0}^{t} p_y(s)(t)\cdot e^{2[m(s)(t)-m(s)(u)]}du + e^{2m(s)(t)}[\varepsilon(t-t_0)+b(s)]\Big]+ \\ +r(s)(t)\Big[\int_{t_0}^{t-\Delta} p_y(s)(t)\cdot e^{2[m(s)(t-\Delta)-m(s)(u)]}du + e^{2m(s)(t-\Delta)}[\varepsilon(t-\Delta-t_0)+b(s)]\Big] = \\ = r(s)(t)[\xi_y(s,t) + \xi_y(s,t-\Delta)] + p_y(s)(t) + \varepsilon.$$

Similarly, by adding (9), we get

$$\|x_{n+1}(s)(t) - y(s)(t)\| \leq \sum_{k=1}^{n+1} p_k(s)(t) \leq \sum_{k\geq 1} p_k(s)(t) \leq \xi_y(s,t).$$

Finally, by passing to the limit as $n \to \infty$, we find (4),(5). □

The next result is a continuous version of Filippov-Ważewski Relaxation theorem.

Theorem 3.2. *Assume that Hypothesis* 2.6 *is satisfied and for any* $(t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, *the set-valued map* $F(t,x,y,.)$ *is continuous.*

Then, for every $\varepsilon > 0$ *and any map* $y(.) : S \to \mathcal{F}([t_0 - \Delta, T], \mathbb{R}^n)$ *such that* $s \to y(s)(.)|_I$ *is continuous, there exists a function* $x(.) : S \to \mathcal{F}([t_0 - \Delta, T], \mathbb{R}^n)$ *such that*
i) for every $s \in S$, $s \to x(s)(.)|_I$ *is continuous,* $x(s)(.)$ *is a solution of (1) and* $x(s)(t) = y(s)(t), \forall t \in [t_0 - \Delta, t_0), s \in S$;
ii) for every $(t,s) \in I \times S$,

$$\|x(s)(t) - y(s)(t)\| \leq \varepsilon + \\ + e^{2m(s,t)}\big[b(s) + \int_{t_0}^{t} d(\dot{y}(s)(u), F(u, y(s)(u), y(s)(u-\Delta), s)\,du\big].$$

Moreover, if we assume, in addition, that there exists a function $q(.) \in L^1(I, \mathbb{R}_+^*)$ *such that*

$$d_H(0_n, F(t,x,y,s)) \leq q(t) \quad a.e.\ t \in I,\ \forall x, y \in \mathbb{R}^n, s \in S$$

then, for any map $x(.) : S \to \mathcal{F}([t_0 - \Delta, T], \mathbb{R}^n)$ *such that for every* $s \in S$, $s \to x(s)(.)|_I$ *is continuous and the function x(s)(.) is a solution of the (relaxed) differential-difference inclusion (2), there exists* $\overline{x}(.) : S \to \mathcal{F}([t_0 - \Delta, T], \mathbb{R}^n)$, *satisfying the following:*
iii) for any $s \in S$, $s \to \overline{x}(s)(.)|_I$ *is continuous,* $\overline{x}(s)(.)$ *is a solution of (1) and* $\overline{x}(s)(t) = x(s)(t)$, $\forall t \in [t_0 - \Delta, t_0)$, $s \in S$;
iv) for every $(t,s) \in I \times S$,

$$\|\overline{x}(s)(t) - x(s)(t)\| < \varepsilon \quad .$$

Proof: In order to prove the first assertion, let us define

$$p(s)(t) = d(\dot{y}(s)(t), F(t, y(s)(t), y(s)(t-\Delta), s))$$

and note that, by Theorem 3.1, it suffices to prove that the function $p(.) : S \to L^1(I, \mathbb{R})$ is continuous at s_0, for any fixed $s_0 \in S$.

Let $\varepsilon > 0$. Following [8], for every $s \in S$, we have

$$\int_{t_0}^{T} |p(s)(t) - p(s_0)(t)|dt \leq$$
$$\leq \int_{t_0}^{T} \Big[\|\dot{y}(s)(t) - \dot{y}(s)(t_0)\| +$$
$$+ d_H(F(t, y(s_0)(t), y(s_0)(t-\Delta), s_0), F(t, y(s)(t), y(s)(t-\Delta), s))\Big] dt \leq$$
$$\leq \int_{t_0}^{T} \Big[\|\dot{y}(s)(t) - \dot{y}(s_0)(t)\| +$$
$$+ r(s)(t)[\|y(s)(t) - y(s_0)(t)\| + \|y(s)(t-\Delta) - y(s_0)(t-\Delta)\|]\Big] dt +$$
$$+ \int_{t_0}^{T} d_H(F(t, y(s_0)(t), y(s_0)(t-\Delta), s_0), F(t, y(s_0)(t), y(s_0)(t-\Delta), s))dt.$$

The map $s \mapsto y(s)(.)$ is continuous at s_0, thus we can find $\delta_1 > 0$ such that $\forall s \in S$, $\|s - s_0\| \leq \frac{\delta_1}{2}$:

$$\int_{t_0}^{T} \Big[\|\dot{y}(s)(t) - \dot{y}(s_0)(t)\| +$$
$$+ r(s)(t)[\|y(s)(t) - y(s_0)(t)\| + \|y(s)(t-\Delta) - y(s_0)(t-\Delta)\|]\Big] dt < \frac{\varepsilon}{2}.$$

Setting

$$p_1(s)(t) = d_H(F(t, y(s_0)(t), y(s_0)(t-\Delta), s_0), F(t, y(s_0)(t), y(s_0)(t-\Delta), s)),$$

it is easy to note that $p_1(.)$ is continuous at s_0. Further, using the condition in (H4$'$), there is a neighborhood $V(s)$ of s_0, such that

$$p_1(s)(t) \leq 2\Big[p_0(s_0)(t) + r(s)(t)\big(\|y(s_0)(t)\| + \|y(s_0)(t-\Delta)\|\big)\Big].$$

As the function defined by the right hand side of the above inequality is continuous from S to $L^1(I, \mathbb{R})$ we can state, in view of Lebesgue dominated convergence Theorem, that $s \to \int_{t_0}^{T} p_1(s)(t)dt$ is continuous at s_0. We can pick a $\delta_2 > 0$ such that, $\forall s \in S$, $\|s - s_0\| \leq \frac{\varepsilon}{2}$:

$$\int_{t_0}^{T} d_H(F(t, y(s_0)(t), y(s_0)(t-\Delta), s_0), F(t, y(s_0)(t), y(s_0)(t-\Delta), s))dt \leq \frac{\varepsilon}{2}.$$

Choose $\delta := \min\{\delta_1, \delta_1\}$. Consequently:

$$\int_{t_0}^{T} \|p(s)(t) - p(s_0)(t)\|dt \leq \varepsilon, \quad \forall \|s - s_0\| \leq \delta.$$

Due to the integrably boundedness of $F(t,.,.,.)$ in the second part of the theorem, then if $\varepsilon > 0$ there is $\delta > 0$ such that

$$\delta[1 + e^{2m(s,T)}(T - t_0 + \int_{t_0}^{T} 2e^{-2m(s,t)}r(s,t)dt] < \varepsilon. \qquad (13)$$

Pick $n \geq 1$ large enough so that for any measurable set $A \subset I$, satisfying $\mu(A) \leq \frac{T-t_0}{n}$, one has

$$\int_A q(t)dt \leq \frac{\delta}{4}.$$

For $j = \overline{1,n}$, set $t_j = t_0 + \frac{j}{n}(T - t_0)$ and $I_j := [t_{j-1}, t_j]$. For all $s \in S$ consider the set-valued maps

$$M_j(s) = \{v \in L^1(I_j, \mathbb{R}^n) : v(t) \in F(t, x(s)(t), x(s)(t - \Delta), s)\, a.e.(I_j)\}.$$

From Lemma 2.2, it follows that $M_j(.)$ is l.s.c.. On the other hand, the maps $x(s)(t_j) - x(s)(t_{j-1})$ are continuous selections of $cl(\int_{I_j} \overline{co}F(t, x(s)(t), x(s)(t - \Delta), s)dt)$ which coincide, by Lemma 2.4, with $cl(\int_{I_j} F(t, x(s)(t), x(s)(t-\Delta), s)dt)$. Applying Lemma 2.5 we obtain the existence of continuous selections $g_j(.) : S \to L^1(I_j, \mathbb{R}^n)$ of $M_j(.)$, such that

$$\| \int_{I_j} g_j(s)(t)dt - \int_{I_j} \dot{x}(s)(t)dt\| < \frac{\delta}{2n},\ \forall s \in S, j = \overline{1,n}.$$

It enables us to consider a function $g(.) : S \to L^1(I, \mathbb{R}^n)$ equal to $g_j(.)$ on I_j. Define $y(.) : S \to \mathcal{F}([t_0 - \Delta, T], \mathbb{R}^n)$ by

$$y(s)(t) = \begin{cases} a(s) + \int_{t_0}^{t} g(s)(u)du, & t \in [t_0, T] \\ x(s)(t), & t \in [t_0 - \Delta, t_0). \end{cases}$$

Let $t \in I$. There exists $j_0 \in 1, ...n$ such that $t \in I_{j_0}$. Then,

$$\|y(s)(t) - x(s)(t)\| = \| \int_{t_0}^{t} [g(s)(u) - \dot{x}(s)(u)]du\| <$$

$$< \sum_{j=1}^{j_0-1} \| \int_{I_j} [g_j(s)(u) - \dot{x}(s)(u)]du\| + \int_{I_j} [\|g_j(s)(u)\| + \|\dot{x}(s)(u)\|]du \leq$$

$$\leq \frac{\delta}{2n}(j_0 - 1) + 2\frac{\delta}{4} < \delta.$$

On the other hand, from Hypothesis (H2) it follows

$$\begin{aligned} & d(\dot{y}(s)(t), F(t, y(s)(t), y(s)(t - \Delta), s)) \leq \\ & \quad \leq d_H(F(t, x(s)(t), x(s)(t - \Delta), s), F(t, y(s)(t), y(s)(t - \Delta), s)) \leq \\ & \quad \leq r(s)(t)[\|x(s)(t) - y(s)(t)\| + \|x(s)(t - \Delta) - y(s)(t - \Delta)\|] \leq \\ & \qquad \leq 2r(s)(t)\delta. \end{aligned}$$

Let $\xi_y(s,t) := e^{2m(s)(t)}\left[\delta(t-t_0)+\int_{t_0}^{t} e^{-2m(s)(u)}\cdot 2r(s)(u)\delta du\right]$. Applying Theorem 3.1 with $\varepsilon := \delta$, we find that there exists a function $\overline{x}(.) : S \to \mathcal{F}([t_0-\Delta, T], \mathbb{R}^n)$ such that $\forall s \in S$, $s \to x(s)(.)|_I$ is continuous, $\overline{x}(s)(.)$ is a solution of (1), satisfies $\overline{x}(s)(t) = x(s)(t), \forall t \in [t_0-\Delta, t_0), \forall s \in S$, $x(s)(t_0) = a(s), \forall s \in S$ and

$$\|\overline{x}(s)(t) - y(s)(t)\| \leq \xi_y(s,t), \forall t \in I, \forall s \in S.$$

Recall that δ is assumed to satisfy (13), hence

$$\begin{aligned}\|\overline{x}(s)(t) - x(s)(t)\| &\leq \|\overline{x}(s)(t) - y(s)(t)\| + \|y(s)(t) - x(s)(t)\| \leq \\ &\leq \xi_y(s,t) + \delta \leq \xi_y(s,T) + \delta < \varepsilon\end{aligned}$$

and the theorem is completely proved. □

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Received: 12.12.2005

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