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The Generalized Criterion of Dieudonné for Valuated Abelian Groups

by
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Abstract

An extension in terms of σ_λ-summable (valuated) groups for a cofinal with ω ordinal λ of the classical Dieudonné criterion (Portugaliae Mathematicae, 1952), concerning the direct sums of p-primary cyclic groups, is established. Specifically, it is proved that if G is an abelian p-group whose length λ is cofinal with ω and if A is a subgroup of G so that A is a σ_λ-summable valuated group using the valuation inherited from the height valuation on G and so that G/A is a σ_λ-summable group, then G is a σ_λ-summable group.

Key Words: σ_λ-summable groups, heights, valuated subgroups.
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1 Introduction

One of the fundamental results for direct sums of cyclic groups in the abelian group theory with numerous valuable applications is the so-named *Kulikov's criterion* [Ku] and its important strengthening due to Dieudonné [Di].

For the reader's convenience and to avoid the concrete referring we give their detailed formulation.

Criterion (L. Y. Kulikov, 1945). *The abelian p-group G is a direct sum of cyclic groups if and only if* $G = \cup_{n<\omega} G_n, G_n \subseteq G_{n+1} \leq G$ *and* $G_n \cap p^n G = 0$, $\forall n \geq 1$.

Criterion (J. A. Dieudonné, 1952). *Suppose G is an abelian p-group with a subgroup A such that G/A is a direct sum of cyclic groups. Then G is a direct sum of cyclic groups if and only if* $A = \cup_{n<\omega} A_n, A_n \subseteq A_{n+1} \leq A$ *and* $A_n \cap p^n G = 0$, $\forall n \geq 1$.

In [Dan] we have recently improved these two criteria to an assertion dealing with a more general class of abelian groups, namely the class of σ-summable groups defined as in [LM], than the direct sums of cyclic groups.

Criterion (P. V. Danchev, 2005). *Let G be an abelian p-group of limit length with a nice subgroup A for which G/A is a σ-summable group. Then G is σ-summable if and only if $A = \cup_{n<\omega} A_n, A_n \subseteq A_{n+1} \leq A$ and, $\forall n \geq 1$, $\exists \alpha_n < length(G) : A_n \cap p^{\alpha_n} G = 0$.*

In particular, if A is a balanced subgroup of the abelian p-group G so that G/A is σ-summable and $length(A) = length(G)$, then G is σ-summable if and only if A is σ-summable.

Remark 1. As it shall be showed in what follows, all calculations can be restricted only on the socle of the full group.

As the title of the paper suggests, it focuses on subgroups of direct sums of p-torsion cyclic groups and their generalizations, viewed as valuated groups. So, our purpose here is to refine the previous necessary and sufficient conditions by the usage of so-called valuated groups, which is a special sort of groups.

All groups considered in the sequel are arbitrary (often valuated) abelian groups. The terminologies and notation not expressly introduced below follow the usage of [Fu]. Throughout, the letter G will always designate an arbitrary abelian group, written additively as is customary when regarding abelian groups, and the letter A a subgroup of G mainly endowed with a valuation induced by the height valuation on G unless we do not give special requirements.

We thus come to the

2 Main Result

Before presenting the central result, we need a brief introductory steps in such ubiquitous groups. And so, following [RW], we systematically recall the concepts necessary for a future application:

An abelian group C is said to be a *valuated group* if it is equipped with a special function, termed as a *valuation*. More concrete, if C is an abelian group and p a prime, a *p-valuation* of C is a function $v_p : C \to \{\text{ordinal numbers}\} \cup \{\infty\}$ satisfying the following conditions $\forall\, a, b \in C$:

1. $v_p(a+b) \geq \min\{v_p(a), v_p(b)\}$, and $v_p(a+b) = \min\{v_p(a), v_p(b)\}$ if $v_p(a) \neq v_p(b)$;
2. $v_p(pa) \geq v_p(a)$, and $v_p(pa) > v_p(a)$ if $v_p(a) \neq \infty$;
3. $v_p(na) = v_p(a)$ if n is an integer not divisible by p.

It is worthwhile noticing that it was shown in ([Ho], Theorem 1) that the third condition from the foregoing definition of p-valuation is decidable from the first two ones, and henceforth it may be dropped off.

A valuated group is an abelian group together with a set of p-valuations v_p, one for each prime number p. This set of p-valuations is known as a valuation, and we say that $v_p(a)$ is the *p-value* of a. If F is a subgroup of the valuated group C, then F becomes a valuated group, called a *valuated subgroup*, by taking the restricted valuation on F.

Notice that if h_p is the *p-height* function of C, then it satisfies the given above three properties. Thereby, any abelian group has a natural valuation which is h_p. More generally, if F is a subgroup of a group C, then the p-height function on C restricts to a valuation on F. We emphasize also that it easily follows from property 2. that $v_p(a) \geq h_p(a)$.

Let now C be a valuated group, and α an ordinal number or the symbol ∞. Then, for every α, the sets

$$C(p^\alpha) = \{c \in C : v_p(c) \geq \alpha\}$$

and

$$p^\alpha C = \{c \in C : h_p(c) \geq \alpha\}$$

are subgroups of C with $p^\alpha C \subseteq C(p^\alpha)$. When the prime p is not in doubt we unambiguously write $C(\alpha)$ for $C(p^\alpha)$. We observe that $C(p^\alpha) = p^\alpha C$ for $v_p = h_p$. Note also that $C(p^\alpha) = C \cap p^\alpha G$ provided that C is a subgroup of a group G. Thus, if C is an isotype subgroup of G, we have that $p^\alpha C = C(p^\alpha)$.

Moreover, let $C(p^\alpha)[p] = C(p^\alpha) \cap C[p]$ where $C[p] = \{c \in C : pc = 0\}$ is the *p-socle* of C. Evidently, $C(p^\alpha)[p] = C[p](p^\alpha)$.

The following definition enlarges the corresponding ones in [LM] and [Dan] pertaining to σ-summable abelian groups.

Definition. Let λ be a cofinal with ω ordinal number. The abelian p-group A is called σ_λ-summable if $A = \cup_{n<\omega} X_n, X_n \subseteq X_{n+1} \leq A$ such that, $\forall n \geq 1$, $\exists \alpha_n < \lambda : X_n \cap A(\alpha_n) = 0$.

We routine observe that if λ is limit of countable cofinality and G is an arbitrary σ_λ-summable group, then any valuated subgroup A of G is also σ_λ-summable. As it stands, because of the inclusion $p^{\alpha_n} A \subseteq A(\alpha_n)$, this class of primary σ_λ-summable groups is smaller than the class of all σ-summable groups, firstly introduced in [LM]. In other words, each σ_λ-summable group A must be σ-summable provided that $\lambda \leq length(A)$, while the converse claim is not always valid. Nevertheless, the σ_λ-summable groups treat the more general conception of a "valuation" than that of "height". That is why every σ-summable group with length not exceeding λ is σ_λ-summable as a valuated group in the sense of the natural height valuation.

Our goal here is to illustrate that a generalized version of the foregoing listed Dieudonné's criterion for so-defined σ_λ-summable p-groups exists; to obtaining the result of Dieudonné we just substitute λ with ω. Regardless of the more global variant of groups considered in [Dan] than these here, the major difference with the cited article is in the approach to attacking the central affirmation. Moreover, as it will be demonstrated below, under the new notion of valuated subgroups and by fixing λ, the restriction on A to be nice in G may be ignored.

By a *valuated vector space* we mean a p-bounded valuated group for some prime integer p. A valuated vector space is *free* if it is a direct sum of valuated cycles. A classical result due to B. Charles asserts that any valuated vector space of length ω or less is isomorphic to the socle of some abelian p-group.

The following technical claim, which is stated only for convenience, is well-known and has a trivial proof; it will therefore be omitted. It is also worthwhile noticing that this claim is a common expansion of ([Fu], vol. I, p. 137, property i)).

Lemma 1. *Let C be an abelian p-group with a subgroup B such that $B \cap pC = pB$ and $C[p] = B[p]$. Then $C = B$.*

The following preliminary technicality is pretty easy to show. It expands the corresponding one in ([Hi], Proposition 1), but however the technique used is similar.

Lemma 2. *If G is an abelian p-group with length (λ) which is cofinal with ω, then G is σ_λ-summable if and only if $G[p]$ is a σ_λ-summable valuated vector space.*

Proof: *"Necessity"*. If G is the union of an ascending sequence of subgroups G_n satisfying the prescribed conditions, we only need to put $E_n = G_n[p]$ in order to demonstrate that $G[p]$ is, by definition, σ_λ-summable as a valuated vector space.

"Sufficiency". Conversely, assume that $G[p]$ is a σ_λ-summable valuated vector space and the ascending chain of subsocles E_n leading up to $G[p]$ is a manifestation of this. Furthermore, let G_0 be a maximal group in G with respect to the property $G_0[p] = E_0$. Inductively, for each $n \geq 0$, we choose $G_{n+1} \leq G$ so that it is maximal with respect to the following two conditions: (1) $G_{n+1} \supseteq G_n$; (2) $G_{n+1}[p] = E_{n+1}$. In this connection, it is not hard to check that $(G_n + E_{n+1})[p] = E_{n+1}$. Hence, the set to which we are employing the classical Zorn's lemma is non-empty. Let now $K = \cup_{n<\omega} G_n$, so K is a subgroup of G. Moreover, the maximality of G_n having the prescribed socle E_n implies that $pG \cap G_n = pG_n$ (see, e.g., [Fu], vol. I, p. 140). Apparently, by taking in both sides the operator $\cup_{n<\omega}$, the group K retains that property, whence $pG \cap K = pK$. Since obviously $G[p] = K[p]$, we infer at once by Lemma 1 that $G = K$. Thus G is the union of the G_n's. But if $\alpha_n < \lambda$ is a bound for the heights as computed in G of the nonzero elements of E_n, then, via the routine argument that for any abelian p-group T we have $T = 0$ only when $T[p] = 0$, it must be a bound for the heights in G of all the nonzero elements of G_n as well. So, G is σ_λ-summable, as affirmed. The proof is completed. □

Note. In ([Hi], p. 3133, lines 9(-) and 10(-)) it seems that there is a printing error, namely that the symbol "G" should be replaced by the symbol "$G[p]$".

We are now ready to proceed by proving the following major statement. It is a true generalization of the foregoing quoted three affirmations.

Theorem 1. *Suppose G is an abelian p-group for which (its length) λ is cofinal with ω, $A \leq G$ and $H = G/A$. If H is a σ_λ-summable group and A is a σ_λ-summable valuated group with a valuation induced by the height valuation on G, then G is σ_λ-summable as a group.*

Proof: Referring to the second Lemma, we reduce the computation to one involving vector space, i.e., in other words, we need only show that $G[p]$ is σ_λ-summable as a valuated vector space.

Our hypotheses imply that $H[p]$ is σ_λ-summable as is $A[p]$ by using the valuation induced via the height function on G.

By definition, we further write down, $A[p] = \cup_{n<\omega} X_n, X_n \subseteq X_{n+1} \leq A[p]$ where $X_n \cap A[p](\alpha_n) = 0$ for some $\alpha_n < \lambda$ and moreover $H[p] = \cup_{n<\omega} Y_n, Y_n \subseteq Y_{n+1} \leq H[p]$ where $Y_n \cap H[p](\beta_n) = 0$ for some $\beta_n < \lambda$. If now Z_n are ascending subgroups of $G[p]$ such that $Z_n \cap A = 0$ and such that $(Z_n \oplus A)/A = Y_n \cap [(G[p] + A)/A]$ and if $\gamma_n = \max\{\alpha_n, \beta_n\}$, then it is plain to see that $G[p] = \cup_{n<\omega}(X_n \oplus Z_n)$ and that $(X_n \oplus Z_n) \cap G[p](\gamma_n) = 0$ along with $\gamma_n < \lambda$. Indeed, given $g \in G[p]$ hence $(g+A)/A \in H[p]$ and thus $(g+A)/A \in Y_m \cap [(G[p]+A)/A] = (Z_m \oplus A)/A$ for some $m \in \mathbb{N}$. Consequently, $g + A \in Z_m \oplus A$ whence $g \in Z_m \oplus A$ and $g \in Z_i \oplus X_i$ for some $i \geq 1$. That is why, all $Z_n \oplus X_n$ generate $G[p]$. Furthermore, for every natural number n, we calculate that $((A \oplus Z_n)/A) \cap ((G[p](\gamma_n) + A)/A) \subseteq Y_n \cap H[p](\gamma_n) \subseteq Y_n \cap H[p](\beta_n) = 0$ and therefore, by what we have already obtained, combined with the modular law from ([Fu], vol. I, p. 13), we deduce $(X_n \oplus Z_n) \cap G[p](\gamma_n) \subseteq (A \oplus Z_n) \cap G[p](\gamma_n) \subseteq A$. Henceforth, since $X_n \subseteq A[p]$ and $Z_n \cap A[p] = 0$, we conclude that $(X_n \oplus Z_n) \cap G[p](\gamma_n) = (X_n \oplus Z_n) \cap A[p](\gamma_n) = (X_n \cap A[p](\gamma_n)) \oplus (Z_n \cap A[p](\gamma_n)) \subseteq (X_n \cap A[p](\alpha_n)) \oplus (Z_n \cap A[p]) = 0$. These two derivations substantiate our claim. Therefore, this argues that $G[p]$ is really σ_λ-summable, which owing to Lemma 2 proves the desired result. The proof of the theorem is finished. □

As an immediate consequence, we formulate the following.

Corollary 1. *Let G be an abelian p-group, λ an ordinal cofinal with ω, A an isotype subgroup of G and $H = G/A$. If both A and H are σ_λ-summable groups, then G is a σ_λ-summable group. In particular, if λ is of countably cofinality, G is a σ_λ-summable group if and only if A is a σ_λ-summable group provided H is σ_λ-summable.*

As another direct application of the foregoing main theorem, we state the following new consequence concerning the well-explored and very attractive class of totally projective groups (for other special applications of that kind see, for example, [Da] and [Dan]).

Corollary 2. *Let G be an abelian p-group such that its length λ is cofinal with ω and such that A is its countable subgroup (of limit length). If G/A is totally projective of length cofinal with ω which length does not exceed λ, then G is σ_λ-summable.*

Proof: First of all, since A is countable of limit length, we can represent it in the following manner: $A = \cup_{n<\omega} A_n, A_n \subseteq A_{n+1} \leq A$ where, for every $n \geq 1$, A_n is finite; thereby $A_n \cap p^{\alpha_n} G = A_n \cap A(\alpha_n) = 0$ for some existing ordinals

$\alpha_n < length(G) = \lambda$. Hence A must be of necessity σ_λ-summable. Apparently, the length restriction on G/A gives that $length(G/A) \leq length(G) = \lambda$. On the other hand, each totally projective group of length, at most λ, which is cofinal with ω is known to be σ_λ-summable as a group having the traditional height valuation. Henceforth, the Theorem works. This completes the proof. □

Remark 2. It is well-known that, under the above circumstances on A and G/A, the whole group G needs not be totally projective; for instance when A is not nice in G because the niceness of A in G insures that G must be totally projective. However, by the last satisfactory description, it is necessarily σ_λ-summable.

Notice also that, by following the same idea as in [LM], it may be proved that if A is an abelian p-group of countable limit length λ, then A is totally projective if and only if $A/p^\alpha A$ is σ_λ-summable for all $\alpha < \lambda$; so imitating [Dan] other valuable applications of Theorem 1 to totally projective groups can be successfully deduced.

Correction: In [Da] there is a misprint, namely on p. 228 in Example the citation [7] should be deleted.

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