Bull. Math. Soc. Sc. Math. Roumanie
Tome 49(97) No. 2, 2006, 171–175

Visser's inequality and its sharp refinement

by
V.K. Jain

Abstract

For a polynomial $p(z) = \sum_{k=0}^{n} c_k z^k$ of degree n, with $\max_{|z|=1} |p(z)| = M$, Visser had obtained

$$|c_0| + |c_n| \leq M.$$

Using certain integral inequality for a polynomial, we have suggested a different proof of Visser's inequality (in a new but equivalent form)

$$|\alpha||c_0| + |\beta||c_n| \leq M\{\max(|\alpha|, |\beta|)\}.$$

Further for polynomial $p(z)$ having all its zeros, either in $|z| \leq 1$ or in $|z| \geq 1$, a sharp refinement

$$|\alpha||c_0| + |\beta||c_n| \leq M\{(|\alpha| + |\beta|)/2\},$$

of new form of Visser's inequality, has also been obtained.

Key Words: Visser's inequality, polynomial, zeros, refinement.
2000 Mathematics Subject Classification: Primary: 30C10, Secondary 30A10.

1 Introduction and statement of results

While suggesting simpler proofs of certain inequalities associated with the polynomial $P(x)$, of degree n and leading coefficient 1, Visser [8] obtained the following analogous inequality for polynomial of a complex variable

Theorem A. *If $p(z) = \sum_{k=0}^{n} c_k z^k$ is a polynomial of degree n with arbitrary complex coefficients, then*

$$|c_0| + |c_n| \leq \max_{|z|=1} |p(z)|. \tag{1}$$

There is equality if and only if $p(z) = a_0 + a_n z^n$.

In the literature, there exist many generalizations and improvements [7, 5, 4, 3, 6], of Visser's inequality (1). In this note, we have used the following interesting result due to Arestov [1, 2].

Theorem B. *Let* ϕ *be a nondecreasing function defined on* $(0,\infty)$*, absolutely continuous in any finite interval* $[\alpha,\beta]$*,* $(\subset (0,\infty))$*, and such that* $u\phi'(u)$ *is nondecreasing. Further, let*

$$\gamma = (\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_n),\ (\in C^{n+1}),$$

be such that the polynomial

$$\Lambda_\gamma(z) = \sum\nolimits_{k=0}^{n} C(n,k)\gamma_k z^k$$

has all of its n *zeros in the* $|z| \leq 1$ *or in* $|z| \geq 1$*, with the convention that if the precise degree* m *of the polynomial* $\Lambda_\gamma(z)$ *is less than* n*, then* $z = \infty$ *will be a zero of* $\Lambda_\gamma(z)$ *of order* $(n-m)$*. Then for any polynomial* $P(z) = \sum_{k=0}^{n} a_k z^k$ *of degree at most* n

$$\int_0^{2\pi} \phi(|\Lambda_\gamma P(e^{it})|)dt \leq \int_0^{2\pi} \phi(D_n(\Lambda_\gamma)|P(e^{it})|)dt,$$

where

$$\begin{aligned} D_n(\Lambda_\gamma) &= \max(|\gamma_0|, |\gamma_n|), \\ \Lambda_\gamma P(z) &= \sum\nolimits_{k=0}^{n} \gamma_k a_k z^k \end{aligned}$$

and suggested a different proof of Visser's inequality (in a new but equivalent form), leading to the result

Theorem 1. *Let* $p(z) = \sum_{k=0}^{n} c_k z^k$ *be a polynomial of degree* n *with*
$\max_{|z|=1} |p(z)| = M.$
Then for arbitrary complex numbers α *and* β

$$|\alpha||c_0| + |\beta||c_n| \leq M\{\max(|\alpha|, |\beta|)\}. \tag{2}$$

Equality holds in (2) *for* $p(z) = Mz^n$*,* $|\alpha| \leq |\beta|$*.*

We, then used another interesting result due to Arestov [2], namely

Theorem C. *Let* ϕ *be a nondecreasing function defined on* $(0,\infty)$*, absolutely continuous in any finite interval* $[\alpha,\beta]$*,* $(\subset (0,\infty))$*, and such that* $u\phi'(u)$ *is nondecreasing. Further, let*

$$\gamma = (\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_n),\ (\in C^{n+1}),$$

be such that the polynomial

$$\Lambda_\gamma(z) = \sum\nolimits_{k=0}^{n} C(n,k)\gamma_k z^k$$

has all of its n zeros in $|z| \le 1$. *Then for any polynomial* $P(z) = \sum_{k=0}^{n} a_k z^k$ *of degree at most n, having all its zeros in* $|z| \ge 1$

$$\int_0^{2\pi}\int_0^{2\pi} \phi(|(1+e^{i\theta})\Lambda_\gamma P(e^{it})|)dtd\theta \le \int_0^{2\pi}\int_0^{2\pi} \phi(|(\gamma_0 + e^{i\theta}\overline{\gamma}_n)P(e^{it})|)dtd\theta,$$

where

$$\Lambda_\gamma P(z) = \sum\nolimits_{k=0}^{n} \gamma_k a_k z^k,$$

and obtained the following refinements of inequality (2).

Theorem 2. *Let* $p(z) = \sum_{k=0}^{n} c_k z^k$ *be a polynomial of degree n, having no zeros in* $|z| < 1$, *with* $\max_{|z|=1}|p(z)| = M$. *Then for complex numbers* α *and* β, *with* $|\alpha| \le |\beta|$

$$|\alpha||c_0| + |\beta||c_n| \le M\{(|\alpha| + |\beta|)/2\}. \tag{3}$$

Equality holds in (3) for $p(z) = (M/2)(\lambda + \mu z^n)$, *with* $|\lambda| = |\mu| = 1$.

On applying Theorem 2 to the polynomial $z^n p(1/z)$, we obtain

***Corollary* 1.** *Let* $p(z) = \sum_{k=0}^{n} c_k z^k$ *be a polynomial of degree n, having all its zeros in* $|z| \le 1$, *with* $\max_{|z|=1}|p(z)| = M$. *Then for complex numbers* α *and* β, *with* $|\alpha| \ge |\beta|$

$$|\alpha||c_0| + |\beta||c_n| \le M\{(|\alpha| + |\beta|)/2\}. \tag{4}$$

Equality holds in (4) for $p(z) = (M/2)(\lambda + \mu z^n)$, *with* $|\lambda| = |\mu| = 1$.

2 Proofs of the theorems.

Proof of Theorem 1. On applying Theorem B to the polynomial $p(z)$, with

$$\phi(u) = u^s, \ (s > 0),$$

$$\gamma_0 = \alpha, \gamma_1 = \gamma_2 = \ldots = \gamma_{n-1} = 0, \gamma_n = \beta,$$

we get

$$\int_0^{2\pi} |\alpha c_0 + \beta c_n e^{int}|^s dt \le \{\max(|\alpha|, |\beta|)\}^s \int_0^{2\pi} |p(e^{it})|^s dt,$$

i.e.

$$\Big(\frac{1}{2\pi}\int_0^{2\pi} |\alpha c_0 + \beta c_n e^{int}|^s dt\Big)^{1/s} \le \{\max(|\alpha|, |\beta|)\}\Big(\frac{1}{2\pi}\int_0^{2\pi} |p(e^{it})|^s dt\Big)^{1/s},$$

which implies, by letting $s \to \infty$,

$$\max_{|z|=1} |\alpha c_0 + \beta c_n z^n| \le \{\max(|\alpha|, |\beta|)\} M,$$

and inequality (2) follows.

Proof of Theorem 2. On applying Theorem C to the polynomial $p(z)$ with

$$\phi(u) = u^s, (s > 0),$$

$$\gamma_0 = \alpha, \gamma_1 = \gamma_2 = \ldots = \gamma_{n-1} = 0, \gamma_n = \beta,$$

we get

$$\Big(\int_0^{2\pi} |1 + e^{i\theta}|^s d\theta\Big)\Big(\int_0^{2\pi} |\alpha c_0 + \beta c_n e^{int}|^s dt\Big)$$

$$\le \Big(\int_0^{2\pi} |\alpha + \overline{\beta} e^{i\theta}|^s d\theta\Big)\Big(\int_0^{2\pi} |p(e^{it})|^s dt\Big),$$

i.e.

$$\Big(\frac{1}{2\pi}\int_0^{2\pi} |\alpha c_0 + \beta c_n e^{int}|^s dt\Big)^{1/s} \le \frac{\Big(\frac{1}{2\pi}\int_0^{2\pi} |\alpha + \overline{\beta} e^{i\theta}|^s d\theta\Big)^{1/s}}{\Big(\frac{1}{2\pi}\int_0^{2\pi} |1 + e^{i\theta}|^s d\theta\Big)^{1/s}} \Big(\frac{1}{2\pi}\int_0^{2\pi} |p(e^{it})|^s dt\Big)^{1/s},$$

which implies, by letting $s \to \infty$,

$$\max_{|z|=1} |\alpha c_0 + \beta c_n z^n| \le \Big(\frac{\max_{|z|=1} |\alpha + \overline{\beta} z|}{\max_{|z|=1} |1 + z|}\Big) M,$$

and inequality (3) follows.

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Received: 05.12.2004
Revised: 10.02.2006

Mathematics Department,
I.I.T. Kharagpur - 721302,
INDIA.
E-mail: `vkj@maths.iitkgp.ernet.in`