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On the coefficients of entire multiple Dirichlet series of several complex variables

by
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Abstract

A necessary and sufficient condition is found so that Goldberg order of a multiple Dirichlet series defining an entire function remains unaltered under rearrangements of coefficients of the series.

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1 Introduction and notations

We denote the complex and real $n-$space by $\mathbb{C}^n$ and $\mathbb{R}^n$ respectively. We indicate the elements

$$(s_1, s_2, \ldots, s_n), (Res_1, Res_2, \ldots, Res_n),$$

$$(\sigma_1, \sigma_2, \ldots, \sigma_n),$$

$$(m_1, m_2, \ldots, m_n)$$

etc. of $\mathbb{C}^n$ by their corresponding unsuffixed symbols s, Res, σ, m etc. and make use of the standard notations of the single variable which are easy to undersand from the context.

For $x, y \in \mathbb{C}^n$, we define

$$x = (x_1, x_2, \ldots, x_n),$$

$$y = (y_1, y_2, \ldots, y_n),$$

$$xy = (x_1 y_1, x_2 y_2, \ldots, x_n y_n),$$

$$\|x\| = x_1 + x_2 + \cdots + x_n,$$

$$x + r = (x_1 + r, x_2 + r, \ldots, x_n + r)$$

for $r \in \mathbb{R}$.

Also, we use

$$a_{\pi(m)} = a_{\pi(m_1)\pi(m_2)\ldots\pi(m_n)},$$

$$\pi(m) = (\pi(m_1), \pi(m_2), \ldots, \pi(m_n)),$$

and

$$\|\pi(m)\| = \pi(m_1) + \pi(m_2) + \cdots + \pi(m_n).$$

Let us consider the multiple Dirichlet series

$$f(s_1, s_2, \ldots, s_n) =$$

$$= \sum_{m_1, m_2, \ldots, m_n = 1}^{\infty} a_{m_1, m_2, \ldots, m_n} \exp\left(s_1 \lambda_{1_{m_1}} + s_2 \lambda_{2_{m_2}} + \cdots + s_n \lambda_{n_{m_n}}\right)$$

that is

$$f(s) = \sum_{m=1}^{\infty} a_m \exp \|s \lambda_{n_{m_n}}\|, \tag{1}$$

where $s_j = \sigma_j + it_j$, $j = 1, 2, \ldots, n$; $a_m \in \mathbb{C}$; $\lambda_{n_{m_n}}$ denotes the real-tuple

$$\left(\lambda_{1_{m_1}}, \lambda_{2_{m_2}}, \ldots, \lambda_{n_{m_n}}\right)$$

$0 \le \lambda_{p_1} \le \lambda_{p_2} \le \lambda_{p_k} \to \infty$ as $k \to \infty$ for $p = 1, 2, \ldots, n$.

Janusauskas [1] had shown that if there exists a tuple $p > \overline{0} = (0, 0, \ldots, 0)$ such that

$$\lim_{\|m\| \to \infty} \sup \frac{\sum_{k=1}^{\infty} \log m_k}{\|p \lambda_{n_{m_n}}\|} = 0 \tag{2}$$

then the domain of absolute convergence of the series (1) coincides with its domain of convergence.

Throughout we shall consider only those multiple Dirichlet series whose coordinate of associated abscissas of convergence will be all of finite or infinite, but not both.

2 Known results

In this section we state some known results in the form of lemmas which will be needed in the sequel.

Lemma 1 *(Sarkar [2]) The necessary and sufficient condition that the series (1) satisfiying (2) to be entire is that*

$$\lim_{\|m\|\to\infty} \frac{\log|a_m|}{\|\lambda_{n_{m_n}}\|} = -\infty. \tag{3}$$

If F stands for the family of all multiple Dirichlet series of the form (1) satisfiyng (2) and (3), then $f \in F$ denotes an entire function over $\mathbb{C}^n$ and we have:

Lemma 2 *(Sarkar [2]) The Goldberg order $\rho(D)$ of f with respect to the region $D = \{s : s \in \mathbb{C}^n, Res = \sigma \leq \lambda, \lambda \in \mathbb{R}^n\}$ is given by*

$$\lim_{\|m\|\to\infty} \sup \frac{\|\lambda_{n_{m_n}}\| \log \|\lambda_{n_{m_n}}\|}{-\log\{|a_m|\Phi_D(m)\}} = \rho(D) \equiv \rho, \qquad 0 \leq \rho \leq \infty, \tag{4}$$

where $\Phi_D(m) = \sup_{s\in D} \left|\exp\|s\lambda_{n_{m_n}}\|\right| = \exp\|\lambda\lambda_{n_{m_n}}\|$.

3 Main Results

Lemma 3 *Let $\pi : \mathbb{N} \to \mathbb{N}$ denote a rearrangement of all non-negative integers by which the integers m_i are replaced by the integers*

$$\pi(m_i) = m_i + o(m_i), \qquad i = 1, 2, \ldots, n,$$

then the rearranged multiple Dirichlet series

$$f_1(s) = \sum_{m=1}^{\infty} a_{\pi(m)} \exp\|s\lambda_{n_{m_n}}\|$$

represents an entire function.

Proof: From (3) it is known that the multiple Dirichlet series $f(s)$ represents an entire function if and only if

$$|a_m|^{\frac{1}{\|\lambda_{n_{m_n}}\|}} \to 0 \quad \text{as} \quad \|m\| \to \infty. \tag{5}$$

This gives, for any $1 > \varepsilon > 0$

$$|a_m|^{\frac{1}{\|\lambda_{n_{m_n}}\|}} < \varepsilon$$

for all large values of $\|m\|$.

Since distinct values of the pair m correspond to distinct values of the pair $\pi(m)$ and vice versa, it follows that

$$|a_{\pi(m)}|^{\frac{1}{\|\lambda_{n_{\pi(m_n)}}\|}} < \varepsilon \tag{6}$$

for all large values of $\|m\|$.

From the given condition, we get

$$\|\pi(m)\| = \|m\| + o(\|m\|)$$

and so for all large values of $\|m\|$,

$$1 - \delta < \frac{\|\pi(m)\|}{\|m\|} < 1 + \delta, \qquad \delta > 0. \tag{7}$$

From (6) and (7), we get, by choosing $\delta > 0$ sufficiently small

$$|a_{\pi(m)}|^{\frac{1}{\|\lambda_{n_{m_n}}\|}} < 2\varepsilon$$

for all large values of $\|m\|$.

Hence $f_1(s)$ represents an entire function. □

Theorem 1 *A necessary and sufficient condition that a rearrangement $\pi : \mathbb{N} \to \mathbb{N}$ keeps unaltered the Goldberg order $\rho(D)$ of an entire multiple Dirichlet series*

$$f(s) = \sum_{m=1}^{\infty} a_m \exp \|s\lambda_{n_{m_n}}\|$$

is that $\pi(m_i) = m_i + o(m_i)$, $i = 1, 2, \ldots, n$.

Proof: If $f(s)$ is an entire multiple Dirichlet series then, by Lemma 1, the rearranged series

$$f_1(s) = \sum_{m=1}^{\infty} a_{\pi(m)} \exp \|s\lambda_{n_{m_n}}\|$$

also represents an entire multiple Dirichlet series.

Let $\rho_1(D)$ be the Goldberg order of $f_1(s)$. Then

$$\lim_{\|m\|\to\infty} \sup \frac{\|\lambda_{n_{m_n}}\| \log \|\lambda_{n_{m_n}}\|}{-\log\{|a_{\pi(m)}|\Phi_D(m)\}} = \rho_1(D) \equiv \rho_1. \tag{8}$$

Sufficient part

Let $\pi(m_i) = m_i + o(m_i)$, $i = 1, 2, \ldots, n$. We want to show that $\rho_1 = \rho$. To this end, first we supose that $0 < \rho < \infty$. Then for any $\varepsilon > 0$ and sufficiently large values of $\|m\|$,

$$|a_m|\Phi_D(m) < \|\lambda_{n_{m_n}}\|^{-\frac{\|\lambda_{n_{m_n}}\|}{(\rho+\varepsilon)}}.$$

Since $\|\pi(m)\| \to \infty$ with $\|m\|$ and distinct values of pair m correspond to distinct values of the pair $\pi(m)$ and vice versa, we have for large m that

$$|a_{\pi(m)}|\Phi_D(\pi(m)) < \|\lambda_{n_{\pi(m_n)}}\|^{-\frac{\|\lambda_{n_{m_n}}\|}{(\rho+\varepsilon)}}.$$

Since $\frac{\|\pi(m)\|}{\|m\|} \to 1$, for sufficiently large $\|m\|$ we have

$$|a_{\pi(m)}|\Phi_D(m) < \|\lambda_{n_{m_n}}\|^{-\frac{\|\lambda_{n_{m_n}}\|}{(\rho+\varepsilon)}}. \tag{9}$$

Further, since $\|\pi(m)\| \to \infty$ with $\|m\|$, it follows that for infinitely many $\|m\|$

$$|a_{\pi(m)}|\Phi_D(\pi(m)) > \|\lambda_{n_{\pi(m_n)}}\|^{-\frac{\|\lambda_{n_{\pi(m_n)}}\|}{(\rho-\varepsilon)}}.$$

Choosing $\varepsilon > 0$ suitably, this give for infinetly many m

$$|a_{\pi(m)}|\Phi_D(m) > \|\lambda_{n_{m_n}}\|^{-\frac{\|\lambda_{n_{m_n}}\|}{(\rho-\varepsilon)}}. \tag{10}$$

From (9) and (10) we get

$$\lim_{\|m\|\to\infty} \sup \frac{\|\lambda_{n_{m_n}}\| \log \|\lambda_{n_{m_n}}\|}{-\log\{|a_{\pi(m)}|\Phi_D(m)\}} = \rho,$$

so that

$$\rho_1 = \rho.$$

The above argument with suitable alterations solve the case $\rho = 0$ and $\rho = \infty$.

Necessary Part

Let $\rho_1 = \rho$. Then for a given $\varepsilon > 0$

$$\|\lambda_{n_{m_n}}\| \log \|\lambda_{n_{m_n}}\| < (\rho + \varepsilon) \log\{|a_{\pi(m)}|\Phi_D(m)\}^{-1},$$

or

$$\|\lambda_{n_{m_n}}\| \log \|\lambda_{n_{m_n}}\| - (\rho + \varepsilon)\lambda\|\lambda_{n_{m_n}}\| < (\rho + \varepsilon) \log |a_{\pi(m)}|^{-1}, \tag{11}$$

for large $\|m\|$ and so for $\|\pi(m)\|$, while

$$\|\lambda_{n_{\pi(m_n)}}\| \log \|\lambda_{n_{\pi(m_n)}}\| - (\rho + \varepsilon)\lambda\|\lambda_{n_{\pi(m_n)}}\| > (\rho - \varepsilon) \log |a_{\pi(m)}|^{-1}, \tag{12}$$

for an infinity of $\|\pi(m)\|$.

From (11) and (12) we get

$$\frac{\|\lambda_{n_{\pi(m_n)}}\| \log \|\lambda_{n_{\pi(m_n)}}\| - (\rho - \varepsilon)\lambda\|\lambda_{n_{\pi(m_n)}}\|}{\|\lambda_{n_{m_n}}\| \log \|\lambda_{n_{m_n}}\| - (\rho + \varepsilon)\lambda\|\lambda_{n_{m_n}}\|} > \frac{\rho - \varepsilon}{\rho + \varepsilon}$$

for an infinity of $\|m\|$. Hence

$$\lim_{\|m\| \to \infty} \inf \frac{\|\lambda_{n_{\pi(m_n)}}\| \log \|\lambda_{n_{\pi(m_n)}}\| - \rho\lambda\|\lambda_{n_{\pi(m_n)}}\|}{\|\lambda_{n_{m_n}}\| \log \|\lambda_{n_{m_n}}\| - \rho\lambda\|\lambda_{n_{m_n}}\|} \geq 1 \tag{13}$$

Interchanging $\|m\|$ and $\|\pi(m)\|$, we get

$$\lim_{\|m\| \to \infty} \inf \frac{\|\lambda_{n_{m_n}}\| \log \|\lambda_{n_{m_n}}\| - \rho\lambda\|\lambda_{n_{m_n}}\|}{\|\lambda_{n_{\pi(m_n)}}\| \log \|\lambda_{n_{\pi(m_n)}}\| - \rho\lambda\|\lambda_{n_{\pi(m_n)}}\|} \geq 1$$

Therefore

$$\lim_{\|m\| \to \infty} \sup \frac{\|\lambda_{n_{\pi(m_n)}}\| \log \|\lambda_{n_{\pi(m_n)}}\| - \rho\lambda\|\lambda_{n_{\pi(m_n)}}\|}{\|\lambda_{n_{m_n}}\| \log \|\lambda_{n_{m_n}}\| - \rho\lambda\|\lambda_{n_{m_n}}\|} \leq 1 \tag{14}$$

Combining (13) and (14) we get

$$\frac{\|\lambda_{n_{\pi(m_n)}}\| \log \|\lambda_{n_{\pi(m_n)}}\| - \rho\lambda\|\lambda_{n_{\pi(m_n)}}\|}{\|\lambda_{n_{m_n}}\| \log \|\lambda_{n_{m_n}}\| - \rho\lambda\|\lambda_{n_{m_n}}\|} \to 1 \quad \text{as} \quad \|m\| \to \infty.$$

This gives

$$\frac{\lambda_{n_{\pi(m_n)}}}{\lambda_{n_{m_n}}} \to 1 \quad \text{as} \quad \|m\| \to \infty.$$

Hence

$$\frac{\pi(m_i)}{m_i} \to 1 \quad \text{as} \quad \|m\| \to \infty.$$

Thus we have

$$\pi(m_i) = m_i + o(m_i), \qquad i = 1, 2, \ldots, n.$$

This completes the proof of the theorem

$\square$

Theorem 2 *Let* $f \in F$ *be of Goldberg order* $\rho \quad (> 0)$ *and*

$$\lim_{r \to \infty} \inf \frac{logM_{f,D}(r)}{\exp(rp)} = t(D) \equiv t, \tag{15}$$

$$\frac{1}{e\rho} \lim_{\|m\| \to \infty} \inf \left[\|\lambda_{n_{m_n}}\| \{|a_m| \Phi_D(m)\}^{\frac{\rho}{\|\lambda_{n_{m_n}}\|}} \right] = \theta(D) \equiv \theta, \tag{16}$$

where

$$M_{f,D}(r) = \sup\{|f(s)| : s \in D + r, r \in \mathbb{R}\},$$

then

$$t \geq \theta \tag{17}$$

Proof: Let $0 < \theta < \infty$. Given $\varepsilon > 0$, we can find N so that

$$\frac{1}{e\rho}\|\lambda_{n_{m_n}}\|\{|a_m|\Phi_D(m)\}^{\frac{\rho}{\|\lambda_{n_{m_n}}\|}} > \theta - \varepsilon \quad \text{for all} \quad \|m\| > N.$$

Also, we have (Sarkar, [2]) for $r \geq r_0$

$$M_{f,D}(r) \geq |a_m|\Phi_D(m)\exp\|r\lambda_{n_{m_n}}\|.$$

Therefore, for $\|m\| > N$

$$\frac{logM_{f,D}(r)}{\exp(r\rho)} \geq \frac{\log|a_m| + \log\Phi_D(m) + \|r\lambda_{n_{m_n}}\|}{\exp(r\rho)} \geq$$

$$\geq \frac{1}{\exp(r\rho)}\{\frac{1}{\rho}\|\lambda_{n_{m_n}}\|\log(e\rho(\theta-\varepsilon)) - \frac{1}{\rho}\|\lambda_{n_{m_n}}\|\log\|\lambda_{n_{m_n}}\| + r\|\lambda_{n_{m_n}}\|\} \geq$$

$$\geq \frac{\rho\theta}{\|\lambda_{n_{m_n}}\|+1}\left(\frac{1}{\rho}\|\lambda_{n_{m_n}}\|\log(e\rho(\theta-\varepsilon)) - \frac{1}{\rho}\|\lambda_{n_{m_n}}\|\log(\rho\theta)\right) =$$

$$\frac{\theta}{1+\frac{1}{\|\lambda_{n_{m_n}}\|}}\{1+\log\left(1-\frac{\varepsilon}{\theta}\right)\} \sim$$

$$\sim \theta\{1+\log\left(1-\frac{\varepsilon}{\theta}\right)\}$$

Therefore

$$t \geq \theta$$

which obviously holds when $\theta = 0$. If $\theta = \infty$, the argument shows that $t = \infty$. This proves the theorem. □

References

[1] JANUSAUSKAS A.I, Elementary theorems on convergence of double Dirichlet series, Dokl. Akad. Nauk. SSSR, 234(1977), 610-614.

[2] SARKAR P.K., On the Goldberg order and Goldberg type of an entire function of several complex variables represented by multiple Dirichlet series, Indian J.Pure Appl. Math., 13(10)(1982), 1221-1229.

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