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Proximal continuity of multivalued maps

by
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Abstract

We study the proximal continuity of multivalued maps in proximity spaces and we discuss the relations between proximal continuity and semi-continuity.

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We show some properties of multivalued maps in proximity spaces. We will deal with the following definition:

Definition 1. *(see [3]) A relation δ on the power set of X is called a proximity on X if and only if δ satisfies the axioms:*
(1) $A\delta B$ implies $B\delta A$
(2) $(A \cup B)\delta C$ iff $A\delta C$ or $B\delta C$
(3) $A\delta B$ implies $A \neq \emptyset$ and $B \neq \emptyset$
(4) $A \cap B \neq \emptyset$ implies $A\delta B$
(5) $A \not\delta B$ implies that there exists a subset E such that $A \not\delta E$ and $(X \setminus E) \not\delta B$ for each $A, B, C \subset X$

Let X and Y be nonempty sets.

If $F : X \to Y$ is a multivalued map, then for any $M \subset Y$ we use the standard notations [1]:

$$F^{+}(M) = \{x \in X : F(x) \subset M\},$$

$$F^{-}(M) = \{x \in X : F(x) \cap M \neq \emptyset\}$$

and

$$F(A) = \bigcup_{x \in A} F(x)$$

for any $A \subset X$.

We define the proximal continuity of multivalued maps.
Let $(X, \delta_1), (X, \delta_2)$ be proximity spaces.

Definition 2. *A multivalued map $F : X \to Y$ is called upper (lower) proximally continuous (shortly: $p-$continuous) if and only if*

$$\forall_{A,B \subset Y}[A \not\delta_2 B \Rightarrow F^{+}(A) \delta_1 F^{+}(B)]$$

(resp.

$$\forall_{A,B \subset Y}[A \not\delta_2 B \Rightarrow F^{-}(A) \not\delta_1 F^{-}(B)])$$

A multivalued map $F : X \to Y$ is called p^*-continuous if and only if

$$\forall_{A,B \subset Y}[A \not\delta_2 B \Rightarrow F^{+}(A) \not\delta_1 F^{-}(B)]$$

Remark 1. *For a single-valuated function $f : (X, \delta_1) \to (Y, \delta_2)$ above forms of the proximal continuity are equivalent*

As an immediat consequence of definitions we have:

$$lower \quad p - continuity \Rightarrow p^* - continuity \Rightarrow upper \quad p - continuity$$

and none of these implications is invertable.

Example 1. *Let $\mathbb{R}$ be the real line and let δ be defined by $A \delta B \Leftrightarrow \overline{A} \cap \overline{B} \neq \emptyset$. This relation determines the natural topology. Let a multivalued map $F : (\mathbb{R}, \delta) \to (\mathbb{R}, \delta)$ be defined as follows:*

$$F(x) = \begin{cases} (-\infty, 0) \setminus \{-3, -2\} & for \quad x \in (-\infty, 1], \\ \{-3, -1, 5\} & for \quad x \in (1, 3), \\ (2, 7) \cup \{-5\} & for \quad x \in [3, +\infty) \end{cases}$$

We will show, that F is not p^*- continuous but is upper $p-$continuous.
For instance, there exists two sets

$$C = (-\infty, 0)$$

and

$$D = (1,6)$$

such that $C \not\delta D$ *and* $F^+(C) = (-\infty, 1]$, $F^-(D) = (1, +\infty)$. *Certainly*

$$\overline{F^+(C)} \cap \overline{F^-(D)} = \{1\} \neq \emptyset,$$

so

$$F^+(C) \delta F^-(D).$$

Then F *is not* p^*-*continuous.*

Let us notice, that for each $A, B \subset \mathbb{R}$ *we have* $F(A) \cap F(B) \neq \emptyset$. *Thus*

$$\overline{F(A)} \cap \overline{F(B)} \neq \emptyset,$$

so

$$F(A) \delta F(B).$$

Then F *is upper* $p-$ *continuous.*

Example 2. *Let* $\mathbb{R}$ *be the real line and let* δ *be defined by* $A\delta B \Leftrightarrow A \cap B \neq \emptyset$. *This relation determines the discret topology. Let* $F : (\mathbb{R}, \delta) \to (\mathbb{R}, \delta)$ *be a multivalued map defined by:*

$$F(x) = \begin{cases} (-\infty, 0) \backslash \{-3, -2\} & for \quad x \in (-\infty, 1) \backslash \{1 - \frac{1}{n} : n = 1, 2, ...\}, \\ \{-3\} & for \quad x \in \{1 - \frac{1}{n} : n = 1, 2, ...\}, \\ \{-2\} & for \quad x \in \{1 + \frac{1}{n} : n = 1, 2, ...\}, \\ (2, 7) & for \quad x \in [1, \infty) \backslash \{1 + \frac{1}{n} : n = 1, 2, ...\} \end{cases}$$

We show, that this multivalued map is p^*-*continuous but is not lower* $p-$*continuous.*

Let $C, D \subset Y$ *and* $C \not\delta D \neq \emptyset$, *then* $C \cap D \neq \emptyset$. *Suppose that*

$$F^+(C) \delta F^-(D).$$

Then

$$F^+(C) \cap F^-(D) \neq \emptyset,$$

so exists $x \in \mathbb{R}$ *such that* $x \in F^+(C)$ *and* $x \in F^-(D)$. *Hence* $F(x) \subset C$ *and* $F(x) \cap D \neq \emptyset$, *cosequently* $C \cap D \neq \emptyset$. *Then* $C \delta D$ *and this is a contadiction. All in all,* F *is* p^*- *continuous.*

We are going to show that this map is not lower $p-$*continuous. Let* $C = (-5, -3)$ *and* $D = (-3, 0)$. *Of course* $C \cap D = \emptyset$, *so* $C \not\delta D$. *But*

$$F^{-}(C) = (-\infty, 1)\backslash\{1 - \frac{1}{n} : n = 1, 2, ...\}$$

and

$$F^{-}(D) = F^{-}(C) \cup \{1 + \frac{1}{n} : n = 1, 2, ...\}$$

.

So

$$F^{-}(C) \cap F^{-}(D) \neq \emptyset,$$

then

$$F^{-}(C)\delta F^{-}(D).$$

Corollary 1. *The upper p−continuity does not imply the lower p−continuity.*

Let $(X, \delta_1), (Y, \delta_2)$ be the proximity spaces.

Theorem 1. *For a multivalued map $F : (X, \delta_1) \to (Y, \delta_2)$ the following conditions are equivalent:*
(1) the multivalued map F is upper $p-$continuous,
(2) $\forall_{A,B\subset X}[A\not\delta_1 B \Rightarrow F(A)\delta_2 F(B)]$
(3) $\forall_{A\subset X}\forall_{B\subset Y}[F(A)\not\delta_2 B \Rightarrow A\not\delta_1 F^{+}(B)]$

Proof: We will show the implication (1) $\Rightarrow$ (2).

Suppose, that exists two sets $A, B \subset X$ such that $A\delta_1 B$ and $F(A)\not\delta_2 F(B)$. From (1) we have

$$F^{+}(F(A))\not\delta_1 F^{+}(F(B)).$$

Because $A \subset F^{+}(F(A))$ and $B \subset F^{+}(F(B))$, so $A\not\delta_1 B$ and this is a cotradiction.

Now we suppose that exists the sets $C, D \subset Y$ such that $C\not\delta_2 D$ and

$$F^{+}(C)\delta_1 F^{+}(D).$$

With (2) we have

$$F(F^{+}(C))\delta_1 F(F^{+}(D)).$$

Because $F(F^{+}(C)) \subset C$ and $F(F^{+}(D)) \subset D$ we have $C\delta_2 D$ and this contradiction finishes the proof.

Now we will show the equivalence (1) $\Leftrightarrow$ (3). Let $A \subset X$ and $B \subset Y$ be such that $F(A)\not\delta_2 B$. By the upper $p-$continuity we have

$$F^{+}(F(A))\not\delta_1 F^{+}(B).$$

But $A \subset F^+(F(A))$, then $A \not\delta_1 F^+(B)$. Suppose that F is not upper $p-$conti nuous. Then exists the sets $A_1, A_2 \subset X$ such that $A_1 \delta_1 A_2$ and

$$F(A_1) \not\delta_2 F(A_2).$$

From assumption we have

$$A_1 \not\delta_1 F^+(F(A_2)).$$

Of course $A_2 \subset F^+(F(A_2))$. Then $A_1 \not\delta_1 A_2$. This contradiction finishes the proof.

□

Lemma 1. *If a multivalued map $F : (X, \delta_1) \to (Y, \delta_2)$ is lower $p-$continuous, then*

$$\forall_{A \subset X} \forall_{B \subset Y} [F(A) \not\delta_2 B \Rightarrow A \not\delta_1 F^-(B)]$$

Proof: The proof is as previously □

Theorem 2. *For a multivalued map $F : (X, \delta_1) \to (Y, \delta_2)$ the following conditions are equivalent:*
(1) the multivalued map F is p^*-continuous,
(2) $\forall_{A \subset X} \forall_{B \subset Y} [F(A) \not\delta_2 B \Rightarrow A \not\delta_1 F^-(B)]$

Proof: Let $A \subset X, B \subset Y$ and let $F(A) \not\delta_2 B$. From the assumption

$$F^+(F(A)) \not\delta_1 F^-(B).$$

Because $A \subset F^+(F(A))$, then $A \not\delta_1 F^-(B)$. Now we will show the implication (2) $\Rightarrow$ (1). Let $C, D \subset Y$ and let $C \not\delta_2 D$. If $F^+(C) = \emptyset$, then $F^+(C) \not\delta_1 F^-(D)$. Now let $F^+(C) \neq \emptyset$. Because $F(F^+(C)) \subset C$ and $C \not\delta_2 D$, then $F(F^+(C)) \not\delta_2 D$. From assumption $F^+(C) \not\delta_1 F^-(D)$. □

A multivalued map $F : (\mathbb{R}, \delta_1) \to (\mathbb{R}, \delta_2)$ is always p^*-continuous, where $A \delta_1 B \Leftrightarrow A \cap B \neq \emptyset$ but δ_2 is any proximity relation. Obviously F is upper $p-$continuous too.

Now we will show some properties of the lower semi-continuity.
Let $(X, \delta_1), (Y, \delta_2)$ be proximity spaces.

Theorem 3. *If a multivalued map $F : (X, \delta_1) \to (Y, \delta_2)$ is lower semi-continuous, then*

$$(*) \qquad \forall_{x \in X} \forall_{A \subset X} [x \delta_1 A \Rightarrow F(x) \delta_2 F(A)]$$

Proof: Let $x \in X, A \subset X$ and let $x\delta_1 A$. Then $x \in \overline{A}$. Since F is lower semi-continuous we have $F(\overline{A}) \subset \overline{F(A)}$, hence $F(x) \subset \overline{F(A)}$. Of course $F(x) \cap \overline{F(A)} \neq \emptyset$, so $F(x)\delta_2\overline{F(A)}$, i.e. $F(x)\delta_2 F(A)$. □

This theorem is not invertable; i.e. the condition (*) does not imply the lower semi-continuity.

Example 3. *Let δ be the natural proximity in $\mathbb{R}$ induced by the euclidian metric and let $\mathbb{Z}$ denote the set of all integers. The map $F : (\mathbb{R}, \delta) \to (\mathbb{R}, \delta)$ defined by*

$$F(x) = \begin{cases} [0,1] & \text{for} \quad x \in \mathbb{Z} \\ 0 & \text{for} \quad x \notin \mathbb{Z} \end{cases}$$

is not lower semi-continuous at each $x \in \mathbb{Z}$. Let $x_0 \in \mathbb{Z}$ then for the set $G = (\frac{1}{2}, 3)$ we have $G \cap F(x) \neq \emptyset$, but for each neighbourhood U of x_0 exists $x \in U \backslash \mathbb{Z}$ such that $G \cap F(x_0) = \emptyset$. On the other hand the map F satisfies (*). For instance let $A \subset \mathbb{R}$, $x \in \mathbb{R}$ and let $x\delta A$. Of course $x \in \overline{A}$ and

$$F(A) = \begin{cases} [0,1] & \text{for} \quad A \cap \mathbb{Z} \neq \emptyset \\ 0 & \text{for} \quad A \cap \mathbb{Z} = \emptyset \end{cases}$$

We consider two situations. First, let $x \notin \mathbb{Z}$. Then $F(x) = \{0\}$. So $F(x)\delta F(A)$. Second, let $x \in \mathbb{Z}$. Then $F(x) = [0,1]$ and $F(x)\delta F(A)$.

The standard proof of the following theorem we shall omit.
Let $(X, \delta_1), (Y, \delta_2)$ be proximity spaces.

Theorem 4. *For a multivalued map $F : (X, \delta_1) \to (Y, \delta_2)$ the following conditions are equivalent:*
(1) the map F is lower semi-continuous,
(2) $\forall_{x \in X} \forall_{A \subset X} [x\delta_1 A \Rightarrow \forall_{y \in F(x)} y\delta_2 F(A)]$.

Let X, Y be topological spaces.

Definition 3. *A multivalued map $F : X \to Y$ is called semi-continuous at a point $x_0 \in X$ if and only if for each open set $W \subset Y$ with $F(x_0) \subset W$ there exists a neighbourhood U of x_0 such that $F(x) \cap W \neq \emptyset$ for each $x \in U$.*

Lemma 2. *For a multivalued map $F : X \to Y$ the following conditions are equivalent:*
(1) F is semi-continuous;
(2) $F^+(W) \subset IntF^-(W)$ for each an open set $W \subset Y$;

(3) $\overline{F^+(D)} \subset F^-(D)$ *for each a closed set* $D \subset Y$*;*
(4) $\forall_{B\subset Y} F^+(IntB) \subset IntF^-(B)$*;*
(5) $\forall_{B\subset Y} \overline{F^+(B)} \subset F^-(\overline{B})$*.*

Remark 2. *If a map F is upper semi-continuous or lower semi-continuous then it is semi-continuous.*

Let us notice, that it is not invertable; i.e. semi-continuity does not imply the upper nor lower semi-continuity. In Example 1 the multivalued map F is upper $p-$continuous so it is semi-continuous but F is neither upper nor lower semi-continuous.

Now we will discuss the relations between $p-$continuity and semi-continuity. Let $(X, \delta_1), (Y, \delta_2)$ be the proximity spaces and let $F : X \to Y$ be a multivalued map.

Theorem 5. *If a map* $F : (X, \delta_1) \to (Y, \delta_2)$ *has compact values and F is upper* $p-$*continuous, then* $F : (X, \tau_{\delta_1}) \to (Y, \tau_{\delta_2})$ *is semi-continuous.*

Proof: Suppose that F is not semi-continuous. Exists a set $D \subset Y$ such that $\overline{F^+(D)} \not\subset F^-(\overline{D})$. Let

$$x_0 \in \overline{F^+(D)} \backslash F^-(\overline{D}).$$

Then $F(x_0) \cap \overline{D} = \emptyset$. Since F has compact values we have $F(x_0) \not\delta_2 D$, so $\{x_0\} \not\delta_1 F^+(D)$, i.e. $x_0 \notin \overline{F^+(D)}$ and this contradiction finishes the proof □

This theorem is not invertable.

Example 4. *Let* $\mathbb{R}$ *be the real line and let* δ *be defined by* $A\delta B \Leftrightarrow \overline{A} \cap \overline{B} \neq \emptyset$*. Let a multivalued map* $F : (\mathbb{R}, \delta) \to (\mathbb{R}, \delta)$ *be defined as follows:*

$$F(x) = \begin{cases} [3,5] & for \quad x < 2, \\ [1,4] & for \quad x = 2, \\ [3/2,\ 2] & for \quad x > 2 \end{cases}$$

The map F is semi-continuous but is not upper $p-$*continuous.*

For instance, there exist two sets $C = [3,5]$ *and* $D = [\frac{3}{2}, 2]$ *such that* $C \not\delta D$ *and* $F^+(C) = (-\infty, 2)$*,* $F^+(D) = (2, \infty)$ *so* $F^+(C)\delta F^-(D)$*. Then F is not upper* $p-$*continuous.*

Theorem 6. *Let* $F : (X, \delta_1) \to (Y, \delta_2)$ *be a* p^*-*continuous map. Then*
(1) F is lower semi-continuous;
(2) if F has compact values, then it is a upper semi-continuous.

Proof: We will show the implication (1) $\Rightarrow$ (2).

Let $B, W \subset Y$ be such that $B \not\delta_2 (Y \backslash W)$. With assumption

$$F^+(B) \not\delta_1 F^-(Y \backslash W),$$

so

$$\overline{F^+(B)} \not\delta_1 F^-(Y \backslash W).$$

Then $\overline{F^+(B)} \subset F^+(W)$; i.e. we have shown, that

$$B \not\delta_2 (Y \backslash W) \Rightarrow \overline{F^+(B)} \subset F^+(W).$$

Let us notice, that

$$\overline{F^+(B)} \subset \bigcap \{F^+(W) : B \not\delta_2 (Y \backslash W)\} = F^+(\bigcap \{W : B \not\delta_2 (Y \backslash W)\}) = F^+(\overline{B}).$$

This means, that F is lower semi-continuous.

Now we will show (2) $\Rightarrow$ (1). Let $x_0 \in X$ and let $W \subset Y$ be an open set such that $F(x_0) \subset W$, then

$$F(x_0) \cap (Y \backslash W) = \emptyset.$$

Because $F(x_0)$ is a compact set and $Y \backslash W$ is closed, we obtain

$$F(x_0) \not\delta_2 (Y \backslash W).$$

From assumption we have

$$F^+(F(x_0)) \not\delta_1 F^-(Y \backslash W).$$

Of course $x_0 \in F^+(F(x_0))$, so $x_0 \not\delta_1 F^-(Y \backslash W)$. Hence exists a set $E \subset X$ such that $x_0 \not\delta_1 (X \backslash E)$ and $E \not\delta_1 F^-(Y \backslash W)$. So $x_0 \notin \overline{X \backslash E}$. i.e. $x_0 \in IntE$. Let $x \in IntE$. Then $x \notin \overline{F^-(Y \backslash W)}$, consequently $x \in IntF^+(W) \subset F^+(W)$. Hence $F(x) \subset W$, i.e. F is upper semi-continuous. □

Theorem 7. *If a map $F : (X, \delta_1) \to (Y, \delta_2)$ is lower p-continuous; then it is upper and lower semi-continuous.*

Proof: Let $B, W \subset Y$ be such that $B \not\delta_2 (Y \backslash W)$. With assumption

$$F^-(B) \not\delta_1 F^-(Y \backslash W),$$

so

$$\overline{F^-(B)} \not\delta_1 (X \backslash F^+(W)).$$

Then $\overline{F^-(B)} \subset F^+(W)$; i.e. we have shown that

$$B \not\delta_2 (Y \backslash W) \Rightarrow \overline{F^-(B)} \subset F^+(W)$$

for each $W \subset Y$. Hence

$$\overline{F^+(B)} \subset \overline{F^-(B)} \subset \bigcap \{F^+(W) : B \not\delta_2 (Y \backslash W)\} = F^+(\bigcap \{W : B \not\delta_2 (Y \backslash W)\}) =$$

$$= F^+(\overline{B}) \subset F^-(\overline{B}).$$

That means that $\overline{F^+(B)} \subset F^+(\overline{B})$ and $\overline{F^-(B)} \subset F^-(\overline{B})$ i.e. F is upper and lower semi-continous □

Let us notice that the upper and lower semi-continuity does not imply the p^*-continuity and lower $p-$continuity.

Example 5. *Let* $\mathbb{R}$ *be the real line and let* δ_1, δ_2 *be defined by*

$$A \delta_1 B \Leftrightarrow [dist(A, B) = 0 \wedge A \neq \emptyset \neq B]$$

and

$$A \delta_2 B \Leftrightarrow \overline{A} \cap \overline{B} \neq \emptyset.$$

Both relations determine the same natural topology. Let a multivalued map

$$F : (\mathbb{R}, \delta_1) \to (\mathbb{R}, \delta_2)$$

be defined by $F(x) = \{x\}$ *for each* $x \in \mathbb{R}$.

Evidently this multivalued map F *is upper and lower semi-continuous but* F *is neither* p^*-*continuous nor lower* $p-$*continuous.*

Now we will show that this map is not upper $p-$*continuous. Let*

$$A = \{1, 2, 3, \dots\}, \quad B = \{n - \frac{1}{n} : n \in \mathbb{N}\}.$$

The A *and* B *are closed sets in* $(\mathbb{R}, \delta_1)$. *We denote by* $a_n = n$ *and* $b_n = n - \frac{1}{n}$, *then* $|a_n - b_n| = \frac{1}{n}$, *for each* $n \in \mathbb{N}$. *Of course*

$$dist(A, B) \leq |a_n - b_n| = \frac{1}{n}$$

for $n \in \mathbb{N}$. *So* $dist(A, B) = 0$ *and* $A \neq \emptyset \neq B$ *Then* $A \delta_1 B$. *We have*

$$\overline{F(A)} \cap \overline{F(B)} = \overline{A} \cap \overline{B} = A \cap B = \emptyset.$$

It means that $F(A) \not\delta_2 F(B)$. *So the map* F *is not upper* $p-$*continuous. Obviously* F *is not* p^*- *continuous and lower* $p-$*continuous.*

Let us notice, that the upper $p-$continuity does not depend on the upper and lower semi-continuity. In Example 1 the multivalued map F is upper $p-$continuous but it is neither upper nor lower semi-continuous.

Really, taking the open set $G = (-6, -4) \cup (2, 7)$ in Y the set $F^+(G) = [3, +\infty)$ is not open in X. Similarly for the closed set $E = \{-3, -1, 5\} \subset Y$ the set $F^+(E) = (1, 3)$ is not closed in X. On the other hand, in Example 5 the map F is upper and lower semi-continuous but F is not upper $p-$continuous.

Now we will show some properties of superposition, sum and product of multivalued maps.

Let $(X, \delta_1), (Y, \delta_2), (Z, \delta_3)$ be the proximity spaces. Let $F : (X, \delta_1) \to (Y, \delta_2)$ and $G : (Y, \delta_2) \to (Z, \delta_3)$ be the multivalued maps.

Theorem 8. *If F and G are lower p-continuous (upper $p-$continuous, p^*- continuous) then the superposition $G \circ F$ is lower p-continuous (upper $p-$continuous, p^*-continuous).*

Proof: Let $A, B \subset \mathbb{Z}$ and $A \not\delta_3 B$. Since G is lower p- continuous we have

$$G^-(A) \not\delta_2 G^-(B).$$

So by the lower p-continity of the map F we have

$$F^-(G^-(A)) \not\delta_1 F^-(G^-(B)),$$

that is

$$(G \circ F)^-(A) \not\delta_1 (G \circ F)^-(B).$$

So $G \circ F$ is lower p-continuous. Remaining proofs are as previously. □

Let $(X, \delta_1), (Y, \delta_2)$ be the proximity spaces and let $F, G : X \to Y$ be the multivalued maps.

Theorem 9. *If F and G are upper p-continuous, then $F \cup G$ is upper p-continuous.*

Proof: Let $C, D \subset Y$ and $C \not\delta_2 D$. From the asumption $F^+(C) \not\delta_1 F^+(D)$ and $G^+(C) \not\delta_1 G^+(D)$. Because

$$F^+(C) \cap G^+(C) \subset F^+(C)$$

and

$$F^{+}(C) \cap G^{+}(C) \subset G^{+}(C)$$

then

$$F^{+}(C) \cap G^{+}(C) \not\delta_1 F^{+}(D) \cup G^{+}(D).$$

But

$$(F \cup G)^{+}(C) = F^{+}(C) \cap G^{+}(C),$$

so

$$(F \cup G)^{+}(C) \not\delta_1 F^{+}(D) \cup G^{+}(D).$$

Furthemore

$$(F \cup G)^{+}(D) \subset F^{+}(D) \cup G^{+}(D),$$

then

$$(F \cup G)^{+}(C) \not\delta_1 (F \cup G)^{+}(D),$$

that is $(F \cup G)$ is upper $p-$continuous. □

Corollary 2. *If the maps F and G are lower $p-$continuous (upper $p-$continuous, p^*-continuous) then $F \cup G$ is upper $p-$continuous.*

Theorem 10. *If F and G are p^*-continuous, then $F \cup G$ is p^*-continuous.*

Proof: Let $C, D \subset Y$ and $C \not\delta_2 D$. From the asumption $F^{+}(C) \not\delta_1 F^{-}(D)$ and $G^{+}(C) \not\delta_1 G^{-}(D)$ so

$$F^{+}(C) \cap G^{+}(C) \not\delta_1 F^{-}(D) \cup G^{-}(D).$$

Because

$$(F \cup G)^{+}(C) = F^{+}(C) \cap G^{+}(C)$$

and

$$(F \cup G)^{-}(D) = F^{-}(D) \cup G^{-}(D),$$

so

$$(F \cup G)^{+}(C) \not\delta_1 (F \cup G)^{-}(D)$$

i.e. $(F \cup G)$ is p^*-continuous. □

Corollary 3. *If the maps F and G are lower $p-$continuous or F is lower $p-$continuous and G is p^*-continuous, then $F \cup G$ is p^*-continuous consequently upper $p-$continuous.*

Let us notice, that union of two multivalued maps which are lower $p-$continuous need not be lower $p-$continuous.

Example 6. *Let $\mathbb{R}$ the real line and let δ be defined by $A\delta B \Leftrightarrow A \cap B \neq \emptyset$. Let the multivalued maps $F, G : (\mathbb{R}, \delta) \to (\mathbb{R}, \delta)$ be defined as follows: $F(x) = \{1\}, G(x) = \{2\}$, for $x \in \mathbb{R}$. The maps F, G are lower $p-$continuous. For instance, let $C, D \subset \mathbb{R}$ and $C \not\delta D$, then $C \cap D = \emptyset$.*

We consider two situations. First, let $1 \in C$, then $1 \notin D$. We have

$$F^-(C) = \mathbb{R}, \quad F^-(D) = \emptyset.$$

Hence $F^-(C) \not\delta F^-(D)$. Second, let $1 \notin C$, then $1 \in D$ or $1 \notin D$. If $1 \in D$ then we have

$$F^-(C) = \emptyset, \quad F^-(D) = \mathbb{R},$$

but

$$F^-(C) \not\delta F^-(D).$$

If $1 \notin D$ then we have $F^-(C) = \emptyset$, $F^-(D) = \emptyset$ so $F^-(C) \not\delta F^-(D)$.

And analogous for G. Of course $(F \cup G)(x) = \{1, 2\}$, for each $x \in \mathbb{R}$. We will show that $F \cup G$ is not lower $p-$continuous. Let $C = \{1\}, D = (1\frac{1}{2}, 5)$. Because $C \cap D = \emptyset$ so $C \not\delta D$. But $(F \cup G)^-(C) = \mathbb{R}$ and $(F \cup G)^-(D) = \mathbb{R}$ then

$$(F \cup G)^-(C) \cap (F \cup G)^-(D) = \mathbb{R} \neq \emptyset.$$

So

$$(F \cup G)^-(C) \not\delta (F \cup G)^-(D).$$

That means $(F \cup G)$ is not lower $p-$continuous.

Theorem 11. *If F and G are lower $p-$continuous, then $F \cap G$ is lower $p-$continuous.*

Proof: Let $C, D \subset Y$ and $C \not\delta_2 D$. From assumption

$$F^-(C) \not\delta_1 F^-(D)$$

and

$$G^-(C)\not\delta_1 G^-(D),$$

so

$$(F^-(C) \cap G^-(C))\not\delta_1 F^-(D)$$

and

$$F^-(C) \cap G^-(C)\not\delta_1 G^-(D).$$

Because

$$(F \cap G)^-(C) \subset F^-(C) \cap G^-(C),$$

so

$$(F \cap G)^-(C)\not\delta_1 F^-(D)$$

and

$$(F \cap G)^-(C)\not\delta_1 G^-(D)$$

then

$$(F \cap G)^-(C)\not\delta_1 F^-(D) \cup G^-(D).$$

Of course

$$(F \cap G)^-(D) \subset F^-(D) \cup G^-(D),$$

so

$$(F \cap G)^-(D)\not\delta_1 (F \cap G)^-(C).$$

Then

$$(F \cap G)^-(C)\not\delta_1 (F \cap G)^-(D),$$

i.e. $F \cap G$ is lower $p-$continuous. □

Corollary 4. *If the maps F and G are lower $p-$continuous then $F \cap G$ is p^*-continuous, so upper $p-$continuous.*

Let us notice that product of two multivalued maps which are upper $p-$continuous need not be upper $p-$continuous.

Example 7. *Let* $\mathbb{R}$ *be the real line and let* δ *defined by* $A\delta B \Leftrightarrow \overline{A}\cap\overline{B}\neq\emptyset$. *Let the multivalued maps* $F, G : (\mathbb{R},\delta)\to(\mathbb{R},\delta)$ *be defined as follows:*

$$F(x)=\begin{cases}(-\infty,0)\backslash\{-3,-2\} & \text{for} \quad x\in(-\infty,1]\\ \{-3,-1,5\} & \text{for} \quad x\in(1,+\infty),\end{cases}$$

$$G(x)=\begin{cases}(-5,2] & \text{for} \quad x\in(-\infty,1)\\ [1,8) & \text{for} \quad x\in(1,+\infty),\end{cases}$$

The maps F, G *are upper* $p-$*continuous.*

For instance, for all $A, B\subset\mathbb{R}$ *we have* $F(A)\cap F(B)\neq\emptyset$. *Thus* $\overline{F(A)}\cap\overline{F(B)}\neq\emptyset$, *i.e.* $F(A)\delta F(B)$. *Analogously we have that* G *is upper* $p-$*continuous. Of course:*

$$(F\cap G)(x)=\begin{cases}(-5,0)\backslash\{-3,-2\} & \text{for} \quad x\in(-\infty,1]\\ \{5\} & \text{for} \quad x\in(1,+\infty),\end{cases}$$

$F\cap G$ *is not upper* $p-$*continuous. Really, exists two sets* $A=(-3,1)$ *and* $B=(1,5)$. $A\delta B$ *because* $\overline{A}\cap\overline{B}=\{1\}\neq\emptyset$. *But* $F(A)=(-5,0)\backslash\{-3,-2\}$ *and* $F(B)=\{5\}$. *Hence* $\overline{F(A)}\cap\overline{F(B)}=\emptyset$, *i.e.* $F(A)\not\delta F(B)$. *That means that* $F\cap G$ *is not upper* $p-$*continuous.*

In a topological space (Y,τ) we denote by $S(Y)$ the family of all nonempty subsets of Y. For an open set $U\subset Y$ we write

$$U^{+}=\{B\in S(Y): B\subset U\},\quad U^{-}=\{B\in S(Y): B\cap U\neq\emptyset\}.$$

The families $\mathcal{B}^{+}=\{U^{+}: U\in\tau\}$ and $\mathcal{P}^{-}=\{U^{-}: U\in\tau\}$ form a base and a subbase of the upper and lower Vietoris toplogy respectively. These topologies will be denoted by τ^{+} and τ^{-}. We write $A\in\tau^{-}-\lim A_j$ and $A\in\tau^{+}-\lim A_j$ if the net $\{A_j : j\in J\}$ converges to A in the space $(S(Y),\tau^{-})$ or $(S(Y),\tau^{+})$, respectively.

Let X be a nonempty set and (Y,δ) be a proximity space.

Definition 4. *A net* $\{F_j : j\in J\}$ *of multivalued maps* $F_j : X\to Y$ *is said to be convergent in a sense of Leader to a multivalued map* $F : X\to Y$ *if*

$$\forall_{A\subset X, B\subset Y}[F(A)\not\delta B\Rightarrow\exists_{j_0\in J}\forall_{j\geq j_0}F_j(A)\not\delta B].$$

Let X be topological space, $(Y,\delta)-$the proximity space and let $F, F_j : X\to Y$ be the multivalued maps for each $j\in J$.

Theorem 12. *If a net $\{F_j : j \in J\}$ of maps $F_j : X \to Y$ is convergent in a sense of Leader to a multivalued map $F : X \to Y$ and F has a compact value, then $F \in \tau^+ - \lim F_j$.*

Proof: Let $x_0 \in X$ and let U be an open set such that $F(x_0) \subset U$. Of course $F(x_0) \cap (Y \backslash U) = \emptyset$. Moreover $F(x_0)$ is a compact set, then $F(x_0) \not\delta (Y \backslash U)$. From the Leader convergence exists $j_0 \in J$ such that $F_j(x_0) \not\delta (Y \backslash U)$ for each $j \geq j_0$. So $F_j(x_0) \cap (Y \backslash U) = \emptyset$. That means $F_j(x_0) \subset U$. I.e. $F \in \tau^+ - \lim F_j$. □

Let $(X, \delta_1), (Y, \delta_2)$ be the proximity spaces and let $F, F_j : X \to Y$ be the multivalued maps for each $j \in J$.

Theorem 13. *If a net $\{F_j : j \in J\}$ of maps $F_j : X \to Y$ is convergent in a sense of Leader to a multivalued map $F : X \to Y$ and the maps F_j are upper $p-$continuous then F is upper $p-$continuous.*

Proof: Let $A, B \subset X$ and $A \delta_1 B$. Suppose that $F(A) \not\delta_2 F(B)$. Then exists a set $E \subset Y$ such that $F(A) \not\delta_2 E$ and $(Y \backslash E) \not\delta_2 F(B)$. Since $F_j : X \to Y$ is Leader convergent to $F : X \to Y$ then exists $j_0, j_1 \in J$ such that $F_j(A) \not\delta_2 E$ for each $j \geq j_0$ and $(Y \backslash E) \not\delta_2 F_j(B)$ for each $j \geq j_1$. Of course $F_j(A) \subset (Y \backslash E)$ for all $j \geq j_0$. Hence $F_j(A) \not\delta_2 F_j(B)$ for each $j \geq \max\{j_0, j_1\}$. From assumption each multivalued map F_j is upper $p-$continuous, so $A \not\delta_1 B$ and this contradiction finishes the proof. □

Theorem 14. *If the maps $F_j : X \to Y$ are lower semi-continuous and $\{F_j : j \in J\}$ is convergent in a sense of Leader to a multivalued map $F : X \to Y$ and $F \in \tau^- - \lim F_j$, then F is lower semi-continuous.*

Proof: Let $A \subset X$ and $x_0 \in \overline{A}$. Suppose that $F(x_0) \not\subset \overline{F(A)}$. Then exists $y \in F(x_0)$ such that $y \notin \overline{F(A)}$. That mean $y \not\delta F(A)$. So exists a set $E \subset Y$ such that $y \not\delta E$ and $(Y \backslash E) \not\delta F(A)$. Because $y \not\delta E$, so $y \not\delta \overline{E}$, consequently $y \in Y \backslash \overline{E}$. We have $F(x_0) \cap (Y \backslash \overline{E}) \neq \emptyset$. Because $F \in \tau^- - \lim F_j$, so there exists $j_0 \in J$ with $F_j(x_0) \cap (Y \backslash \overline{E}) \neq \emptyset$ for each $j \geq j_0$. Moreover the Leader convergence of the net $\{F_j : j \in J\}$ to F implies the existence of $j_1 \in J$ with the property $F_j(A) \not\delta (Y \backslash \overline{E})$ for every $j \geq j_1$. Let us fix $j_2 \geq j_i$ for $i \in \{0, 1\}$. We can choose $j \geq j_2$ such that $F_j(x_0) \cap (Y \backslash \overline{E}) \neq \emptyset$. Then exists a point $p \in F_j(x_0) \cap (Y \backslash \overline{E})$. Of course $p \in F_j(\overline{A})$. Because $F_j(A) \subset E$ then $(Y \backslash \overline{E}) \cap F_j(A) = \emptyset$. Since the set $(Y \backslash \overline{E})$ is a neighbourhood of p then $p \notin \overline{F_j(A)}$. In consequence, we have shown $\overline{F_j(A)} \not\subset \overline{F(A)}$ and this contradiction finishes the proof. □

Let (Y, δ) be the proximity space. In the sequel we will consider the set $J = \{\xi : \xi < \Omega\}$, where Ω is the first uncountable ordinal number. By $C^p(F), C^-(F)$ we denote the set of all points at which F is semi-continuous and lower semi-continuous respectively.

Theorem 15. *Let X be a first countable space and let a net $\{F_\xi : \xi \in J\}$ of multivalued maps $F_\xi : X \to Y$ be convergent in the sense of Leader to a multivalued map $F : X \to Y$.*

(1) If F has compact values and $F \in \tau^+ - \lim F_\xi$, then

$$\bigcap_{\alpha<\Omega} \bigcup_{\xi} C^p(F_\xi) \subset C^p(F)$$

(2) If F has compact values and $F \in \tau^- - \lim F_\xi$, then

$$\bigcap_{\alpha<\Omega} \bigcup_{\xi} C^-(F_\xi) \subset C^-(F)$$

where the union is taken under all ξ satisfying $\alpha \leq \xi \leq \Omega$.

Proof: Let $x_0 \notin C^p(F)$ and let $\{U_n : n \in \mathbb{N}\}$ be neighbourhood base of x_0. Exists an open set $W \subset Y$ such that $F(x_0) \subset W$ and exists $x_n \in U_n$ such that $F(x_n) \cap W = \emptyset$ for every $n \in \mathbb{N}$. Because (Y, τ_δ) is a regular space, so exists an open set $V \subset Y$ such that

$$F(x_0) \subset V \subset \overline{V} \subset W.$$

Then $F(x_n) \cap \overline{V} = \emptyset$ for all $n \in \mathbb{N}$. Because F has the compact values, so $F(x_n) \not\delta \overline{V}$. Since $F(x_0) \subset V$ and $F \in \tau^+ - \lim F_\xi$ so exists $\xi_0 < \Omega$ such that $F_\xi(x_0) \subset V$ for $\xi_0 \leq \xi \leq \Omega$. As $F(x_n) \not\delta \overline{V}$, then from the convergence we have

$$\forall_{n\in\mathbb{N}} \exists_{\xi_n<\Omega} \forall_\xi [\xi_n \leq \xi < \Omega \Rightarrow F_\xi(x_n) \not\delta \overline{V}].$$

Thus $F_\xi(x_n) \cap \overline{V} = \emptyset$, then

$$F_\xi(x_n) \subset Y \backslash \overline{V} \subset Y \backslash V,$$

for every $n \in \mathbb{N}$ and $\xi_n \leq \xi \leq \Omega$. From the property of ordinal numbers there exists $\alpha < \Omega$ with $\xi_n \leq \alpha$ for every $n \in \mathbb{N}$. Then $F_\xi(x_0) \subset V$ and $F_\xi(x_n) \subset Y \backslash V$, for each ξ such that $\alpha \leq \xi \leq \Omega$ and for each $n \in \mathbb{N}$. Because $x_n \to x_0$ then $x_0 \notin C^p(F_\xi)$ for ξ such that $\alpha \leq \xi \leq \Omega$ and the proof is finished.

Using similar arguments we can prove (2) □

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