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Common fixed point theorems in metric spaces

by
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Abstract

Results on common fixed point theorems for a class of mappings have been obtained in complete metric spaces. The results extend the theorems of Chatterjea, Iseki, Kannan, Khan, Nešić , Rhoades, Rus and others.

Key Words: *common fixed point, Cauchy sequence and complete metric space.*
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1 Introduction

In [5] we proved the following result.

Theorem A *Let (X,d) be a metric space and T a self-mapping of X satisfying*

$$[1+pd(x,y)]\,d(Tx,Ty) \leq$$

$$\leq p\,[d(x,Tx)d(y,Ty)+d(x,Ty)d(y,Tx)]+$$

$$+q\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}\,[d(x,Ty)+d(y,Tx)]\right\}$$

for all x,y in X, where $p \geq 0$ and $0 < q < 1$. If (X,d) is $T-$orbitally complete, then T has a unique fixed point in X

2 Pairs of mappings

We prove the following theorem for two mappings S and T which extends Theorem A.

Theorem 1. *Let S and T be mappings of a metric space (X,d) into itself satisfying the inequality*

$$[1+pd(x,y)]\,d(Sx,Ty)\leq$$

$$\leq p\,[d(x,Sx)d(y,Ty)+d(x,Ty)d(y,Sx)]+$$

$$+qmax\left\{d(x,y),d(x,Sx),d(y,Ty),\frac{1}{2}\,[d(x,Ty)+d(y,Sx)]\right\} \qquad (1)$$

for all x,y in X, where $p\geq 0$ and $0<q<1$. If (X,d) is $(T,S)-$orbitally complete, then S and T have a unique common fixed point u in X

Proof: Let x_0 be any point of X. Define the sequence $\{x_n\}$ by

$$x_{2n-1}=Sx_{2n-2},x_{2n}=Tx_{2n-1},(n=1,2,3,...)$$

By (1) for $x=x_{2n-2}$ and $y=x_{2n-1}$ we get

$$[1+pd(x_{2n-2},x_{2n-1})]\,d(x_{2n-1},x_{2n})\leq$$

$$p\,[d(x_{2n-2},x_{2n-1})d(x_{2n-1},x_{2n})+d(x_{2n-2},x_{2n})d(x_{2n-1},x_{2n-1})]+$$

$$+q\max\{d(x_{2n-2},x_{2n-1}),d(x_{2n-2},x_{2n-1}),d(x_{2n-1},x_{2n}),$$

$$\frac{1}{2}\,[d(x_{2n-2},x_{2n})+d(x_{2n-1},x_{2n-1})]\}$$

or equivalently

$$d(x_{2n-1},x_{2n})\leq$$

$$\leq q\max\left\{d(x_{2n-2},x_{2n-1}),d(x_{2n-1},x_{2n}),\frac{1}{2}d(x_{2n-2},x_{2n})\right\}$$

The inequality

$$d(x_{2n-1},x_{2n})\leq qd(x_{2n-1},x_{2n})$$

implies $d(x_{2n-1},x_{2n})=0$.

Assume now that $x_n\neq x_{n+1}$ for each $n=0,1,2,\ldots$.

If

$$max\left\{d(x_{2n-2},x_{2n-1}),\frac{1}{2}d(x_{2n-2},x_{2n})\right\}=\frac{1}{2}d(x_{2n-2},x_{2n}),$$

then we have

$$d(x_{2n-1},x_{2n})\leq q\frac{1}{2}d(x_{2n-2},x_{2n})\leq$$

$$\leq\frac{1}{2}q\left[d(x_{2n-2},x_{2n-1})+d(x_{2n-1},x_{2n})\right]$$

and hence

$$d(x_{2n-1},x_{2n})\leq\frac{q}{2-q}d(x_{2n-2},x_{2n-1})\leq qd(x_{2n-2},x_{2n-1})$$

Therefore

$$d(x_{2n-1},x_{2n})\leq qd(x_{2n-2},x_{2n-1}).$$

Similary

$$d(x_{2n-2},x_{2n-1})\leq qd(x_{2n-3},x_{2n-2}),$$

so that

$$d(x_{2n-1},x_{2n})\leq q^2d(x_{2n-3},x_{2n-2})\leq\ldots\leq q^{2n-1}d(x_0,x_1)$$

.

For any $m>n$,

$$d(x_n,x_m)\leq\sum_{k=n}^{m-1}d(x_k,x_{k+1})\leq q^n\frac{d(x_0,x_1)}{1-q}.$$

Hence , $\{x_n\}$ is a Cauchy sequence. Since X is (T,S)-orbitally complete, there exists a point u in X such that $u=\lim_n x_n$.

Using (1) and the triangle inequality we have

$$d(Su,u)\leq d(Su,x_{2n})+d(u,x_{2n})=$$

$$=d(u,x_{2n})+d(Su,Tx_{2n-1})\leq$$

$$\leq d(u,x_{2n})+p\frac{d(u,Su)d(x_{2n-1},x_{2n})+d(u,x_{2n})d(x_{2n-1},Su)}{1+pd(u,x_{2n-1})}+$$

$$+\frac{q}{1+pd(u,x_{2n-1})}max\left\{d(u,x_{2n-1}),d(u,Su),d(x_{2n-1},x_{2n}),\right.$$

$$\left.\frac{1}{2}\left[d(u,x_{2n})+d(x_{2n-1},Su)\right]\right\}\leq$$

$$\leq \frac{d(u,x_{2n})+pd(u,x_{2n})d(u,x_{2n-1})+p\left[d(u,x_{2n})+d(x_{2n-1},u)\right]d(u,Su)}{1+pd(u,x_{2n-1})}+$$

$$+\frac{p\left[d(x_{2n-1},u)+d(u,Su)\right]d(u,x_{2n})+q\left[d(u,x_{2n})+d(x_{2n-1},u)+d(u,Su)\right]}{1+pd(u,x_{2n-1})},$$

which implies

$$d(u,Su)\leq \frac{1}{1-q}\{(1+q)d(u,x_{2n})+2pd(u,x_{2n})\left[d(x_{2n-1},u)+d(u,Su)\right]+$$

$$qd(x_{2n-1},u)\}.$$

Letting n tend to infinity we see that $d(u,Su)=0$. Hence, u is a fixed point of S. Similarly, $Tu=u$.

Suppose that u and v are common fixed points of S and T. Then by (1) it follows

$$\left[1+pd(u,v)\right]d(u,v)\leq pd^2(u,v)+qd(u,v),$$

which implies $u=v$. Therefore, u is a unique common fixed point of S and T. This completes the proof.

□

Corollary 1. *(***Kannan***) Let (X,d) be a complete metric space and $f,g:X\to X$, two mappings for which there exists a number b, $0<b<\frac{1}{2}$, such that*

$$d(f(x),g(y))\leq b\left[d(x,f(x))+d(y,g(y))\right]$$

for all x,y in X. Then f and g have a unique common fixed point.

Corollary 2. *(***Chatterjea** *) Let (X,d) be a complete metric space and $f,g:X\to X$, two mappings for which there exists a number c, $0<c<\frac{1}{2}$, such that*

$$d(f(x),g(y))\leq c\left[d(x,g(y))+d(y,f(x))\right]$$

for all x,y in X. Then f and g have a unique common fixed point.

Corollary 3. *(***Rus***) Let* (X, d) *be a complete metric space and* $f, g : X \to X$*, two mappings for which there exists numbers* $\alpha, \beta, \gamma > 0$*,* $\alpha + 2\beta + 2\gamma < 1$*, such that*

$$d(f(x), g(y)) \leq \alpha d(x, y) + \beta\left[d(x, f(x)) + d(y, g(y))\right] +$$

$$+\gamma\left[d(x, g(y)) + d(y, f(x))\right]$$

for all x, y *in* X*. Then* f *and* g *have a unique common fixed point.*

Corollary 4. *(***Rhoades, Theorem 14***) Let* f *and* g *be mappings of a complete metric space* (X, d) *into itself satisfying*

$$d(f(x), g(y)) \leq$$

$$\leq hmax\left\{d(x, y), d(x, f(x)), d(y, g(y)), \frac{1}{2}\left[d(x, g(y)) + d(y, f(x))\right]\right\}$$

for all x, y *in* X*, where* $0 \leq h < 1$ *and let* x_0 *in* X*. Then* f *and* g *have a unique common fixed point* z *and* $(fg)^n(x_0) \to z$ *and* $(gf)^n(x_0) \to z$*.*

Corollary 5. *(***Khan***) Let* S *and* T *be mappings of a complete metric space* (X, d) *into itself satisfying*

$$d(STx, TSy) \leq$$

$$\leq kmax\left\{d(x, y), d(x, STx), d(y, TSy), \frac{1}{2}\left[d(x, TSy) + d(y, STx)\right]\right\}$$

for all x, y *in* X*, where* $0 \leq k < 1$ *in* X*. Then* S *and* T *have a unique common fixed point .*

Proof: By Theorem 1, u is a unique fixed point of ST and TS. Then

$$ST(Su) = S(TSu) = Su,$$

and so $Su = u$. Similarly, $Tu = u$. So we proved that u is a common fixed point of S and T. □

3 Sequences of mappings

We prove the following:

Theorem 2. *Let (X, d) be a complete metric space and let $\{T_n\}$ be a sequence of mappings of X into itself satisfying*

$$[1 + pd(x, y)]\, d(T_i^k x, T_j^k y) \leq$$

$$\leq p\left[d(x, T_i^k x)d(y, T_j^k y) + d(x, T_j^k x)d(y, T_i^k x)\right] +$$

$$+q \max\{d(x, y), d(x, T_i^k x), d(y, T_j^k y),$$

$$\frac{1}{2}\left[d(x, T_j^k x) + d(y, T_i^k x)\right]\}$$

for every x, y in X and some fixed positive integer k, where $p \geq 0$, $0 < q < 1$ and $i, j = 1, 2, 3, \ldots.$ Then $\{T_n\}$ has a unique common fixed point.

Proof: By Theorem 1, u is a unique fixed point of T_i^k and T_j^k. Then

$$T_i^k(T_i u) = T_i(T_i^k u) = T_i u \qquad (i = 1, 2, 3\ldots)$$

and so $T_i u = u$ $(i = 1, 2, \ldots)$. So we proved that u is a unique common fixed point of $\{T_n\}$. □

Remark 1. *In case $p = 0$ and $k = 1$ in Theorem 2, we obtain Theorem 1 of Iseki.*

Remark 2. *In case $p = 0$ in Theorem 2, we obtain Theorem 20 of Rhoades.*

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