Bull. Math. Soc. Sc. Math. Roumanie
Tome 46(94) No. 3-4, 2003, 143–147

Generalization of a result on the roots of a trinomial equation

by

V.K.Jain

Abstract

It is known that the trinomial equation

$$cz^n - z + 1 = 0,$$

has a root, in both the regions $|z-1| \geq 1$, $|z-1| \leq 1$, $(n \geq 2)$, as well as the exterior of every circle, which passes through the origin, $(n \geq 3)$. We have obtained a generalization, as well as an extension of this result and have shown that the polinomial

$$cz^n - (z-1)^m, c \neq 0,$$

has a zero, in both the regions $|z-1| \geq 1$, $|z-1| \leq 1$, $(1 \leq m \leq n-1)$, as well as, in the exterior of every circle, which passes through the origin, $(1 \leq m \leq n-2)$

Key Words: Zeros of complex polinomials
2000 Mathematics Subject Classification: Primary: 12D10, Secondary: 30C15.

1 Introduction and statement of results

About the location of the roots of the trinomial equation

$$cz^n - z + 1 = 0 \tag{1}$$

we have the following result due to Fejer ([1]) and Szego ([4]):

Theorem A *The trinomial equation*

$$cz^n - z + 1 = 0, \qquad n \geq 2,$$

has a root in both the regions $|z-1| \geq 1$, $|z-1| \leq 1$.

Joyal, Labelle and Rahman [2] obtained additional information about the location of roots of the trinomial equation (1) and proved

Theorem B *The trinomial equation*

$$cz^n - z + 1 = 0, \qquad n \geq 3,$$

has a root outside every circle, which passes through the origin.

In this paper, we have obtained generalization, as well as extension, of both Theorem A and Theorem B. More precisely, we prove

Theorem 1. *The polinomial*

$$p(z) = cz^n - (z-1)^m, \qquad 1 \leq m \leq n-1, \qquad c \neq 0 \tag{2}$$

has at least one zero in $|z-1| \leq 1$ *and at least one zero in* $|z-1| \geq 1$.

Theorem 2. *The polinomial*

$$p(z) = cz^n - (z-1)^m, \qquad 1 \leq m \leq n-2, \qquad c \neq 0 \tag{3}$$

has a zero outside every circle which passes through the origin.

2 Proofs of the theorems

Proof: (Theorem 1) Let us first consider that

$$n = m + 1$$

Then the equation

$$p(\xi + 1) = 0$$

can be rewritten as

$$c\xi^{m+1} + \cdots + c = 0$$

thereby implying that the polynomial $p(\xi+1)$ has at least one zero in $|\xi| \leq 1$ and at least one zero in $|\xi| \geq 1$, and Theorem 1 follows in the particular case we have considered.

Now consider that

$$n > m + 1 \tag{4}$$

We assume that polynomial $p(z)$ has all its zeros in $|z-1| \leq 1$. Therefore, by Gauss - Lucas theorem the polinomial

$$p^{(m)}(z) = cn(n-1)\dots(n-m+1)z^{n-m} - m!$$

will also have all its zeros in $|z-1| \leq 1$. But by (4), we have

$$n - m \geq 2,$$

and therefore $p^{(m)}(z)$ can not have all its zeros in $|z-1| \leq 1$. Thus, we get an absurdity and hence our assumption that $p(z)$ has all its zeros in $|z-1| \leq 1$, should be wrong, thereby leading to the conclusion that $p(z)$ has at least one zero in $|z-1| > 1$ (i.e. in $|z-1| \geq 1$).

Now we will show that the polynomial

$$cz^n - (z-1)^m$$

has at least one zero in $|z-1| \leq 1$, in the present case. In other words, we have to prove that the poynomial

$$c(\xi+1)^n - \xi^m$$

has at least one zero in $|\xi| \leq 1$, i.e. the polynomial

$$c(1+\xi)^n - \xi^{n-m}$$

has at least one zero in $|\xi| \geq 1$, i.e. the polynomial

$$\Phi(t) = ct^n - (t-1)^{n-m}, \tag{5}$$

has at least one zero in $|t-1| \geq 1$. On the contrary, let us assume that the polynomial $\Phi(t)$ has all its zeros in $|t-1| < 1$. By repetead application of Gauss-Lucas theorem [3, Theorem (6.1)], we can say that the polinomial

$$\Phi^{(n-m-1)}(t) = cn(n-1)\dots(m+2)t^{m+1} - (n-m)!(t-1)$$

$$= (n-m)!\left[\{cn(n-1)\dots(m+2)/((n-m)!)\}\, t^{m+1} - t + 1\right],$$

will have all its zeros in $|t-1| < 1$, (contradicting Theorem A for $m = 1$), and the polynomial

$$\Phi^{(n-m)}(t) = (n-m)!\left[\{cn(n-1)\dots(m+2)(m+1)/(n-m)!\}\, t^m - 1\right],$$

will also have all its zeros in $|t-1| < 1$, (contradicting the fact, that for $m > 1$, the polynomial $\Phi^{(n-m)}(t)$ can not have all its zeros in $|t-1| < 1$), thereby implying that, our assumption, that $\Phi(t)$ has all its zeros in $|t-1|<1$ is wrong. This completes the proof of Theorem 1. □

Proof: (Theorem 2)
On the contrary, let us assume that there exists a circle

$$\gamma : |z-\alpha| = |\alpha|,$$

such that the polynomial $p(z)$ has all its zeros in $\overline{I(\gamma)}$. This means, on using

$$z = \alpha\xi,$$

that the polynomial

$$\varphi(\xi) = p(\alpha\xi) = c\alpha^n\xi^n - (\alpha\xi - 1)^m$$

has all its zeros in $|\xi - 1| \leq 1$. By Gauss-Lucas theorem the polynomial

$$\varphi^{(m)}(\xi) = c\alpha^n n(n-1)\ldots(n-m+1)\xi^{n-m} - m!\alpha^m$$

will also have all its zeros in $|\xi - 1| \leq 1$, and Theorem 2 follows, like Theorem 1, after observing that

$$n - m \geq 2,$$

by using (3) □

References

[1] Fejer L., Uber kreigebiete in denin einen wenzel einer algebraischen gleichung liegt, Jahresbericht der deutschen Math. Vereinigling 26(1917), 114-128.

[2] Joyal A., Labelle G., Rahman Q.I., On the location of zeros of polinomials, Canad.Math.Bull. 10(1967), 53-63.

[3] Marden M., Geometry of Polinomials, Mathematical Survey, No.3, Amer.Math.Soc., Providence, Rhode Island, 1966.

[4] SZEGO G., Bemerkungen zu einen satz von J.H.Grace uber die wurzeln algebraicher gleichungen, Math.Z. 13(1992), 28-55.

Received: 11.08.2003

Mathematics Department,
I.I.T., Kharagpur
721302
India