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Finitely modular functions on inf-semi-lattices of elements of a Riesz space

by
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Abstract

We analyze a condition referred as "finite modularity", necessary and sufficient for the modular extension of a group valued function phi from an inf-semi-lattice S of elements of a Riesz space E to the lattice generated by S. If E has a weak order unit, we investigate the modular (respect. linear if phi is vector valued) extension of phi to the oval (respect. vector subspace) generated by S.

Key Words: *Riesz spaces, Inf-semi-lattices, Finitely modular functions.*
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Throughout this paper E denotes a Riesz space and S an inf-semi-lattice of elements of E, namely S is a nonempty subset of E such that $inf(x, y) \in S$ for all $x, y \in S$. A function $\varphi : S \to G$,where G is an abelian group, is said to be modular iff $\varphi(\sup(x, y)) = \varphi(x) + \varphi(y) - \varphi(\inf(x, y))$ for all $x, y \in S$ such that $\sup(x, y) \in S$.

We begin with studying when there exists a modular extension of φ to generated lattice $l(S)$ by S. This extension problem has been considered for a set function defined on a $\cap$-semi-lattice in a nonvoid set (see [3], [4], [5], [9]). We study the extension problem in Riesz spaces following an order of ideas similar to those that have been used from H. KÖNIG in [5] for real set functions. In particular H. KÖNIG has used a definition introduced in [4] from H. GRAMER for set functions, said "general addivity". Here we introduce a definition suggested from Gramer's, which is a natural extension of the modularity. We say that a function $\varphi : S \to G$ is *finitely modular* iff

$$\varphi(\sup_{i\in I} x_i) = \sum_{\emptyset\neq J\subset I} (-1)^{card(J)-1} \cdot \varphi(\inf_{i\in J} x_i) \qquad (1)$$

for all finite non empty family $(x_i)_{i\in I}$ in S such that $\sup_{i\in I} x_i \in S$.

The finite modularity, which is in general stronger than modularity, is a necessary and sufficient condition to obtain the existence of a modular extension of φ to $l(S)$ (see Theorem 2). Also we study the increase of the above modular extension under the hypothesis that G is an ordered group (see Theorem 2). Moreover the σ-continuity of the same extension has been caracterized if E is Dedekind-σ-complete and φ is with values in another Riesz space (see: Theorem 3).

Finally if E has a weak order unity and elements of S are unitary, then we study the existence of the unique modular extension of φ to the generated oval by S. Further, when φ is vector valued, $0 \in S$, and $\varphi(0) = 0$, we study the existence of the unique linear extension of φ to generated vector subspace of E by S. Using some results of [1] and [7], we prove that in these mentioned cases the finite modularity is necessary and sufficient for the existence of the above unique extension of φ (see Theorem 4 and Theorem 5).

Throughout this work we will use some definitions and results contents in [2] and [7].

Now we start by proving one lemma.

Lemma 1. *Each modular function on a lattice of elements of E is finitely modular.*

Proof: Let L be a lattice of elements of E and let ψ be a modular function on L with value in G. We have to prove that for all of the finite non empty families $(x_i)_{i\in I}$ in L, we have

$$\psi(\sup_{i\in I} x_i) = \sum_{\emptyset\neq J\subset I} (-1)^{card(J)-1} \cdot \psi(\inf_{i\in J} x_i). \qquad (2)$$

Let $n := card(I)$.

The cases $n = 1, 2$ are obvious. Let (2) be true for $n \geq 1$ and let $(x_i)_{i\in I}$ be in L such that $card(I) = n+1$. If $l \in I$, then $J \mapsto J \cup \{l\}$ is an one to one map of $\{J \subset I\backslash\{l\} : J \neq \emptyset\}$ onto $\{J \subset I : l \in J \neq \{l\}\}$ and for all $J \subset I\backslash\{l\}$, $J \neq \emptyset$, we have

$$(-1)^{card(J)} \cdot \psi(\inf_{i\in J} \inf(x_i, x_l)) = (-1)^{card(J\cup\{l\})-1} \cdot \psi(\inf_{i\in J\cup\{l\}} x_i).$$

It follows that

$$\psi(\sup_{i\in I} x_i) = \psi(\sup_{i\in I\backslash\{l\}} x_i) + \psi(x_l) - \psi(\sup_{i\in I\backslash\{l\}} \inf(x_i, x_l))$$

$$= \sum_{\emptyset\neq J\subset I\backslash\{l\}} (-1)^{card(J)-1} \cdot \psi(\inf_{i\in J} x_i) + \psi(x_l)$$

$$+ \sum_{J \subset I, l \in J \neq \{l\}} (-1)^{card(J)-1} \cdot \psi(\inf_{i \in J} x_i)$$

$$= \sum_{\emptyset \neq J \subset I} (-1)^{card(J)-1} \cdot \psi(\inf_{i \in J} x_i).$$

□

By Lemma 1, if $L = E$ and ψ is identity map of E, it follows the

Theorem 1. *If $(x_i)_{i \in I}$ is a finite nonempty family in E, then*

$$\sup_{i \in I} x_i = \sum_{\emptyset \neq J \subset I} (-1)^{card(J)-1} \cdot \inf_{i \in J} x_i. \tag{3}$$

Now, let S be an inf-semi-lattice in E, G an abelian group, and $\varphi : S \to G$. The generated lattice $l(S)$ by S is the set of all $\sup_{i \in I} x_i$, where $(x_i)_{i \in I}$ is a finite nonempty family in S. We observe that, if ψ is a modular extension of φ to $l(S)$, then as a consequence of Lemma 1, we have

$$\psi(\sup_{i \in I} x_i) = \sum_{\emptyset \neq J \subset I} (-1)^{card(J)-1} \cdot \varphi(\inf_{i \in J} x_i)$$

for all of the finite non empty families $(x_i)_{i \in I}$ in S. Hence it follows the uniqueness of the modular extension of φ. In order to get some conditions for the existence, we introduce the map

$$\tilde{\varphi}((x_i)_{i \in I}) := \sum_{\emptyset \neq J \subset I} (-1)^{card(J)-1} \cdot \varphi(\inf_{i \in J} x_i) \tag{4}$$

where $(x_i)_{i \in I}$ is a finite non empty family in S. In particular we put

$$\tilde{\varphi}(x_p, \ldots, x_q) := \tilde{\varphi}((x_k)_{p \leq k \leq q})$$

for all finite sequence $(x_k)_{p \leq k \leq q}$ in S with $p, q \in \mathbb{N}$ and $p \leq q$.

We shall need two lemmas.

Lemma 2. *We have:*

i) if $1 \leq n \in \mathbb{N}$ and $(x_k)_{0 \leq k \leq n}$ is a finite sequence in S, then

$$\tilde{\varphi}(x_0, \ldots, x_n) = \tilde{\varphi}(x_0) + \tilde{\varphi}(x_1, \ldots, x_n) - \tilde{\varphi}(\inf(x_0, x_1), \ldots, \inf(x_0, x_n));$$

ii) if $1 \leq n \in \mathbb{N}$, $(x_k)_{0 \leq k \leq n}$ is a finite sequence in S, and there exists $k = 1, \ldots, n$ such that $x_0 \leq x_k$, then

$$\tilde{\varphi}(x_0, \ldots, x_n) = \tilde{\varphi}(x_1, \ldots, x_n);$$

iii) if $1 \leq m, n \in \mathbb{N}$ and $(x_k)_{1\leq k\leq n}$ and $(y_h)_{1\leq h\leq m}$ are finite sequences in S, then

$$\tilde{\varphi}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \tilde{\varphi}(x_1, \ldots, x_n) + \tilde{\varphi}(y_1, \ldots, y_m)$$

$$-\tilde{\varphi}(\inf(x_1, y_1), \ldots, \inf(x_1, y_m), \ldots, \inf(x_n, y_1), \ldots, \inf(x_n, y_m));$$

iv) if $1 \leq m, n \in \mathbb{N}$ and $(x_k)_{1\leq k\leq n}$ and $(y_h)_{1\leq h\leq m}$ are finite sequences in S, then

$$\tilde{\varphi}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \sum_{\emptyset \neq J \subset \{1,\ldots,n\}} (-1)^{card(J)-1} \cdot \tilde{\varphi}(\inf_{k\in J} x_k, y_1, \ldots, y_m).$$

Proof: i) Since $J \mapsto J\backslash\{0\}$ is one to one map from $\{J \subset \{0, \ldots .n\} : 0 \in J \neq \{0\}\}$ onto $\{J \subset \{1, \ldots, n\} : J \neq \emptyset\}$, we have that

$$\tilde{\varphi}(\inf(x_0, x_1), \ldots, \inf(x_0, x_n)) = \sum_{\emptyset \neq J \subset \{1,\ldots,n\}} (-1)^{card(J)-1} \cdot \varphi(\inf_{k\in J} \inf(x_0, x_k)) =$$

$$= \sum_{J \subset \{0,\ldots,n\}, 0\in J \neq \{0\}} (-1)^{card(J\backslash\{0\})-1} \cdot \varphi(\inf_{k\in J\backslash\{0\}} \inf(x_0, x_k)) =$$

$$= -\sum_{J \subset \{0,\ldots,n\}, 0\in J \neq \{0\}} (-1)^{card(J)-1} \cdot \varphi(\inf_{k\in J} x_k)$$

and, hence,

$$\tilde{\varphi}(x_0, \ldots, x_n) = \sum_{\emptyset \neq J \subset \{0,\ldots,n\}} (-1)^{card(J)-1} \cdot \varphi(\inf_{k\in J} x_k) =$$

$$= \varphi(x_0) + \sum_{\emptyset \neq J \subset \{0,\ldots,n\}, 0\notin J} (-1)^{card(J)-1} \cdot \varphi(\inf_{k\in J} x_k) +$$

$$+ \sum_{J \subset \{0,\ldots,n\}, 0\in J \neq \{0\}} (-1)^{card(J)-1} \cdot \varphi(\inf_{k\in J} x_k) =$$

$$= \tilde{\varphi}(x_0) + \tilde{\varphi}(x_1, \ldots, x_n) - \tilde{\varphi}(\inf(x_0, x_1), \ldots, \inf(x_0, x_n)).$$

ii) We suppose that $x_0 \leq x_1$. Using i) for $n = 1$ we have

$$\tilde{\varphi}(x_0, x_1) = \tilde{\varphi}(x_0) + \tilde{\varphi}(x_1) - \tilde{\varphi}(\inf(x_0, x_1)) = \tilde{\varphi}(x_1).$$

After, if $n \geq 2$, then

$$\tilde{\varphi}(x_0, x_1, \ldots, x_n) - \tilde{\varphi}(x_1, \ldots, x_n) =$$

$$= \tilde{\varphi}(x_0) - \tilde{\varphi}(x_0, \inf(x_0, x_2), \ldots, \inf(x_0, x_n)) =$$

$$= \tilde{\varphi}(x_0) - \tilde{\varphi}(x_0) - \tilde{\varphi}(\inf(x_0, x_2), \ldots, \inf(x_0, x_n)) +$$

$$+ \tilde{\varphi}(\inf(x_0, x_2), \ldots, \inf(x_0, x_n)) = 0.$$

iii) By i), iii) is obvious for $n = 1$. Let the same be true for $n \geq 1$ and let be $(x_k)_{1\leq k\leq n}$ in S. Applying i) we deduce that

$$\tilde{\varphi}(x_0, x_1, \ldots, x_n) + \tilde{\varphi}(y_1, \ldots, y_m) - \tilde{\varphi}(x_0, x_1, \ldots, x_n, y_1, \ldots, y_m) =$$

$$= \tilde{\varphi}(x_1, \ldots, x_n) - \tilde{\varphi}(\inf(x_0, x_1), \ldots, \inf(x_0, x_n)) +$$

$$+ \tilde{\varphi}(y_1, \ldots, y_m) - \tilde{\varphi}(x_1, \ldots, x_n, y_1, \ldots, y_m) +$$

$$+ \tilde{\varphi}(\inf(x_0, x_1), \ldots, \inf(x_0, x_n), \inf(x_0, y_1), \ldots, \inf(x_0, y_m)) =$$

$$= \tilde{\varphi}(\inf(x_1, y_1), \ldots, \inf(x_1, y_m), \ldots, \inf(x_n, y_1), \ldots, \inf(x_n, y_m)) +$$

$$+ \tilde{\varphi}(\inf(x_0, y_1), \ldots, \inf(x_0, y_m)) -$$

$$- \tilde{\varphi}(\inf(x_0, x_1, y_1), \ldots, \inf(x_0, x_1, y_m), \ldots, \inf(x_0, x_n, y_1), \ldots, \inf(x_0, x_n, y_m)) =$$

$$= \tilde{\varphi}(\inf(x_0, y_1), \ldots, \inf(x_0, y_m)) - \tilde{\varphi}(x_0) +$$

$$+ \tilde{\varphi}(x_0, \inf(x_1, y_1), \ldots, \inf(x_1, y_m), \ldots, \inf(x_n, y_1), \ldots, \inf(x_n, y_m)).$$

Furthermore by ii) and i), it follows that

$$\tilde{\varphi}(x_0, \inf(x_1, y_1), \ldots, \inf(x_1, y_m), \ldots, \inf(x_n, y_1), \ldots, \inf(x_n, y_m)) =$$

$$= \tilde{\varphi}(x_0, \inf(x_0, y_1), \ldots, \inf(x_0, y_m), \inf(x_1, y_1),$$

$$\ldots, \inf(x_1, y_m), \ldots, \inf(x_n, y_1), \ldots, \inf(x_n, y_m)) =$$

$$= \tilde{\varphi}(x_0) + \tilde{\varphi}(\inf(x_0, y_1), \ldots, \inf(x_0, y_m), \ldots, \inf(x_n, y_1), \ldots, \inf(x_n, y_m)) -$$

$$- \tilde{\varphi}(\inf(x_0, y_1), \ldots, \inf(x_0, y_m), \inf(x_0, x_1, y_1), \ldots, \inf(x_0, x_1, y_m),$$

$$\ldots, \inf(x_0, x_n, y_1), \ldots, \inf(x_0, x_n, y_m)) =$$

$$= \tilde{\varphi}(x_0) + \tilde{\varphi}(\inf(x_0, y_1), \ldots, \inf(x_0, y_m), \ldots, \inf(x_n, y_1), \ldots, \inf(x_n, y_m)) -$$

$$- \tilde{\varphi}(\inf(x_0, y_1), \ldots, \inf(x_0, y_m))$$

and, hence,

$$\tilde{\varphi}(x_0, x_1, \ldots, x_n) + \tilde{\varphi}(y_1, \ldots, y_m) - \tilde{\varphi}(x_0, x_1, \ldots, y_1, \ldots, y_m) =$$

$$= \tilde{\varphi}(\inf(x_0, y_1), \ldots, \inf(x_0, y_m), \ldots, \inf(x_n, y_1), \ldots, \inf(x_n, y_m)).$$

iv) We suppose $m = 1$ and observe that

$$\sum_{k=1}^{n} \Big(\sum_{\emptyset \neq J \subset \{1, \ldots, n\}, card(J) = k} (-1)^{k-1} \Big) = \sum_{k=1}^{n} (-1)^{k-1} \cdot \binom{n}{k}$$

and, hence,

$$\sum_{\emptyset \neq J \subset \{1, \ldots, n\}} (-1)^{card(J)-1} = 1. \qquad (5)$$

By (4), (5) and i), we obtain

$$\tilde{\varphi}(x_1,\dots,x_n,y_1) =$$
$$= \tilde{\varphi}(x_1,\dots,x_n) + \varphi(y_1) - \tilde{\varphi}(\inf(y_1,x_1),\dots,\inf(y_1,x_n)) =$$
$$= \sum_{\emptyset\neq J\subset\{1,\dots,n\}} (-1)^{card(J)-1}\cdot(\varphi(\inf_{k\in J} x_k) + \varphi(y_1) - \varphi(\inf_{k\in J}\inf(y_1,x_k))) =$$
$$= \sum_{\emptyset\neq J\subset\{1,\dots,n\}} (-1)^{card(J)-1}\cdot(\varphi(\inf_{k\in J} x_k) + \varphi(y_1) - \varphi(\inf(\inf_{k\in J} x_k, y_1))) =$$
$$= \sum_{\emptyset\neq J\subset\{1,\dots,n\}} (-1)^{card(J)-1}\cdot(\tilde{\varphi}(\inf_{k\in J} x_k, y_1).$$

Let iv) be true for $m \geq 1$ and let be $(y_h)_{1\leq h\leq m+1}$ in S. Using again i), it follows

$$\tilde{\varphi}(x_1,\dots,x_n,y_1,\dots,y_m,y_{m+1}) =$$
$$= \tilde{\varphi}(x_1,\dots,x_n,y_1,\dots,y_m) + \varphi(y_{m+1}) -$$
$$-\tilde{\varphi}(\inf(x_1,y_{m+1}),\dots,\inf(x_n,y_{m+1}),\inf(y_1,y_{m+1}),\dots,\inf(y_m,y_{m+1})) =$$
$$= \sum_{\emptyset\neq J\subset\{1,\dots,n\}} (-1)^{card(J)-1}\cdot(\tilde{\varphi}(\inf_{k\in J} x_k, y_1,\dots,y_m) + \varphi(y_{m+1}) -$$
$$-\tilde{\varphi}(\inf(\inf_{k\in J} x_k, y_{m+1}), \inf(y_1,y_{m+1}),\dots,\inf(y_m,y_{m+1}))) =$$
$$= \sum_{\emptyset\neq J\subset\{1,\dots,n\}} (-1)^{card(J)-1}\cdot\tilde{\varphi}(\inf_{k\in J} x_k, y_1,\dots,y_m,y_{m+1})$$

and iv) is verified.

□

Lemma 3. *Let be φ finitely modular, $1 \leq n \in \mathbb{N}$, $(x_k)_{1\leq k\leq n}$ in S, $y \in S$, and $y \leq \sup_{1\leq k\leq n} x_k$. Then*

$$\tilde{\varphi}(y,x_1,\dots,x_n) = \tilde{\varphi}(x_1,\dots,x_n). \qquad (6)$$

Proof: From i) of Lemma 2 it results

$$\tilde{\varphi}(y,x_1,\dots,x_n) - \tilde{\varphi}(x_1,\dots,x_n) =$$
$$= \varphi(y) - \tilde{\varphi}(\inf(y,x_1),\dots,\inf(y,x_n)).$$

Hence, being $\sup_{1\leq k\leq n}\inf(y,x_k) = y$ and φ finitely modular, we obtain

$$\tilde{\varphi}(\inf(y,x_1),\dots,\inf(y,x_n)) = \varphi(y).$$

From this it follows (6). □

Now, we may prove the following

Theorem 2. *There exists a modular extension φ' of φ to $l(S)$ iff φ is finitely modular. In this case φ' is unique. Furthermore if G is an order group, then φ' is increasing iff for all $x \in S$ and for all finite non empty family $(x_i)_{i\in I}$ in S, it results*

$$\sup_{i\in I} x_i \leq x \Rightarrow \tilde{\varphi}((x_i)_{i\in I}) \leq \varphi(x). \qquad (7)$$

Proof: Let φ' a modular extension of φ to $l(S)$. Let $x \in l(S)$ and $(x_i)_{i\in I}$ a finite non empty family in S such that $x = \sup_{i\in I} x_i$. From (4) and Lemma 1 we obtain

$$\varphi'(x) = \sum_{\emptyset \neq J \subset I} (-1)^{card(J)-1} \cdot \varphi(\inf_{i\in J} x_i).$$

Moreover, for (4),

$$\varphi'(x) = \tilde{\varphi}((x_i)_{i\in I}). \qquad (8)$$

Hence φ is finitely modular, while φ' is unique.

On the contrary, we suppose φ finitely modular. Let be $1 \leq n, m \in \mathbb{N}$ and $(x_k)_{1\leq k\leq n}$ and $(y_h)_{1\leq h\leq m}$ in S such that $\sup_{1\leq k\leq n} x_k = \sup_{1\leq h\leq m} y_h$. Being $y_1 \leq \sup_{1\leq k\leq n} x_k$, by Lemma 3 we have

$$\tilde{\varphi}(y_1, x_1, \ldots, x_n) = \tilde{\varphi}(x_1, \ldots, x_n). \qquad (9)$$

After, because $y_2 \leq \sup_{1\leq k\leq n} x_k \leq \sup(y_1, x_1, \ldots, x_n)$, using again the Lemma 3, it follows $\tilde{\varphi}(y_2, y_1, x_1, \ldots, x_n) = \tilde{\varphi}(y_1, x_1, \ldots, x_n)$. From this and (9) we have

$$\tilde{\varphi}(y_2, y_1, x_1, \ldots, x_n) = \tilde{\varphi}(x_1, \ldots, x_n).$$

Going on in the same manner, we prove that

$$\tilde{\varphi}(y_1, \ldots, y_m, x_1, \ldots, x_n) = \tilde{\varphi}(x_1, \ldots, x_n).$$

Hence $\tilde{\varphi}(x_1, \ldots, x_n) = \tilde{\varphi}(y_1, \ldots, y_m)$. Then it is possible to define the function $\varphi' : l(S) \to G$ by (8) where $x \in l(S)$ and $(x_i)_{i\in I}$ is a finite non empty family in S such that $x = \sup_{i\in I} x_i$. Obviously φ' is an extension of φ to $l(S)$. In order to prove that φ' is modular, let be $x, y \in l(S)$, $1 \leq n, m \in \mathbb{N}$, $(x_k)_{1\leq k\leq n}$ and $(y_h)_{1\leq h\leq m}$ in S such that $x = \sup_{1\leq k\leq n} x_k$ and $y = \sup_{1\leq h\leq m} y_h$. Thus we have

$$\sup(x, y) = \sup(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

and

$$\inf(x, y) = \sup(\inf(x_1, y_1), \ldots, \inf(x_1, y_m), \ldots, \inf(x_n, y_1), \ldots, \inf(x_n, y_m)).$$

Using iii) of Lemma 2, it results

$$\varphi'(\sup(x, y)) + \varphi'(\inf(x, y)) = \tilde{\varphi}(x_1, \ldots, x_n, y_1, \ldots, y_m) +$$

$$+\tilde{\varphi}(\inf(x_1, y_1), \ldots, \inf(x_1, y_m), \ldots, \inf(x_n, y_1), \ldots, \inf(x_n, y_m)) =$$
$$= \tilde{\varphi}(x_1, \ldots, x_n) + \tilde{\varphi}(y_1, \ldots, y_m) = \varphi'(x) + \varphi'(y).$$

Now we suppose that G is ordered and φ' is increasing. Let be $1 \le n \in \mathbb{N}$, $(x_k)_{1\le k\le n}$ in S and $x \in S$ such that $\sup_{1\le k\le n} x_k \le x$. Then

$$\tilde{\varphi}(x_1, \ldots, x_n) = \varphi'(\sup_{1\le k\le n} x_k) \le \varphi'(x) = \varphi(x)$$

and (7) is true.

On the contrary we suppose (7) true. Let $1 \le n \in \mathbb{N}$, $(x_k)_{1\le k\le n}$ in S, and $y \in S$. Then, by i) of Lemma 2, it results

$$\tilde{\varphi}(y, x_1, \ldots, x_n) - \tilde{\varphi}(x_1, \ldots, x_n) = \varphi(y) - \tilde{\varphi}(\inf(y, x_1), \ldots, \inf(y, x_n)).$$

From this and by (7), being $\sup_{1\le k\le n} \inf(y, x_k) \le y$, we have

$$\tilde{\varphi}(\inf(y, x_1), \ldots, \inf(y, x_n)) \le \varphi(y).$$

Hence $\tilde{\varphi}(x_1, \ldots, x_n) \le \tilde{\varphi}(y, x_1, \ldots, x_n)$.

Moreover, if $x, y \in l(s)$, $x \le y$, $1 \le m, n \in \mathbb{N}$ and $(x_k)_{1\le k\le n}$ and $(y_h)_{1\le h\le m}$ are in S such that $x = \sup_{1\le k\le n} x_k$ and $y = \sup_{1\le h\le m} y_h$, then we have

$$\varphi'(x) = \tilde{\varphi}(x_1, \ldots, x_n) \le \tilde{\varphi}(y_1, \ldots, y_m, x_1, \ldots, x_n) = \varphi'(\sup(x, y)) = \varphi'(y).$$

□

At the last, we suppose E Dedekind σ-complete and F another Riesz space. Let still S be an inf-semi-lattice in E and $\varphi : S \to F$. We prove the following

Theorem 3. *There exists a modular extension φ' of φ to $l(S)$ such that, for all increasing sequence (x_n) in $l(S)$ and for all $x \in l(S)$, it results*

$$\sup_n x_n = x \Rightarrow (o) - \lim_n \varphi'(x_n) = \varphi'(x) \tag{10}$$

iff for all sequence (x_n) in S and for all $x \in S$

$$\sup_n x_n = x \Rightarrow (o) - \lim_n \tilde{\varphi}(x_0, \ldots, x_n) = \varphi(x). \tag{11}$$

Proof: We suppose that there exists a modular extension φ' of φ to $l(S)$ satisfying (10). Let be (x_n) a sequence in S and $x \in S$ such that $\sup_n x_n = x$. Thus $\sup_n(\sup_{0\le k\le n} x_k) = x$. Hence, by (10),

$$(o) - \lim_n \tilde{\varphi}(x_0, \ldots, x_n) = (o) - \lim_n \varphi'(\sup_{0\le k\le n} x_k) = \varphi'(x) = \varphi(x).$$

On the contrary, now we suppose (11) true for all sequence (x_n) in S and for all $x \in S$. Let $(x_k)_{0\le k\le n}$ in S and $x \in S$ such that $x = \sup_{0\le k\le n}$. For all $p \in \mathbb{N}$ we put

$$y_p := \begin{cases} x_p, & \text{if} \quad 0 \le p \le n; \\ x_n, & \text{if} \quad n+1 \le p. \end{cases}$$

For all $p \ge n$, using ii) of Lemma 2, we have

$$\tilde{\varphi}(y_0, \dots, y_n, \dots, y_p) = \tilde{\varphi}(x_0, \dots, x_n)$$

and, hence,

$$(o) - \lim_p \tilde{\varphi}(y_0, \dots, y_n, \dots, y_p) = \tilde{\varphi}(x_0, \dots, x_n)$$

Being $\sup_p y_p = x$, from (11) it follows that $\tilde{\varphi}(x_0, \dots, x_n) = \varphi(x)$. Then φ is finitely modular and, hence, there exists a modular extension φ' of φ to $l(S)$ (see Theorem 2).

Finally, in order to prove (10), let (x_n) be an increasing sequence in $l(S)$ and $x \in l(S)$ such that $\sup_n x_n = x$. In connection with (x_n), there exists a sequence (y_l) in S and a tightly increasing sequence (k_n) in $\mathbb{N}$, such that, for all $n \in \mathbb{N}$, $x_n = \sup_{0\le l\le k_n} y_l$. It follows that, for all $n \in \mathbb{N}$, $y_n \le x_n$ and, hence, $y_n \le x$. Thus, there exists $\sup_n y_n \in E$. Furthermore, being $x \in l(S)$, there exists $(z_h)_{0\le h\le p}$ in S such that

$$x = \sup_{0\le h\le p} z_h. \tag{12}$$

Let, now, $\emptyset \ne J \subset \{0, \cdots, p\}$. Hence

$$\inf_{h\in J} z_h \le x = \sup_n (\sup_{0\le l\le k_n} y_l) \le \sup_n y_n \le x$$

and

$$\inf_{h\in J} z_h = \inf(\inf_{h\in J} z_h, \sup_n y_n) = \sup_n \inf(\inf_{h\in J} z_h, y_n).$$

Therefore, by use of (11),

$$(o) - \lim_n \tilde{\varphi}(\inf(\inf_{h\in J} z_h, y_0), \dots, \inf(\inf_{h\in J} z_h, y_n)) = \varphi(\inf_{h\in J} z_h).$$

From this and for i) of Lemma 2,

$$(o) - \lim_n (\tilde{\varphi}(\inf_{h\in J} z_h, y_0, \dots, y_n) - \tilde{\varphi}(y_0, \dots, y_n)) = 0.$$

Otherwise, for (5),

$$(o) - \lim_n (\sum_{\emptyset\ne J\subset\{0,\dots,p\}} (-1)^{card(J)-1} \cdot \tilde{\varphi}(\inf_{h\in J} z_h, y_0, \dots, y_n) - \tilde{\varphi}(y_0, \dots, y_n)) = 0.$$

Hence, using iv) of Lemma 2,

$$(o) - \lim_n (\tilde{\varphi}(z_0, \ldots, z_p, y_0, \ldots, y_n) - \tilde{\varphi}(y_0, \ldots, y_n)) = 0.$$

Consequently, applying Theorem 2, (12) and Lemma 3, it results

$$\varphi'(x) = \tilde{\varphi}(z_0, \ldots, z_p) = (o) - \lim_n \tilde{\varphi}(y_0, \ldots, y_n) = (o) - \lim_n \varphi'(x_n).$$

□

Remark 1. *If*

$$(\forall x, y \in S)((\exists z \in S)(x, y \leq z) \Rightarrow \sup(x, y) \in S), \tag{13}$$

then φ is modular iff it is finitely modular. Moreover, in this case, if G is ordered, then φ is increasing iff the modular extension φ' of φ to $l(S)$ is increasing. Finally, if E is Dedekind σ-complete and φ is with value in another Riesz space, then for all increasing sequence (x_n) in $l(S)$ and for all $x \in l(S)$, we have

$$\sup_n x_n = x \Rightarrow (o) - \lim_n \varphi'(x_n) = \varphi'(x)$$

iff for all sequence (x_n) in S and for all $x \in S$ we have

$$(o) - \lim_n x_n = n \Rightarrow (o) - \lim_n \varphi(x_n) = \varphi(x).$$

Let, now, E be a Riesz space with a weak order unity $\mathbf{1}$ and let $U(E)$ be the set of unitary elements of E (see [2] and [7]). Furthermore, let S be an inf-semilattice in E with $S \subset U(E)$. Let $o(S)$ be the generated oval by S and $r(S)$ be the generated ring by S (see [7]). Evidently it results $l(S) \subset o(S) \subset r(S)$ and, hence, $o(S) = o(l(S))$ and $r(S) = r(o(S))$. If $0 \in S$, then $o(S) = r(S)$ (see [7]). If, again, φ is a map from S to an abelian group G, then we have the following

Theorem 4. *There exists a modular extension φ'' of φ to $o(S)$ iff φ is finitely modular. In this case φ'' is unique.*

Proof: Let φ be finitely modular and let φ' be the modular extension of φ to $l(S)$ (see Theorem 2). Thus, as a consequence of Theorem 3.1 in [7], there exists an unique modular extension φ'' of φ' (and, hence, of φ) to $o(S)$.

On the contrary, if φ'' is a modular extension of φ to $o(S)$, then the restriction of φ'' to $l(S)$ is a modular extension of φ to $l(s)$. Hence, by Theorem 2, φ is finitely modular. □

Remark 2. *Let φ be finitely modular and φ'' its modular extension to $o(S)$ (see Theorem 4). If $x \in o(S)$, by Theorem 2.5 in [7], there exists a finite non empty family $(x_i)_{i\in I}$ in $l(S)$ and $(\alpha_i)_{i\in I}$ in $\mathbb{Z}$, such that $\sum_{i\in I}\alpha_i = 1$ and $x = \sum_{i\in I}\alpha_i \cdot x_i$. Thus (see Theorem 3.1 in [7])*

$$\varphi''(x) = \sum_{i\in I}\alpha_i \cdot \varphi'(x_i). \qquad (14)$$

Remark 3. *Let A be a ring of elements in E and G an abelian group. A function φ in A in G such that $\varphi(0) = 0$ is finitely modular (otherwise modular) iff it is additive. Indeed, if $x, y \in U(E)$, then $y = \sup(\inf(y, \mathbf{1} - x), \inf(y, x))$ and $\sup(x, y) = \sup(x, \inf(y, \mathbf{1} - x))$.*

From Theorem 4 we deduce the following

Corollary 1. *If $0 \in S$ and $\varphi(0) = 0$, then φ is finitely modular iff there exists an (unique) additive extension of φ to $r(S)$.*

Now, for all $X \subset E$, we denote with $v(X)$ the generated vector subspace of E by X. The following lemma holds.

Lemma 4. *We have that*

$$v(S) = v(r(S)). \qquad (15)$$

Proof: Let be $x \in l(S)$ and let be $(x_i)_{i\in I}$ a finite non empty family in S such that $x = \sup_{i\in I} x_i$. Since, from Theorem 1, we have $x = \sum_{\emptyset\neq J\subset I}(-1)^{card(J)-1} \cdot \inf_{i\in J} x_i$. It follows that $x \in v(S)$. Consequently

$$l(S) \subset v(S). \qquad (16)$$

Further, as a consequence of Theorem 2.4 in [7], we obtain that $r(l(S)) \subset v(l(S))$. Then $r(S) \subset v(l(S))$ and also $v(r(S)) \subset v(l(S))$. Hence $v(l(S)) = v(r(S))$. From this, being for (16) $v(l(S)) \subset v(S) \subset v(r(S))$, it follows that $v(S) = v(l(S)) = v(r(S))$. □

Remark 4. *We have $r(S) \subset v(S)$ and, hence,*

$$v(S) = v(l(S)) = v(o(S)) = v(r(S)). \qquad (17)$$

Finally let F be a real vector space and φ a function of S in F.

Theorem 5. *If $0 \in S$ and $\varphi(0) = 0$, then there exists a linear extension L_φ of φ to $v(S)$ iff φ is finitely modular. In this case L_φ is unique.*

Proof: Let φ be finitely modular and let φ'' be the unique modular extension (hence additive for Remarque 3) of φ to $r(S)$ (see Corollary of Theorem 4). Using Prop. 1, pag 875 in [1], it follows the existence of an unique linear function L_φ of $v(r(S))$ in F such that

$$(\forall x \in r(S))(L_\varphi(x) = \varphi''(x)).$$

Thus, by the previous lemma, L_φ results a linear extension of φ to $v(S)$. Moreover, if L another linear extension of φ a $v(S)$, then the restriction of L to $r(S)$ results additive, while the restriction of L to S is equal to φ. Consequently the restriction of L to $r(S)$ is equal to φ'' and, hence $L = L_\varphi$. Finally, if there exists a linear extension of φ to $v(S)$, then its restriction to $r(S)$ is the (unique) additive extension of φ to $r(S)$. Hence, by Corollary of Theorem 4, φ results finitely modular. $\square$

Example 1. *1) Let f_1, f_2, f_3 be the real functions on $[-1, 1]$ such that for all $x \in [-1, 1]$*

$$f_1(x) := \begin{cases} -2x - 1 & \text{if} \quad -1 \leq x \leq -1/2; \\ (2x+3)/3 & \text{if} \quad -1/2 \leq x \leq 1. \end{cases}$$

$$f_2(x) := \begin{cases} -x & \text{if} \quad -1 \leq x \leq 0; \\ 1 & \text{if} \quad x = 0 \\ x & \text{if} \quad 0 \leq x \leq 1 \end{cases}$$

and

$$f_3(x) := \begin{cases} (1-2x)/3 & \text{if} \quad -1 \leq x \leq -1/2; \\ 2x - 1 & \text{if} \quad -1/2 \leq x \leq . \end{cases}$$

Moreover, let

$$S := \{f_1, f_2, f_3, \inf(f_1, f_2), \inf(f_1, f_3), \inf(f_2, f_3), \inf(f_1, f_2, f_3), \sup(f_1, f_2, f_3)\}.$$

It is easy to prove that such S is an inf-semi-lattice in the Riesz space $\mathbb{R}^{[-1,1]}$. S is not a lattice. Furthermore $0 \notin S$, S does no t verify the condition (13), and each function on S with value in an abelian group G is modular. Finally, if $G \neq \{0\}$ and $a \in G \setminus \{0\}$, then the function $\varphi : S \to G$ defined $\varphi(\inf_{i \in J} f_i) := a$ for all $\emptyset \neq J \subset \{1, 2, 3\}$ and $\varphi(\sup(f_1, f_2, f_3)) := a + a$, is not modular.

Example 2. *If $a, b \in \mathbb{R}$, $a < b$, then the real function $f \mapsto \int_a^b f(x)dx$ is finitely modular on all inf-semi-lattice of integrable functions in $[a, b]$ (see: Theorem 1).*

Example 3. *Let S be the set of real positive functions on $[0, 1]$ such that the set $Z(f)(:= \{x \in [0, 1] : f(x) = 0\})$ results non empty and countable. S is an inf-semi-lattice in the Riesz space $\mathbb{R}^{[0,1]}$) and it is not a lattice. Let, again, G be an abelian group with $G \neq \{0\}$ and $a \in G \setminus \{0\}$. We consider the function $\varphi : S \to G$ such that for all $f \in S$*

$$\varphi(f) := \begin{cases} 0 & \text{if} \quad Z(f) \quad \text{is countable;} \\ a & \text{if otherwise.} \end{cases}$$

Now let $f, g \in S$ such that

$$f(x) := \begin{cases} 0 & \text{if} \quad x = 0 \quad \text{or if} \quad (\exists n \in \mathbb{N})(x = 1/(2n+1)); \\ 1 & \text{if otherwise.} \end{cases}$$

and

$$g(x) := \begin{cases} 0 & \text{if} \quad x = 0 \quad \text{or if} \quad (\exists n \in \mathbb{N})(x = 1/(2n+1)); \\ 1 & \text{if otherwise.} \end{cases}$$

It results that $\sup(f, g) \in S, \varphi(\sup(f, g)) = a, \varphi(f) = \varphi(g) = \varphi(\inf(f, g)) = 0$, *and hence φ is not modular.*

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