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Gaps and Densities

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Abstract

If $A \subseteq N = \{1, 2, \dots, n, \dots\}$ then $n+1, n+2, \dots, n+k$ $(n \geq 0)$ is called the gap in the set A provided that $\{n+1, n+2, \dots, n+k\} \cap A = \emptyset$. In the paper we shall study a relationship between the gaps in sets $A \subseteq N$ and densities of sets A.

Key Words: *gap, gap density, uniform density, Baire's space, porosity.*
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Introduction and Notations

It is well-known (see [7], p. 23, Ex. 4) that the sequence

$$n! + 2, n! + 3, \dots, n! + n \quad (n \geq 2) \tag{1}$$

contains no prime number. Therefore (1) is a gap in the set P of all primes.

By the definition of the gap in a set it could be expected that the size of gaps and densities of sets will be related.

We recall the concept of the uniform and asymptotic density of a set (see [1]).

If $A \subseteq N, t \geq 0, s \in N$, then $A\,(t+1, t+s)$ denotes the number of elements from the set A lying in the interval $[t+1, t+s]$, i.e.

$$A\,(t+1, t+s) = |A \cap [t+1, t+s]\,|$$

($|M|$ denotes the cardinality of the set M).

Set

$$\alpha_s = \min_{t \geq 0} A\,(t+1, t+s)\,, \;\; \alpha^s = \max_{t \geq 0} A\,(t+1, t+s)$$

$$\beta_s = \liminf_{t \to \infty} A\,(t+1, t+s)\,, \;\; \beta^s = \limsup_{t \to \infty} A\,(t+1, t+s)\,.$$

Then

$$\underline{u}(A) = \lim_{s\to\infty} \frac{\alpha_s}{s} = \lim_{s\to\infty} \frac{\beta_s}{s},$$

$$\overline{u}(A) = \lim_{s\to\infty} \frac{\alpha_s}{s} = \lim_{s\to\infty} \frac{\beta_s}{s}$$

exist.

The numbers $\underline{u}(A), \overline{u}(A)$ are called the lower and upper uniform density of A, respectively.

Futher we put

$$\underline{d}(A) = \liminf_{s\to\infty} \frac{A(1,s)}{s}, \ \overline{d}(A) = \limsup_{s\to\infty} \frac{A(1,s)}{s}.$$

The numbers $\underline{d}(A), \overline{d}(A)$ are called the lower and upper asymptotic density of A, respectively.

If $\underline{u}(A) = \overline{u}(A)$ then $u(A)$ denotes their common value and it is called the uniform density of A. If $\underline{d}(A) = \overline{d}(A)$ then $d(A)$ denotes their common value and it is called the asymptotic density of A.

We have

$$0 \le \underline{u}(A) \le \underline{d}(A) \le \overline{d}(A) \le \overline{u}(A) \le 1 \tag{2}$$

(see [1]).

The outstanding point of our investigation in the first section of the paper are results established in [2], [4] concerning the sequences (1) and

$$(n!)^2 + k, \ n \ge 2, \ 2 \le k \le n. \tag{3}$$

In [4] the sequence (3) is shown to be a gap in the set of all powers of primes. In [2] it is proved that for $n > 1$ and $2 \le k \le n$ the number $n! + k$ has either a prime factor $p > n$ or it is a power of k where k is a prime, $k > \frac{n}{2}$. Further for $n \ge 6, 2 \le k \le n$ the number $n! + k$ has more than one prime divisor and at least one of them is greater than n.

In Section 1 we shall give an effective construction of gaps in the set of integers with bounded number of prime divisors.

In Section 2 we shall study a relationship between the gap density introduced in [3] and both asymptotic and uniform densities of sets $A \subseteq N$.

In Section 3 we shall investigate the structure of Baire's metric space X of all sequences $M = m_1 < m_2 < \dots$ of natural numbers (see [8], p. 95) from the point of view of the gap density of its subsets. We denote by M such a sequence and simultaneously the set $\{m_1, m_2, \dots\}$. For its more detail study the notion of porosity of a set in a metric space will be used in Section 4, too (see [10], [11]). Let us recall the concept of porosity of a set.

Let (Y, d) be a metric space. If $y \in Y$ then $B(y, \delta)$ denotes for $\delta > 0$ a ball in Y, i.e. $B(y, \delta) = \{x \in Y : d(x, y) < \delta\}$.

Let $M \subseteq Y, y \in Y, \delta > 0$. Set

$$\gamma(y, \delta, M) = \sup\{t > 0 : \exists z \in B(y, \delta) : [B(z, t) \subseteq B(y, \delta)] \wedge [B(z, t) \cap M = \emptyset]\}.$$

If such a t does not exist we put $\gamma(y, \delta, M) = 0$.

We define

$$\underline{p}(y, M) = \liminf_{\delta \to 0_+} \frac{\gamma(y, \delta, M)}{\delta}, \ \overline{p}(y, M) = \limsup_{\delta \to 0_+} \frac{\gamma(y, \delta, M)}{\delta}.$$

The numbers $\underline{p}(y, M)$ and $\overline{p}(y, M)$ are called the lower and upper porosity of the set M at $y \in Y$, respectively.

If $\underline{p}(y, M) = \overline{p}(y, M)$ then $p(y, M)$ denotes their common value and it is called the porosity of the set M at y.

Obviously we have $\underline{p}(y, M), \overline{p}(y, M), p(y, M) \in [0, 1]$.

A set M is called porous (c-porous) at $y \in Y$ provided that $\overline{p}(y, M) > 0$ ($\overline{p}(y, M) \geq c > 0$).

A set M is called very porous (very c-porous) at $y \in Y$ if $\underline{p}(y, M) > 0$ ($\underline{p}(y, M) \geq c > 0$).

A set M is called strongly porous at $y \in Y$ if $p(y, M) = 1$.

A set M is called σ-porous, σ-very porous and σ-strongly porous at $y \in Y$ provided that $M = \cup_{n=1}^{\infty} M_n$ and each of the sets M_n ($n = 1, 2, \ldots$) is porous, very porous and strongly porous, respectively at y.

In a similar way the concepts of σ-c-porous, σ-very c-porous set are defined.

If a set M is porous at every $y \in Y$ then it is nowhere-dense in Y and if a set M is σ-porous at every $y \in Y$ then it is a set of the first Baire category in Y (see [10], [11]).

1 Gaps in the sets $D^{(s)}$

We say that a set $A \subseteq N$ has gaps of arbitrary length provided that for each $k \in N$ thete exists $n \in N$ such that

$$n + 1, n + 2, \ldots, n + k$$

is a gap in A. The property "to have gaps of arbitrary length "is closely related to the uniform density.

Theorem 1.1 *If $\underline{u}(A) = 0$ then the set A has gaps of arbitrary length.*

Corollary 1.1 *If $\underline{d}(A) = 0$ then the set A has the gaps of arbitrary length* (see (2)).

Proof of Theorem 1.1. We proceed indirectly. Suppose that A has not gaps of arbitrary length.Then there exists such $k_0 \in N$ that for each $t \geq 0$ the sequence

$$t+1, t+2, \dots, t+k_0$$

contains a number from A.

Thus

$$A(t+1, t+k_o) \geq 1 \tag{4}$$

for each $t \geq 0$.

Let $s > k_0$. Then there exists $j \in N$ so that

$$jk_0 \leq s < (j+1)k_0. \tag{5}$$

For an arbitrary $t \geq 0$ a simple estimation gives from (4), (5):

$$\begin{aligned} A(t+1, t+s) &\geq A(t+1, t+k_0) + \ldots + A(t+(j-1)k_0+1, t+jk_0) \\ &\geq j > \frac{s}{k_0} - 1. \end{aligned}$$

From this we obtain

$$\alpha_s = \min_{t \geq 0} A(t+1, t+s) > \frac{s}{k_0} - 1,$$

$$\frac{\alpha_s}{s} > \frac{1}{k_0} - \frac{1}{s}.$$

Therefore

$$\underline{u}(a) = \lim_{s \to \infty} \frac{\alpha_s}{s} \geq \frac{1}{k_0} > 0,$$

a contradiction to the assumption of the theorem. □

In this way, every set $A \subseteq N$ with zero lower uniform density has gaps of arbitrary length. Theorem 1.1 guarantees their existence. In what follows we focus our attention on the effective construction of gaps in the sets $D^{(s)}$, where $D^{(s)}$ $(s \in N)$ denotes in agreement with [5] the set of all $n \in N$ the standard form of which contains no more than s distinct primes. Clearly $D^{(1)}$ is just the set of all powers of primes. For this set the gaps of arbitrary length were already described in [4] (see (3)). We shall extend this result to all sets $D^{(s)}$.

It is well-known that $d(D^{(s)}) = 0$ (see [5], p. 255, Theorem 11.8). The existence of required gaps already follows from our Theorem 1.1.

The next theorem enables us to give an effective construction of the gaps of arbitrary length for every set $D^{(s)}$ $(s \geq 2)$.

Theorem 1.2 *Let*

$$B+1, B+2, \dots, B+m \ (B \in N) \tag{6}$$

be a gap in the set $D^{(s)}$ $(s \geq 1)$. *Put* $A = B + m$. *Then*

$$(A!)^2 + B + 1, (A!)^2 + B + 2, \dots, (A!)^2 + B + m \tag{6*}$$

is a gap in the set $D^{(s+1)}$.

Proof. We proceed indirectly. Suppose that a number of the form (6*) belongs to $D^{(s+1)}$, e.g. let $(A!)^2 + k \in D^{(s+1)}$, $B + 1 \le k \le B + m$.

Thus

$$(A!)^2 + k = p_1^{\alpha_1} \cdots p_s^{\alpha_s} p_{s+1}^{\alpha_{s+1}}, \tag{7}$$

where p_j $(j = 1, 2, \dots, s, s+1)$ are mutually distinct primes, $\alpha_i \ge 0$ $(i = 1, 2, \dots, s, s+1)$.

Since k divides the left-hand side of (7), it divides the right-hand side, too. So

$$k = p_1^{\beta_1} \cdots p_s^{\beta_s} p_{s+1}^{\beta_{s+1}},\ \beta_j \ge 0\ (j = 1, 2, \dots, s, s+1).$$

If $\beta_j = 0$ for some j, $1 \le j \le s+1$ then k belongs to $D^{(s)}$, thereby contradicting that (6) is a gap in $D^{(s)}$.

Hence $\beta_j > 0$ $(j = 1, 2, \dots, s, s+1)$ and so $\alpha_j > 0$ $(j = 1, 2, \dots, s, s+1)$ as well, since k divides the right-hand side of (7).

Observe that the inequality $\beta_j \ge \alpha_j$ does not hold for all $j = 1, 2, \dots, s, s+1$. In the opposite case $(A!)^2 = 0$. So there exists an i, $1 \le i \le s+1$ with $\beta_i < \alpha_i$. Dividing both sides of (7) by $p_i^{\beta_i}$ we get

$$C \cdot A! + p_1^{\beta_1} \cdots p_{i-1}^{\beta_{i-1}} p_{i+1}^{\beta_{i+1}} \cdots p_{s+1}^{\beta_{s+1}} = p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - \beta_i} p_{i+1}^{\alpha_{i+1}} \cdots p_{s+1}^{\alpha_{s+1}}, \tag{8}$$

where $C = \frac{A!}{p_i^{\beta_i}}$ is an integer.

The prime p_i divides both the right-hand side of (8) and $C \cdot A!$ but it does not divide the second summand on the left-hand side, thereby we get a contradiction. □

2 Gap density and densities d, u

In this section we shall be concerned with a relationship between the gap density and densities d, u.

The gap density was introduced in [3] as follows:

If $A \subseteq N$, A is an infinite set then $l(A, n)$ denotes the length of a maximal gap after the number n in A, i.e. $l(A, n) = v$ if $\{n+1, n+2, \dots, n+v\} \cap A = \emptyset$ and $n + v + 1 \in A$.

If $n + 1 \in A$ then we put $l(A, n) = 0$.

The gap density $\lambda(A)$ is defined as follows:

$$\lambda(A) = \limsup_{n \to \infty} \left(1 + \frac{1}{n} l(A, n)\right).$$

Evidently, $1 \le \lambda(A) \le +\infty$.

It can be easily seen that if

$$A = \{a_1 < a_2 < \ldots < a_n < a_{n+1} < \ldots\}$$

then

$$\lambda(A) = \limsup_{k\to\infty} \frac{a_{k+1}}{a_k}$$

(see [3]).

The gap density was introduced by G. Grekos and B. Volkmann in connection with their study of the set

$$S(A) = \{(\underline{d}(B), \overline{d}(B)) \in R^2 : B \subseteq A\}.$$

The following proposition illustrates the relationship of the asymptotic density to the gap density.

Proposition 2.1 *If $\underline{d}(A) > 0$, then $\lambda(A) \le \frac{\overline{d}(A)}{\underline{d}(A)}$.*

Corollary 2.1 *If $\lambda(A) = +\infty$, then $\underline{d}(A) = 0$.*

Corollary 2.2 *If $d(A)$ exists and $d(A) > 0$, then $\lambda(A) = 1$.*

From this corollary and from relation (2) follows the following:

Corollary 2.3 *If $u(A)$ exists and $u(A) > 0$, then $\lambda(A) = 1$.*

Proof of Proposition 2.1. Let $A = \{a_1 < a_2 < \ldots\}$. Then

$$\lambda(A) = \limsup_{k\to\infty} \frac{a_{k+1}}{a_k} = \limsup_{k\to\infty} \frac{k/a_k}{k/a_{k+1}}. \tag{9}$$

We recall that the asymptotic densities of the set A verify

$$\overline{d}(A) = \limsup \frac{k}{a_k},\ \underline{d}(A) = \liminf \frac{k}{a_k}.$$

Let $0 < \varepsilon < \underline{d}(A)$. There is an integer $k_0 = k_0(\varepsilon)$ such that, for each $k \ge k_0$, we have

$$\frac{k}{a_k} \le \overline{d}(A) + \varepsilon$$

and

$$\frac{k+1}{a_{k+1}} \ge \underline{d}(A) - \varepsilon.$$

This combined with (9) gives

$$\lambda(A) \leq \frac{\overline{d}(A) + \varepsilon}{\underline{d}(A) - \varepsilon}.$$

Letting ε tend to zero, we get the proposition. □

Remark 2.1 A slightly more general formulation, using the convention $0 \times \infty = 0$, is the following one:

"For any set $A \subset N$, we have $\underline{d}(A) \times \lambda(A) \leq \overline{d}(A)$".

Now we prove that all values of upper, lower asymptotic densities and gap density, compatible with the above inequality, may appear. At first we prove this for sets having zero density, because these constructions will then be used in our general theorem (Proposition 2.3).

Proposition 2.2 *Let*

$$S_0 = \{A \subset N : A \text{ is infinite and d(A)} = 0\}.$$

Then

$$\lambda(S_0) = \{\lambda(A) : A \in S_0\} = [1, +\infty].$$

Proof. For each $\gamma \in [1, +\infty]$, we shall define a set A_γ such that $d(A_\gamma) = 0$ and $\lambda(A_\gamma) = \gamma$. We distinguish three cases:

Case 1: $\gamma = 1$.

We set $A = \{a_1, a_2, \ldots\}$, where

$$a_1 = 1, a_2 = 2, a_k = [k \log k] \quad (k \geq 3).$$

Then $d(A) = \lim_{k\to\infty} \frac{k}{a_k} = 0$. Further

$$\begin{aligned} l(A, a_n) &= |\{[n \log n] + 1, [n \log n] + 2, \ldots, [(n+1)\log(n+1)] - 1\}| \\ &= [(n+1)\log(n+1)] - 1 - [n \log n] \\ &\leq (n+1)\log(n+1) - n \log n. \end{aligned}$$

Set $g(t) = t \log t$ $(t > 1)$. In virtue of Lagrange's mean value theorem we have

$$g(n+1) - g(n) = g'(y_n), \; n < y_n < n+1,$$

and so

$$\frac{l(A, a_n)}{a_n} \leq \frac{\log y_n + 1}{a_n} \leq \frac{\log(n+1) + 1}{[n \log n]} \to 0$$

if $n \to \infty$. Therefore $\lambda(A) = 1$.

Case 2: $1 < \gamma < +\infty$.

We set $A_\gamma = \{[\gamma^n] : n \in N\}$. Obviously $d(A_\gamma) = 0$ and

$$\lambda(A_\gamma) = \lim_{n\to\infty} \frac{[\gamma^{n+1}]}{[\gamma^n]} = \gamma.$$

Case 3. $\gamma = +\infty$.

We put $A_{+\infty} = \{n^n : n \in N\}$. We have $d(A_{+\infty}) = 0$ and $\lambda(A_{+\infty}) = +\infty$ because

$$\frac{(n+1)^{n+1}}{n^n} = (n+1)(1+\frac{1}{n})^n \to +\infty$$

as n tends to infinity. □

Now we formulate and prove the general property.

Proposition 2.3 *Let α, β be two real numbers, $0 \le \alpha \le \beta \le 1$. Let $\gamma \in [1, +\infty]$. Suppose $\alpha\gamma \le \beta$* (with the convention $0 \times (+\infty) = 0$). Then there is a set $A \subset N$ such that

$$\underline{d}(A) = \alpha,\ \overline{d}(A) = \beta \text{ and } \lambda(\mathrm{A}) = \gamma.$$

Proof. We consider two cases.

Casse 1. $\alpha = 0$.

If $\beta = 0$, then Proposition 2.2 provides the desired set. Suppose $0 < \beta$. Recall that γ may be any real number ≥ 1 or $+\infty$. Consider the set A_γ corresponding to that value of γ, constructed in Proposition 2.2. That is, $A_\gamma \subset N$ is such that $d(A_\gamma) = 0$ and $\lambda(A_\gamma) = \gamma$. We shall find a set $A \supset A_\gamma$ such that $\underline{d}(A) = 0$, $\overline{d}(A) = \beta$ and $\lambda(A) = \gamma$.

Let $A_\gamma = \{a_1 < a_2 < \ldots\}$. We shall define two sequences (p_i) and (q_i) of integers

$$0 < p_1 < q_1 < p_2 < q_2 < \ldots$$

and the set A will be of the form

$$A = \bigcup_{i=1}^{+\infty} ((p_i, q_i] \cap A_\gamma) \cup ((q_i, p_{i+1}] \cap N).$$

Thus A_γ is a subset of A. Also the sequences (p_i) and (q_i) are such that, for all $i \geq 1$,

$$|(p_i, q_i] \cap A_\gamma| \geq 2.$$

This implies that the gap density of A remains equal to γ.

We define the sequences (p_i) and (q_i) by induction. Suppose that these sequences have been defined up to the index i and that

$$\frac{A(q_i)}{q_i} < \frac{1}{i}.$$

We suppose also that $\frac{1}{i} < \frac{\beta}{2}$. For i smaller then $\frac{2}{\beta}$ we may suppose that the p_i's and the q_i's have been chosen arbitrarily.

Putting into A consecutive integers

$$q_i + 1, q_i + 2, \ldots$$

the function

$$\frac{A(x)}{x}, \ x = q_i + 1, q_i + 2, \ldots$$

increases tending to one. We stop at $x = p_{i+1}$ such that

$$\left| \frac{A(p_{i+1})}{p_i} - \beta \right| < \frac{1}{i+1}.$$

After p_{i+1} the sequence A is for a while identical to A_γ. We require that

$$|(p_{i+1}, q_{i+1}] \cap A_\gamma| \geq 2$$

and that

$$\frac{A(q_{i+1})}{q_{i+1}} < \frac{1}{i+1}.$$

The last condition is verified for q_{i+1} large enough because $d(A_\gamma) = 0$.

Case 2: $\alpha > 0$.

In this case γ is necessarily a real number, $1 \leq \gamma \leq \frac{\beta}{\alpha}$. Actually, if $\alpha = 1$, then $A = N$ is OK. So let $\alpha < 1$. If $\beta = \alpha$, then $\gamma = 1$. By Corollary 2.1, any set A with $d(A) = \alpha$ verifies the required conditions. For instance, the set

$$S = \left\{ \left[\frac{k}{\alpha} \right] : k \in N \right\}.$$

Now suppose that $\alpha < \beta$. We define the set A by induction. Suppose that $A \cap [0, p_i]$ has been determined and that

$$\left| \frac{A(p_i)}{p_i} - \alpha \right| < \frac{1}{i}.$$

We choose an integer $q_i > p_i$ such that, by taking

$$A \cap (p_i, q_i] = \{p_i + 1, \dots, q_i\},$$

one has

$$\left| \frac{A(q_i)}{q_i} - \beta \right| < \frac{1}{i}.$$

The next, after q_i, element of A will be $r_i = 1 + [\gamma q_i]$. Then take in A the elements

$$\left[\frac{r_i}{\alpha}\right], \left[\frac{r_i+1}{\alpha}\right], \dots, \left[\frac{r_i+h}{\alpha}\right] =: p_{i+1}$$

where p_{i+1} is such that

$$\left| \frac{A(p_{i+1})}{p_{i+1}} - \alpha \right| < \frac{1}{i+1}.$$

Thus the induction step is accomplished. □

Remark 2.2 In [3], some more general results are proved. Our Proposition 2.1 follows from the Lemma in p. 144 of [3]. Also Proposition 2.3 is included in Theorem 1, p. 134 of [3]. However, as the object in [3] was to find relations between the gap density $\lambda(A)$ and the hole density set $S(A)$, the proof are more complicated, each one taking a few pages.

In our further considerations we shall use the notion of a dyadic value (Dualwert) of an infinite set $A \subseteq N$. If $A = \{a_1 < a_2 < \dots\}$ is an infinite set then the number $\rho(A) = \sum_{k=1}^{\infty} 2^{-a_k} \in (0,1]$ is called the dyadic value of A. In this way a one-to-one mapping $\rho : \mathcal{U} \to (0,1]$ onto the interval $(0,1]$ is defined, where $\mathcal{U}$ denotes the system of all infinite subsets of N.

If $\mathcal{S} \subseteq \mathcal{U}$ then we put $\rho(\mathcal{S}) = \{\rho(A) : A \in \mathcal{S}\}$. The magnitude of the set $\rho(\mathcal{S}) \subseteq (0,1)$ provides a tool for measuring the size of the class $\mathcal{S}$ (see [6], p. 189-195).

We say that almost all sets $A \subseteq N$ have a property (V) provided that the Lebesgue measure of the set $\rho(\mathcal{S}(V))$ is 1, where $\mathcal{S}(V)$ denotes the class of all $A \in \mathcal{U}$ having the propery (V). It is well-known that almost all sets $A \in \mathcal{U}$ have the asymptotic density $\frac{1}{2}$ (see [6], p. 190). In what follows $\mu(M)$ denotes the Lebesgue measure of the set M.

Corollary 2.2 suggests the investigation of the class

$$\mathcal{S}^{(1)} = \{A \in \mathcal{U} : \lambda(A) = 1\}$$

from the standpoint of dyadic values.

Proposition 2.4 *We have*

$$\mu\left(\rho\left(\mathcal{S}^{(1)}\right)\right) = 1,$$

i.e. almost all sets $A \in \mathcal{U}$ *have the gap density* $\lambda(A) = 1$.

Corollary 2.4 *Set $C\mathcal{S}^{(1)} = \mathcal{U} \setminus \mathcal{S}^{(1)}$. Then $\mu\left(\rho\left(C\mathcal{S}^{(1)}\right)\right) = 0$.*

Proof of Proposition 2.4. Set

$$\mathcal{V}^{(\frac{1}{2})} = \left\{A \in \mathcal{U} : d(A) = \frac{1}{2}\right\}.$$

Then by Corollary 2.2 $\mathcal{V}^{(\frac{1}{2})} \subseteq \mathcal{S}^{(1)}$ holds.

Therefore

$$\rho(\mathcal{V}^{(\frac{1}{2})}) \subseteq \rho(\mathcal{S}^{(1)}). \tag{10}$$

Since $\mu(\rho(\mathcal{V}^{(\frac{1}{2})})) = 1$ (see [6], p. 190, Theorem 1) according to (10) $\mu(\rho(\mathcal{S}^{(1)})) = 1$, too. □

In what follows we shall study the sets $A \subseteq N$ with bounded gaps. Put

$$\mathcal{T}_m = \{A \in \mathcal{U} : (\forall n) l(A, n) \leq m\} \ (m = 1, 2, \ldots),$$

$$\mathcal{T} = \bigcup_{m=1}^{\infty} \mathcal{T}_m. \tag{11}$$

Then

$$\mathcal{T} = \{A \in \mathcal{U} : \limsup_{n\to\infty} l(A, n) < +\infty\}.$$

Proposition 2.5 *We have*

$$\mu\left(\rho(\mathcal{T})\right) = 0.$$

Corollary 2.5 *Take into account that*

$$C\mathcal{T} = \{A \in \mathcal{U} : \limsup_{n\to\infty} l(A, n) = +\infty\}.$$

That is $\mu\left(\rho(C\mathcal{T})\right) = 1$ and so for almost all $A \in \mathcal{U}$ we have

$$\limsup_{n\to\infty} l(A, n) = +\infty.$$

Proof of Proposition 2.5. According to (11) we have $\rho(\mathcal{T}) = \cup_{m=1}^{\infty} \rho(\mathcal{T}_m)$. Hence it suffices to prove that $\mu(\rho(\mathcal{T}_m)) = 0$ for all $m = 1, 2, \ldots$.

For, let $m \in N$. Denotes by N_2 the set of all dyadic normal numbers lying in the interval $(0, 1]$. Then we have

$$\rho(\mathcal{T}_m) \subseteq (0, 1] \setminus N_2. \tag{12}$$

Indeed, if $A \in \mathcal{T}_m$ then $\rho(A)$ does not contain any block consisting of $m+1$ 0's. Therefore $\rho(A) \notin N_2$. But $\mu(N_2) = 1$ (see [6], p. 193, Theorem 2b). Therefore from (12) we obtain $\mu\left(\rho(\mathcal{T}_m)\right) = 0$. □

3 Baire's space $\mathcal{U}$ and gap density

Denote by $\mathcal{U}$ the set of all increasing sequences of natural numbers. We define the Baire metric ρ on $\mathcal{U}$ as follows:

If $M = \{m_1 < m_2 < \ldots\}$, $H = \{h_1 < h_2 < \ldots\}$ belong to $\mathcal{U}$ then $\rho(M,H) = 0$ if $M = H$ and $\rho(M,H) = 1/\min\{n : m_n \neq h_n\}$ if $M \neq H$ (see [8], p. 95).

The metric space $(\rho, \mathcal{U})$ is complete.

Then λ, $\lambda(A)$ being the gap density of a set $A \subseteq N$, $A \in \mathcal{U}$ can be considered as a real function defined on $\mathcal{U}$. We shall study the structure of the space $\mathcal{U}$ from the standpoint of this function.

In the first place let us observe that it suffices to restrict ourselves to the investigation of the behaviour of the sequence

$$\left(\frac{l(M,n)}{n}\right)_{n=1}^{\infty}, \ M \in \mathcal{U} \tag{13}$$

where $l(M,n) = 0$ if $n+1 \in M$ and $l(M,n) = v$ if $\{n+1, \ldots, n+v\} \cap M = \emptyset$ and $n + v + 1 \in M$.

Obviously

$$\limsup_{n\to\infty} \frac{l(M,n)}{n} = \lambda(M) - 1$$

whenever the left-hand side is a finite number, further

$$\lambda(M) = +\infty \iff \limsup_{n\to\infty} \frac{l(M,n)}{n} = +\infty.$$

We shall deal with limit points of the sequence (13) more precisely. An application of Baire categories explains the structure of the space $\mathcal{U}$ from viewpoint of infiniteness of the gap density.

Theorem 3.1 *The set*

$$\mathcal{R} = \{A \in \mathcal{U} : \lambda(A) = +\infty\}$$

is a residual G_δ-set in $\mathcal{U}$.

Corollary 3.1 *a) The set*

$$\mathcal{K} = \{A \in \mathcal{U} : \lambda(A) < +\infty\}$$

is a set of the first category in $\mathcal{U}$.
b)Each of the sets $\mathcal{S}^{(1)}$, $\mathcal{V}^{(\frac{1}{2})}$ is the set of the first category in $\mathcal{U}$.

Proof of Theorem 3.1. For fixed $k, K \in N$ set

$$W(k, K) = \left\{ M \in \mathcal{U} : \frac{l(M, k)}{k} > K \right\}.$$

We shall show that $W(k, K)$ is an open set in $\mathcal{U}$.

Let $M = \{m_1 < m_2 < \ldots\}$ belong to $W(k, K)$. Then there exists an index i so that $m_i \leq k \leq m_{i+1}$ (we can restrict ourselves to $k \geq m_1$). Let us construct the ball (in $\mathcal{U}$):

$$B\left(M, \frac{1}{i+1}\right) = \left\{ H \in \mathcal{U} : \rho(M, H) < \frac{1}{i+1} \right\}.$$

If $H \in B\left(M, \frac{1}{i+1}\right)$, $H = \{h_1 < h_2 < \ldots\}$ then $h_j = m_j$ $(j = 1, 2, \ldots, i+1)$. This implies $l(H, k) = l(M, k)$ and so

$$\frac{l(H, k)}{k} > K,$$

i.e. $H \in W(k, K)$.

We shall prove that

$$\mathcal{R} = \bigcap_{K=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{k \geq n}^{\infty} W(k, K). \tag{14}$$

Indeed, if $M \in \mathcal{R}$ then there exists a sequence $k_1 < k_2 < \ldots < k_r < \ldots$ with

$$\lim_{r \to \infty} \frac{l(M, k_r)}{r} = +\infty.$$

But then for each $K \geq 1$ there exists a $k_r \geq n$ such that

$$\frac{l(M, k_r)}{k_r} > K,$$

i.e. M belongs to the right-hand side of (14).

Conversly, suppose that M belongs to the right-hand side of (14).

Choose $K = 1,\ n = 1$. Then there exists $k_1 \geq n = 1$ so that

$$\frac{l(M, k_1)}{k_1} > 1.$$

Choose $K = 2,\ n = k_1 + 1$. Then there exists $k_2 \geq k_1 + 1$ so that

$$\frac{l(M, k_2)}{k_2} > 2, \text{ and so on.}$$

By induction we construct the sequence of natural numbers $k_1 < k_2 < \ldots < k_r < \ldots$ such that

$$\frac{l(M, k_r)}{k_r} > r, \ (r = 1, 2, \ldots). \tag{15}$$

From (15) we get

$$\limsup_{k \to \infty} \frac{l(M, k)}{k} = +\infty$$

and so $\lambda(M) = +\infty$.

Since $W(k, K)$ $(k, K \in N)$ are open sets, on account of (14) $\mathcal{R}$ is a G_δ-set in $\mathcal{U}$.

To finish the proof it suffices to show that $\mathcal{R}$ is a dense set in $\mathcal{U}$ (see [9], p. 49).

Take an arbitrary ball $B(A, \varepsilon)$, $A \in \mathcal{U}$, $A = \{a_1 < a_2 < \ldots\}$, $\varepsilon > 0$.

It suffices to show that

$$B(A, \varepsilon) \cap \mathcal{R} \neq \emptyset. \tag{16}$$

For, choose $p \in N$ so that $\frac{1}{p} < \varepsilon$.

Define $H = \{h_1 < h_2 < \ldots\}$ as follows: $h_j = a_j$ $(j = 1, 2, \ldots, p)$. Let $t \in N$ be chosen in such a way that $(p+t)^3 > a_p$. Set $h_{p+1} = (p+1+t)^3, h_{p+2} = (p+2+t)^3, \ldots$.

Then

$$H \in B\left(A, \frac{1}{p}\right) \subseteq B(A, \varepsilon).$$

Further it is easy to check that

$$\lim_{n \to \infty} \frac{l(H, n)}{n} = +\infty.$$

Hence $H \in \mathcal{R}$ and so (16) holds. □

Since for $A = \{a_1 < a_2, \ldots\} \in \mathcal{U}$ we have

$$\lambda(A) = \limsup_{n \to \infty} \frac{a_{n+1}}{a_n}$$

(see [3]), we shall study the structure of the space $\mathcal{U}$ from the standpoint of the set of limits points of the sequence $\left(\frac{a_{n+1}}{a_n}\right)_{n=1}^{\infty}$.

If $(b_n)_{n=1}^{\infty}$ is a sequence of real numbers then $(b_n)_n'$ denotes the set of all its limit points.

Theorem 3.2 *The set $\mathcal{R}^*$ of all $A = \{a_1 < a_2, \ldots\} \in \mathcal{U}$ for which*

$$\left(\frac{a_{n+1}}{a_n}\right)_n' = [1, +\infty]$$

is a residual set in $\mathcal{U}$.

Proof. We shall realize the proof in several steps.

Step 1. Let $r \in (1, +\infty)$, $r = \frac{p}{q} \in Q$ (a rational number). We shall prove that the set $W(r)$ of all $A \in \mathcal{U}$, $A = \{a_1 < a_2, \dots\}$ for which $r \in \left(\frac{a_{n+1}}{a_n}\right)_n^{'}$ is a residual set in $\mathcal{U}$.

Step 2. Then the set $\bigcap_{r \in Q \cap (1,+\infty)} W(r)$ is a residual set in $\mathcal{U}$, too. (Since the intersection of a countable system of residual sets is a residual set.)

Step 3. It is well-known that the set of all limit points of a sequence is closed. Therefore if $A \in \bigcap_{r \in Q \cap (1,+\infty)} W(r)$ then

$$\left(\frac{a_{n+1}}{a_n}\right)_n^{'} = [1, +\infty],$$

$$A = \{a_1 < a_2 < \dots\}.$$

We shall detail our procedure.

Step 1. Set

$$W(r, k, n) = \left\{ A \in \mathcal{U} : \left| \frac{a_{n+1}}{a_n} - r \right| < \frac{1}{k} \right\}, \tag{17}$$

$$W(r) = \bigcap_{k=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} \left\{ A \in \mathcal{U} : \left| \frac{a_{n+1}}{a_n} - r \right| < \frac{1}{k} \right\}. \tag{17'}$$

The fact that $W(r, k, n)$ is an open set in $\mathcal{U}$ can be verified in a similar way we proceeded in the proof of Theorem 3.1 to show that $W(k, K)$ is an open set in $\mathcal{U}$. But then (17), (17') imply that $W(r)$ is a G_δ-set in $\mathcal{U}$.

We shall check that the set $W(r)$ is a dense set in $\mathcal{U}$:

Let $P = \{p_1 < p_2 < \dots\} \in \mathcal{U}$, $\varepsilon > 0$. Construct the ball $B(P, \varepsilon)$. We shall show that

$$B(P, \varepsilon) \cap W(r) \neq \emptyset.$$

Let $s \in N$, $\frac{1}{s} < \varepsilon$. Define $H = \{h_1 < h_2 < \dots\}$ as follows:

$$h_j = p_j \ (j = 1, \dots, s).$$

Choose an integer t_1, so that $t_1 q > h_s$ $(= p_s)$.

Set

$$h_{s+1} = t_1 q, \ h_{s+2} = t_1 p.$$

Then $h_{s+2} > h_{s+1}$, as p > q and $\frac{h_{s+2}}{h_{s+1}} = r = \frac{p}{q}$.

Choose $t_2 \in N$ so that $t_2 q > h_{s+2}$.
Set

$$h_{s+3} = t_2 q, \;\; h_{s+4} = t_2 p.$$

Then $h_{s+4} > h_{s+3}$, and $\frac{h_{s+4}}{h_{s+3}} = r \; (= \frac{p}{q})$, and so on.

By induction we obtain $H = \{h_1 < h_2 < \ldots\}, \;\; H \in B(P, \varepsilon)$ with

$$\frac{h_{s+2j}}{h_{s+2j-1}} = \frac{p}{q} = r \;\; (j = 1, 2, \ldots),$$

i.e. $\frac{p}{q} = r \in \left(\frac{h_{n+1}}{h_n}\right)'_n$. Therefore $H \in W(r)$. This ends the Step 1 of the proof.

Step 2. According to Step 1 the set $\bigcap_{r \in Q \cap (1, +\infty)} W(r)$ is a residual set in $\mathcal{U}$.

Step 3. If $A \in \bigcap_{r \in Q \cap (1,+\infty)} W(r), \;\; A = \{a_1 < a_2 < \ldots\}$ then the set $\left(\frac{a_{n+1}}{a_n}\right)'_n$ contains all rational numbers lying in the interval $(1, +\infty)$. Since this set is closed,we have

$$\left(\frac{a_{n+1}}{a_n}\right)'_n = [1, +\infty]. \quad \square$$

4 Applications of porosity to the study of the function $\lambda : \mathcal{U} \to [1, +\infty]$

In this section the outstanding point of our investigation will be Theorem 3.1. According to this theorem the set

$$\begin{aligned} C\mathcal{R} &= \{A \in \mathcal{U} : \lambda(A) < +\infty\} \\ &= \left\{A \in \mathcal{U} : \limsup_{n \to \infty} \frac{a_{n+1}}{a_n} < +\infty\right\} \end{aligned}$$

is a set of the first category in $\mathcal{U}$.

By Proposition 2.5 (in our notation we have $\mathcal{T} = C\mathcal{R}$) we have $\mu(\rho(\mathcal{T})) = 0$, i.e. $C\mathcal{R}$ is a null-system from the Lebesgue measure viewpoint. Hence it is natural to examine the set $C\mathcal{R}$ in terms of porosity.

Theorem 4.1 *The set $\mathcal{T} = C\mathcal{R} = \{A \in \mathcal{U} : \lambda(A) < +\infty\}$ is a σ-strongly porous set at each point of $\mathcal{U}$.*

Proof. As

$$\lambda(A) = \limsup_{n \to \infty} \frac{a_{n+1}}{a_n}, \;\; A = \{a_1 < a_2 < \ldots\},$$

the set $\mathcal{T}$ can be expressed in the form

$$\mathcal{T} = \bigcup_{m=1}^{\infty} \mathcal{T}_m,$$

where

$$\mathcal{T}_m = \left\{ A \in \mathcal{U} : (\forall n) \frac{a_{n+1}}{a_n} \leq m \right\} \ (m = 1, 2, \ldots).$$

Therefore it suffices to prove that each of the sets $\mathcal{T}_m$ is strongly porous at each point of $A \in \mathcal{U}$, i.e. $p(A, \mathcal{T}_m) = 1$.

Let $B(A, \varepsilon)$ be an arbitrary ball, $\varepsilon > 0$. Choose $v \in N$ so that

$$\frac{1}{v+1} \leq \varepsilon < \frac{1}{v} \tag{18}$$

(we already suppose that $\varepsilon < 1$).

Construct the ball $B\left(A, \frac{1}{v+1}\right) \subseteq B(A < \varepsilon)$.

We define the sequence $D = \{d_1 < d_2 < \ldots\}$ as follows:

$$d_n = a_n, \ n = 1, 2, \ldots, v+1.$$

Further we choose

$$\begin{aligned} d_{v+2} &= (m+1)a_{v+1}, \\ d_{v+3} &= d_{v+2} + 1, \\ d_{v+4} &= d_{v+3} + 1, \ \text{a.s.o.} \end{aligned}$$

Obviously we have

$$D \in B\left(A, \frac{1}{v+1}\right)$$

and

$$\frac{d_{v+2}}{d_{v+1}} = m + 1 > m.$$

If $H = \{h_1 < h_2 < \ldots\} \in B\left(D, \frac{1}{v+2}\right)$ then

$$h_{v+2} = d_{v+2}, h_{v+1} = d_{v+1}$$

and so

$$\frac{h_{v+2}}{h_{v+1}} = m + 1 > m.$$

Therefore $H \notin \mathcal{T}_m$ and so

$$B\left(D, \frac{1}{v+2}\right) \cap \mathcal{T}_m = \emptyset.$$

Further, it can be easily verified that

$$B\left(D, \frac{1}{v+2}\right) \subseteq B(A, \varepsilon).$$

But then

$$\begin{aligned} \gamma(A, \varepsilon, \mathcal{T}_m) &\geq \frac{1}{v+2}, \\ \frac{\gamma(A, \varepsilon, \mathcal{T}_m)}{\varepsilon} &\geq \frac{1/(v+2)}{1/v} = \frac{v}{v+2}. \end{aligned}$$

If $\varepsilon \to 0_+$ then $v \to +\infty$ and the above inequality implies $p(A, \mathcal{T}_m) = 1$. □

Theorem 4.1. is a best possible result. Namely, it is easy to see that the set $\mathcal{T}$ is dense in $\mathcal{U}$.

We shall examine the porosity character of the set $\mathcal{T} = \{A = \{a_1 < a_2 < \ldots\} : \lambda(A) < +\infty\}$ in the space $(\mathcal{U}, d)$, where d is Fréchet's metric,

$$d(Z, Y) = \sum_{k=1}^{\infty} 2^{-k} \frac{|z_k - y_k|}{1 + |z_k - y_k|}, Z = (z_k)_{k=1}^{\infty}, Y = (y_k)_{k=1}^{\infty} \in \mathcal{U}.$$

Lemma 4.1 *(i) If $z_j = y_j$ $(j = 1, 2, \ldots, k)$ then $d(Z, Y) < 2^{-k}$.*
(ii) If $d(Z, Y) < 2^{-k-1}$ then $z_j = y_j$ $(j = 1, 2, \ldots, k)$.

Proof. (i) A simple estimates gives

$$\begin{aligned} d(Z, Y) &= \sum_{j=1}^{\infty} 2^{-j} \frac{|z_j - y_j|}{1 + |z_j - yj|} = \sum_{j=k+1}^{\infty} 2^{-j} \frac{|z_j - y_j|}{1 + |z_j - yj|} \\ &< 2^{-k-1} + 2^{-k-2} + \ldots = 2^{-k}. \end{aligned}$$

(ii) Let $z_j \neq y_j$ for some j, $1 \leq j \leq k$.
Then

$$d(Z, Y) \geq 2^{-j} \frac{1}{1+1} = 2^{-j-1} \geq 2^{-k-1}. \quad \square$$

By help of Lemma 4.1 we obtain the next result.

Theorem 4.2 *In the Fréchet space $(\mathcal{U}, d)$ the following assertions are true:*
(a) The set $\mathcal{T}$ is σ-very $\frac{1}{8}$- porous at each point $A \in \mathcal{U}$.
(b) The set $\mathcal{T}$ is σ-strongly porous at each point $A \notin \mathcal{T}$.

Proof. (a) Construct the ball $B(A, \delta)$, $\delta > 0$ (in $\mathcal{U}$). We can already suppose that $\delta < 1$ and so we can choose $l \in N$ with

$$2^{-l-1} \leq \delta < 2^{-l}.$$

Let $m \in N$. Define a sequence $D = \{d_1 < d_2 < \ldots\}$ as follows:

$$\begin{aligned} d_j &= a_j \ (j = 1, 2, \ldots, l+1) \\ d_{l+2} &= (m+1) d_{l+1}, \\ d_{l+3} &= d_{l+2} + 1, \\ d_{l+4} &= d_{l+3} + 1, \text{ a.s.o.} \end{aligned}$$

Obviously

$$\frac{d_{l+2}}{d_{l+1}} = m + 1 > m,$$

i.e. $D \notin \mathcal{T}_m$.

By Lemma 4.1 (i) we have

$$D \in B(A, 2^{-l-1}) \subseteq B(A, \delta).$$

Further if $H = \{h_1 < h_2 < \ldots\} \in B(D, 2^{-l-3})$ then again by Lemma 4.1 (ii) we obtain

$$h_{l+2} = d_{l+2}, \ h_{l+1} = d_{l+1}$$

and so $H \notin \mathcal{T}_m$.

Since $B(D, 2^{-l-3}) \subseteq B(A, \delta)$ we obtain

$$\gamma(A, \delta, \mathcal{T}_m) \geq 2^{-l-3}.$$

By the choice of δ we conclude $\underline{p}(A, \mathcal{T}_m) \geq 2^{-3}$ $(m = 1, 2, \ldots)$.

(b) Let $A = \{a_1 < a_2 < \ldots\} \notin \mathcal{T}_m$. Then for each $m \in N$ there exists $l \in N$ such that

$$\frac{a_{l+1}}{a_l} > m \tag{19}$$

(because $A \notin \mathcal{T}_m$).

Choose $\delta > 0$. We can already suppose that $\delta < 2^{-l-2}$.

Take $Y \in B(A, \delta)$. Then Y belongs to $B(A, 2^{-l-2})$ as well, and so $y_j = a_j$ $(j = 1, 2, \ldots, l+1)$. But then by (19) $\frac{y_{l+1}}{y_l} > m$ holds, i.e. $Y \notin \mathcal{T}_m$.

Therefore

$$B(A, \delta) \cap \mathcal{T}_m = \emptyset.$$

But then for each $\delta < 2^{-l-2}$ we have

$$\gamma(A, \delta, \mathcal{T}_m) = \delta \ (m = 1, 2, \ldots).$$

From this we get $p(A, \mathcal{T}_m) = 1$ $(m = 1, 2, \ldots)$. □

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