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On the $\Psi-$Instability of a Nonlinear Volterra Integro-Differential System

by
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Abstract

We prove necessary and sufficient conditions for $\Psi-$ instability of the solutions of a linear homogeneous differential system and sufficient conditions for $\Psi-$ instability of the solutions of a nonlinear Volterra integro-differential system.

Key Words: $\Psi-$ *stability,* $\Psi-$ *instability.*
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1 Introduction

The purpose of our paper is to provide sufficient conditions for $\Psi-$ instability of trivial solution of a nonlinear Volterra integro-differential system of the form

$$x' = A(t)x + \int_0^t F(t,s,x(s))ds \tag{1}$$

and a linear system of the form

$$x' = [A(t) + B(t)]\, x \tag{2}$$

as a perturbed equations of

$$y' = A(t)y \tag{3}$$

We investigate conditions on the fundamental matrix Y(t) for linear equation (3) and on the functions B(t) and F(t,s,x) under which the trivial solution of (1), (2) or (3) is $\Psi-$ unstable on R_+. Here, Ψ is a matrix function. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of solutions.

The problem of $\Psi-$ stability for systems of ordinary differential equations has been studied by many authors, as e.g. O. Akinyele [1], [2], A. Constantin [4], [5],

T. Hallam [11], J. Kuben [13], J. Morchalo [16]. In these papers, the function Ψ is a scalar continuous function (and increasing, differentiable and bounded in [1] or nondecreasing and such $\Psi(t) \geq 1$ in [4]).

In our papers [8], [9], [10], we have proved sufficient conditions for various types of $\Psi-$ stability of trivial solution of the equations (1), (2) and (3). In these papers, the function Ψ is a matrix function.

Recent works for stability of solutions of (1) have been by C. Avramescu [3], by T. Hara, T. Yoneyama and T. Itoh [12], by V. Lakshmikantham and M. Rama Mohana Rao [14], by W.E. Mahfoud [15] and others.

Coppel's [7] paper deals with the instability of solutions of systems of differential equations.

2 Definitions, notations and hypotheses

Let R^n denote the Euclidean n - space. For $x = (x_1, x_2, \dots x_n)^T \in R^n$, let

$$\| x \| = \max\{| x_1 |, | x_2 |, \dots | x_n |\}$$

be the norm of x. For an n×n matrix $A = (a_{ij})$, we define the norm $| A |$ of A by

$$| A | = \sup_{\|x\| \leq 1} \| Ax \|$$

; it is well-known that

$$| A | = \max_{1 \leq i \leq n} \sum_{j=1}^{n} | a_{ij} |.$$

In the above equations, we consider that A(t) is a continuous n×n matrix on $R_+ = [0, \infty)$ and $F : D \times R^n \to R^n$,

$$D = \{(t, s) \in R^2 \mid 0 \leq s \leq t < \infty\},$$

is a continuous n - vector such that $F(t,s,0) = 0$ for $(t,s) \in D$.

Let $\Psi_i : R_+ \to (0, \infty)$, i = 1, 2, ... n, be continuous functions and

$$\Psi = \mathrm{diag}[\Psi_1, \Psi_2, \dots \Psi_n].$$

A matrix P is said to be a projection if $P^2 = P$. If P is a projection, then so is I - P. Two such projections, whose sum is I and hence whose product is 0, are said to be supplementary.

Definition 1. *The trivial solution of (1) is said to be $\Psi-$ stable on R_+, if for every $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution x(t) of (1) which satisfies the inequality $\| \Psi(t_0)x(t_0) \| < \delta(\varepsilon, t_0)$ exists and satisfies the inequality $\| \Psi(t)x(t) \| < \varepsilon$ for all $t \geq t_0$.*

Otherwise, we say that the trivial solution of (1) is $\Psi-$ unstable on R_+.

Definition 2. *We say that $\varphi : R_+ \to R^n$ is $\Psi-$ bounded on R_+ if $\Psi(t)\varphi(t)$ is bounded on R_+.*

Remark 1. *For $\Psi_i = 1$, $i = 1, 2,...n$, we obtain the notions of just stability , instability, resp. boundedness.*

3 Preliminaries

In this section, we give two Lemmas which be useful in the proofs of our theorems below.

Lemma 1. *Let $Y(t)$ be an invertible matrix which is a continuous function of t on R_+ and let P be a projection.*

If there exist a continuous function $\varphi : R_+ \to (0, \infty)$ and a positive constant M such that

$$\int_0^t \varphi(s) \mid \Psi(t)Y(t)PY^{-1}(s)\Psi^{-1}(s) \mid ds \ \leq \ M, \text{ for all } t \ \geq \ 0,$$

and

$$\int_0^{+\infty} \varphi(s)ds = +\infty$$

then, there exists a constant $N > 0$ such that

$$\mid \Psi(t)Y(t)P \mid \ \leq \ Ne^{-M^{-1}\int_0^t \varphi(s)ds}, \text{ for all } t \ \geq \ 0.$$

Consequently,

$$\lim_{t\to\infty} \ \mid \Psi(t)Y(t)P \mid \ = 0.$$

Proof: If $P = 0$, the conclusion is obvious.

For $P \neq 0$, let $q(t) = \mid \Psi(t)Y(t)P \mid^{-1}$, for $t \geq 0$. From the identity

$$\left(\int_0^t \varphi(s)q(s)ds\right) \Psi(t)Y(t)P =$$

$$= \int_0^t \varphi(s)\Psi(t)Y(t)PY^{-1}(s)\Psi^{-1}(s)\Psi(s)Y(s)Pq(s)ds,$$

for $t \geq 0$, it follows that

$$\left(\int_0^t \varphi(s)q(s)ds\right) \mid \Psi(t)Y(t)P \mid \leq$$

$$\leq \int_0^t \varphi(s) \mid \Psi(t)Y(t)PY^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)Y(s)P \mid q(s)ds, \ t \ \geq \ 0.$$

Thus, the scalar function $h(t) = \int_0^t \varphi(s)q(s)ds$ satisfies the inequality

$$h(t)q^{-1}(t) \leq M, \text{ for all } t \geq 0.$$

We have

$$h'(t) = \varphi(t)q(t) \geq M^{-1}h(t)\varphi(t), \text{ for all } t \geq 0.$$

It follows that

$$h(t) \geq h(t_1)e^{M^{-1}\int_{t_1}^t \varphi(s)ds}, \text{ for all } t \geq t_1 > 0$$

and hence

$$\mid \Psi(t)Y(t)P \mid = q^{-1}(t) \leq Mh^{-1}(t) \leq$$

$$Mh^{-1}(t_1)e^{-M^{-1}\int_{t_1}^t \varphi(s)ds}, \text{ for } t \geq t_1 > 0$$

Because $\mid \Psi(t)Y(t)P \mid$ is a continuous function on $[0, t_1]$, it follows that there exists a positive constant N such that

$$\mid \Psi(t)Y(t)P \mid \leq Ne^{-M^{-1}\int_0^t \varphi(s)ds}, \text{ for all } t \geq 0.$$

Consequently,

$$\lim_{t\to\infty} \mid \Psi(t)Y(t)P \mid = 0.$$

The proof is complete. □

Lemma 2. *Let Y(t) be an invertible matrix which is a continuous function of t on R_+ and let P be a projection.*

If there exists a constant $M > 0$ such that

$$\int_t^\infty \mid \Psi(t)Y(t)PY^{-1}(s)\Psi^{-1}(s) \mid ds \leq M, \text{ for all } t \geq 0,$$

then, for any vector $x_0 \in R^n$ such that $Px_0 \neq 0$,

$$\limsup_{t\longrightarrow\infty} \| \Psi(t)Y(t)Px_0 \| = +\infty.$$

Proof: Let q(t) = $\| \Psi(t)Y(t)Px_0 \|^{-1}$ for t $\geq$ 0. From the identity

$$\left(\int_t^T \mathrm{q(s)ds}\right) \Psi(\mathrm{t})\mathrm{Y(t)Px_0} =$$

$$= \int_t^T \Psi(\mathrm{t})\mathrm{Y(t)PY^{-1}(s)}\Psi^{-1}(\mathrm{s})\Psi(\mathrm{s})\mathrm{Y(s)Px_0q(s)ds}$$

for T $\geq$ t $\geq$ 0, it follows that

$$\left(\int_t^T \mathrm{q(s)ds}\right) \| \Psi(\mathrm{t})\mathrm{Y(t)Px_0} \| \leq$$

$$\leq \int_t^T | \Psi(\mathrm{t})\mathrm{Y(t)PY^{-1}(s)}\Psi^{-1}(\mathrm{s}) | \| \Psi(\mathrm{s})\mathrm{Y(s)Px_0} \| \mathrm{q(s)ds},$$

for all T $\geq$ t $\geq$ 0.

Thus, the scalar function q satisfies the inequality

$$\mathrm{q^{-1}(t)} \int_t^T \mathrm{q(s)ds} \leq \mathrm{M}, \text{ for all } \mathrm{T} \geq \mathrm{t} \geq 0.$$

It follows that $\int_t^\infty q(s)ds$ exists. Thus, lim $\inf_{t \longrightarrow \infty}$ q(t) = 0, which implies

$$\limsup_{t \longrightarrow \infty} \| \Psi(\mathrm{t})\mathrm{Y(t)Px_0} \| = +\infty.$$

The proof is complete. □

4 Ψ− Instability of linear equations

The purpose of this section is to study the Ψ− instability of trivial solution of the liniar equations (2) and (3).

The conditions for Ψ− instability of the trivial solution of (3) can be expressed in terms of a fundamental matrix for (3).

Theorem 1. *Let Y(t) be a fundamental matrix for (3).*

Then, the trivial solution of (3) is Ψ− unstable on R_+ if and only if there is a projection P such that | Ψ(t)Y(t)P | is unbounded on R_+.

Proof: If the trivial solution of (3) is Ψ− unstable on R_+, then | Ψ(t)Y(t) | is unbounded on R_+(see Theorem 1, [8]).

Suppose, conversely, that | Ψ(t)Y(t)P | is unbounded on R_+. We may reason by r.a.a. If the trivial solution of (3) is Ψ− stable on R_+, then, for $\varepsilon > 0$ and $\mathrm{t_0} \geq 0$, there exists a $\delta = \delta(\varepsilon,\mathrm{t_0}) > 0$ such that any solution x(t) of (3) which satisfies the inequality $\| \Psi(\mathrm{t_0})x(\mathrm{t_0}) \| < \delta(\varepsilon, t_0)$ exists and satisfies the inequality $\| \Psi(\mathrm{t})\mathrm{x(t)} \in\| < \varepsilon$ for all t $\geq$ $\mathrm{t_0}$.

Let $t_0 \geq 0$ and $x_0 \in R^n$ such that $| \Psi(t_0)Y(t_0)P | \neq 0$ and

$$\| x_0 \| < \delta \times | \Psi(t_0)Y(t_0)P |^{-1} = \delta_0.$$

We have

$$\| \Psi(t_0)Y(t_0)Px_0 \| < \delta.$$

It follows that

$$| \Psi(t)Y(t)Px_0 | < \varepsilon$$

for all $t \geq t_0$.

Thus, for $u \in R^n$, $\| u \| \leq 1$, we have $| \Psi(t)Y(t)P\delta_0 u | < \varepsilon$ for all $t \geq t_0$.

It follows that $| \Psi(t)Y(t)P | \leq \varepsilon\delta_0^{-1}$ for all $t \geq t_0$. This is contradictory.

The proof is complete. □

Remark 2. *In the same manner as in just instability, we can speak about $\Psi-$ instability of a linear equation (3).*

Theorem 2. *If there exist a projection $P \neq 0$ and a positive constant M such that the fundamental matrix Y(t) for (3) satisfies the inequality*

$$\int_t^\infty | \Psi(t)Y(t)PY^{-1}(s)\Psi^{-1}(s) | \, ds \ \leq \ M, \text{ for all } t \ \geq \ 0,$$

then, the trivial solution of (3) is $\Psi-$ unstable on R_+.

Proof: Suppose the contrary. Therefore, for every $\varepsilon > 0$ and any $t_0 \geq 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution y(t) of (3) which satisfies the inequality $\| \Psi(t_0)y(t_0)\| < \delta(\varepsilon, t_0)$ exists and satisfies the inequality $\| \Psi(t)y(t)\| < \varepsilon$ for all $t \geq t_0$.

Without loss of generality, we may assume that $Y(0) = I$ (else, we replace Y(t) with $Y(t)Y^{-1}(0)$ and P with $Y(0)PY^{-1}(0)$).

For $t_0 = 0$, we can choose $y(0) \in R^n$ such that $(I - P)y(0) = 0$ and $0 < \| \Psi(0)y(0) \| < \delta(\varepsilon, 0)$. Then, $\| \Psi(t)y(t) \| < \varepsilon$ for all $t \geq 0$.

On the other hand,

$$\Psi(t)y(t) = \Psi(t)Y(t)Y^{-1}(0)y(0) = \Psi(t)Y(t)Py(0).$$

From Lemma 2, it follows that

$$\limsup_{t\longrightarrow\infty} \| \Psi(t)y(t) \| = \infty,$$

which is contradictory.

The proof is complete. □

Example 1. Consider the linear equation

$$x' = Ax$$

where A is an n×n real constant matrix.

Suppose that the characteristic roots of A are divided into two sets Λ_1 and Λ_2 such that every root in the first set has real part less than α and every root in the second has real part greater than β, where $\alpha < \beta$ are given. We can represent the vector space R^n as the direct sum of two subspaces X_1, X_2 invariant under A such that all characteristic roots of the restriction of A in X_i belong Λ_i (i = 1, 2). The corresponding projections P_1, P_2 of R^n onto X_1, X_2 commute with A, are supplementary and there exists a constant $K > 0$ such that

$$|\, e^{tA}P_1 \,| \;\le\; Ke^{\alpha t},\ \text{for } t \;\ge\; 0 \tag{4}$$

$$|\, e^{tA}P_2 \,| \;\le\; Ke^{\beta t},\ \text{for } t \;\le\; 0. \tag{5}$$

It will be noted that $P_i \neq 0$ if Λ_i is not empty ([6], Chapter III).
Suppose that α and β are given such that $\Lambda_2 \neq \emptyset$.
Let $\gamma \in [\alpha, \beta)$ be and $\Psi(t) = e^{-\gamma t}I_n$. Then, we have

$$|\, \Psi(t)e^{tA}P_1e^{-sA}\Psi^{-1}(s) \,| \;\le\; Ke^{(\alpha - \gamma)(t - s)},\ \text{for } t \ge s,$$

$$|\, \Psi(t)e^{tA}P_2e^{-sA}\Psi^{-1}(s) \,| \;\le\; Ke^{(\beta - \gamma)(t - s)},\ \text{for } t \le s.$$

Thus, from the Theorem 2, it follows that the equation $x' = Ax$ is $\Psi-$ unstable on R_+.

Example 2. Consider the linear equation

$$x' = A(t)x$$

where A(t) is an n×n continuous periodic matrix of period τ.

From the Floquet's Theorem, it follows that a fundamental matrix Y(t) for the periodic linear equation above with $Y(0) = I$ can be represented in the form

$$Y(t) = P(t)e^{tL},$$

where L is a constant matrix and P(t) is a periodic matrix with period τ.

Suppose that the characteristic roots of L are divided into two sets Λ_1 and Λ_2 such that every root in the first set has real part less than α and every root in the second has real part greater than β, where $\alpha < \beta$ are given. We can represent the vector space R^n as the direct sum of two subspaces X_1, X_2 invariant under $C = Y(\tau) = e^{\tau L}$ and hence under $L = \tau^{-1}\log C$, such that every characteristic root of the restriction of C to X_1 (X_2) has absolute value less than $e^{\tau\alpha}$ (greater than $e^{\tau\beta}$). Then all characteristic roots of the restriction of L to X_i belong to

Λ_i. Since the projections P_1, P_2 of R^n onto X_1, X_2 commute with L, it follows from (4) and (5) that there exists a constant $K > 0$ such that

$$| Y(t)P_1Y(s) | \leq Ke^{\alpha(t-s)}, \text{ for } t \geq s,$$

$$| Y(t)P_2Y(s) | \leq Ke^{\beta(t-s)}, \text{ for } t \leq s.$$

It will be noted that P_i is real and $P_i \neq 0$ if Λ_i is not empty ([6], Chapter III).

Suppose that α and β are given such that $\Lambda_2 \neq \emptyset$.

Let $\gamma \in [\alpha, \beta)$ and $\Psi(t) = e^{-\gamma t}I_n$. Then, we have

$$| \Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s) | \leq Ke^{(\alpha-\gamma)(t-s)}, \text{ for } t \geq s,$$

$$| \Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s) | \leq Ke^{(\beta-\gamma)(t-s)}, \text{ for } t \leq s.$$

Thus, from the Theorem 2, it follows that the equation $x' = A(t)x$ is $\Psi-$ unstable on R_+.

Theorem 3. *Suppose that:*

1. There exist supplementary projections P_1, P_2, $P_2 \neq 0$, and a positive constant K such that the fundamental matrix Y(t) of the equation (3) satisfies the condition

$$\int_0^t | \Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s) | \, ds + \int_t^\infty | \Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s) | \, ds \leq K$$

for all $t \geq 0$.

2. B(t) is a continuous $n \times n$ matrix function for $t \geq 0$ and satisfies one of following conditions:

i. $M = \sup_{t \geq 0} | \Psi(t)B(t)\Psi^{-1}(t) |$ is a sufficiently small number ($MK < 1$).

ii. $\lim_{t \to \infty} | \Psi(t)B(t)\Psi^{-1}(t) | = 0$.

Then, the linear equation (2) is $\Psi-$ unstable on R_+.

Proof: In the case i), suppose the contrary. Therefore, for every $\varepsilon > 0$ and any $t_0 \geq 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution x(t) of (2) which satisfies the inequality $\| \Psi(t_0)x(t_0) \| < \delta$ exists and satisfies the inequality $\| \Psi(t)x(t) \| < \varepsilon$ for all $t \geq t_0$.

Without loss of generality, we may assume that $Y(0) = I$. For $t_0 = 0$, we can choose $x(0) \in R^n$ such that $P_1x(0) = 0$ and $0 < \| \Psi(0)x(0) \| < \delta(\varepsilon, 0)$. Then, $\| \Psi(t)x(t) \| < \varepsilon$ for all $t \geq 0$.

We consider the function

$$y(t) = x(t) - \int_0^t Y(t)P_1Y^{-1}(s)B(s)x(s)ds + \int_t^\infty Y(t)P_2Y^{-1}(s)B(s)x(s)ds,$$

for $t \geq 0$.

For $v \geq t \geq 0$ we have

$$|\int_t^v Y(t)P_2Y^{-1}(s)B(s)x(s)ds \; | \;\; \leq$$

$$\leq \;\; | \Psi^{-1}(t) | \int_t^v | \Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s) \; || \; \Psi(s)B(s)\Psi^{-1}(s) \; ||| \; \Psi(s)x(s) \; \| \, ds \;\; \leq$$

$$\leq \;\; | \Psi^{-1}(t) \; | \, M\varepsilon \int_t^v | \; \Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s) \; | \; ds.$$

It follows that the integral

$$\int_t^\infty Y(t)P_2Y^{-1}(s)B(s)x(s)ds$$

is convergent.

Thus, the function $y(t)$ exists for $t \geq 0$.

Clearly, the function $y(t)$ is continuously differentiable on R_+ and we have

$$y'(t) = x'(t) - \int_0^t Y'(t)P_1Y^{-1}(s)B(s)x(s)ds - Y(t)P_1Y^{-1}(t)B(t)x(t) +$$

$$+ \int_t^\infty Y'(t)P_2Y^{-1}(s)B(s)x(s)ds - Y(t)P_2Y^{-1}(t)B(t)x(t) =$$

$$= A(t)x(t) + B(t)x(t) -$$

$$- A(t)\left(\int_0^t Y(t)P_1Y^{-1}(s)B(s)x(s)ds - \int_t^\infty Y(t)P_2Y^{-1}(s)B(s)x(s)ds\right) -$$

$$- B(t)x(t) = A(t)y(t).$$

Thus, the function $y(t)$ is a solution on R_+ of the linear equation (3).

Since

$$\| \Psi(t)y(t) \| \;\; \leq \;\; \| \Psi(t)x(t) \| +$$

$$+ \int_0^t | \; \Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s) \; || \; \Psi(s)B(s)\Psi^{-1}(s) \; ||| \; \Psi(s)x(s) \; \| \, ds \; +$$

$$+\int_t^{\infty} \mid \Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s) \parallel \Psi(s)B(s)\Psi^{-1}(s) \mid\parallel \Psi(s)x(s) \parallel ds \le$$

$$\le \varepsilon\,(\ 1 + MK\ ),\ \text{for } t\ \ge\ 0,$$

it follows that the solution y(t) is $\Psi-$ bounded on R_+.

On the other hand,

$$y(t) = Y(t)Y^{-1}(0)y(0) = Y(t)[P_1y(0) + P_2y(0)] = Y(t)P_2y(0).$$

If $P_2y(0) \neq 0$, from the Lemma 2 it follows that

$$\limsup_{t\longrightarrow\infty} \parallel \Psi(t)Y(t)P_2(0) \parallel = \infty,$$

which is contradictory.

Thus, $P_2y(0) = 0$ and then y(t) = 0 on R_+.

Thereafter,

$$x(t) = \int_0^t Y(t)P_1Y^{-1}(s)B(s)x(s)ds - \int_t^{\infty} Y(t)P_2Y^{-1}(s)B(s)x(s)ds, \text{ for } t\ \ge\ 0.$$

It follows that

$$\parallel \Psi(t)x(t) \parallel\ \ \le\ \ MK\sup_{t\ge0} \parallel \Psi(t)x(t) \parallel,\ \text{for } t\ \ge\ 0,$$

which is contradictory.

Thus, the linear equation (2) is $\Psi-$ unstable on R_+.

ii). The proof is similar to that of i).

The proof of the theorem is complete. □

Remark 3. *We can show furter that if x(t) is a $\Psi-$ bounded solution of the equation (2), then* $\lim t \to \infty \parallel \Psi(t)x(t) \parallel = 0$ *and there exists a constant $N > 0$, independent of B, such that*

$$\parallel \Psi(t)x(t) \parallel\ \ \le\ \ (\ 1 - MK\ )^{-1} N \parallel P_1x(0) \parallel,\ \text{for } t\ \ge\ 0.$$

If the linear equation (3) is only $\Psi-$ unstable, then the linear equation (2) can't be $\Psi-$ unstable. This is shown by the next example transformed after an example due to O. Perron [17].

Example 3. Let a, b $\in$R such that $1 \leq 2a < 1 + e^{-\pi}$, $b \neq 0$ and

$$A(t) = \begin{pmatrix} \sin\ln(t+1) + \cos\ln(t+1) - 2a & be^{-a(t+1)} \\ 0 & -a \end{pmatrix}.$$

Then, a fundamental matrix for the homogeneous equation (3) is

$$Y(t) = \begin{pmatrix} be^{(t+1)[\sin\ln(t+1) - 2a]} \int_1^{t+1} e^{-s\ \sin\ln s} ds & e^{(t+1)[\sin\ln(t+1) - 2a]} \\ e^{-a(t+1)} & 0 \end{pmatrix}.$$

Let

$$\Psi(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{a(t+1)} \end{pmatrix}.$$

We have

$$\Psi(t)Y(t) = \begin{pmatrix} be^{(t+1)[\sin\ln(t+1) - 2a]} \int_1^{t+1} e^{-s\ \sin\ln s} ds & e^{(t+1)[\sin\ln(t+1) - 2a]} \\ 1 & 0 \end{pmatrix}.$$

Let $\alpha \in (0, \frac{\pi}{2})$ be such that $\cos\alpha > (2a-1)e^{\pi}$. Let $t_n = e^{(2n-\frac{1}{2})\pi}$ be for n = 1, 2... .For $t_n \leq s \leq t_n e^{\alpha}$ we have $s \cos\alpha \leq -s \sin\ln s \leq s$ and hence

$$e^{t_n e^{\pi}(\sin\ln t_n e^{\pi} - 2a)} \int_1^{t_n e^{\pi}} e^{-s\sin\ln s}\ ds >$$

$$e^{t_n e^{\pi}(\sin\ln t_n e^{\pi} - 2a)} \int_{t_n}^{t_n e^{\alpha}} e^{-s\sin\ln s} ds >$$

$$> e^{t_n e^{\pi}(1-2a)} \int_{t_n}^{t_n e^{\alpha}} e^{s\cos\alpha} ds =$$

$$e^{t_n[(1-2a)e^{\pi} + \cos\alpha\}} \left(e^{t_n(e^{\alpha}-1)\cos\alpha} - 1\right)(\cos\alpha)^{-1} \to \infty$$

This shows that $|\ \Psi(t)Y(t)\ |$ is unbounded on R_+.
It follows that the equation (3) is Ψ− unstable on R_+.
If we take

$$B(t) = \begin{pmatrix} 0 & -be^{-a(t+1)} \\ 0 & 0 \end{pmatrix},$$

then, a fundamental matrix for the perturbed equation (2) is

$$\widetilde{Y}(t) = \begin{pmatrix} e^{(t+1)[\sin\ln(t+1) - 2a]} & 0 \\ 0 & e^{-a(t+1)} \end{pmatrix}.$$

We have

$$\Psi(t)\widetilde{Y}(t) = \begin{pmatrix} e^{(t\,+\,1)[\sin\ln(t\,+\,1)\,-\,2a]} & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that $\mid \Psi(t)\widetilde{Y}(t) \mid \leq 1$ for all $t \geq 0$.

It follows that the equation (2) is not $\Psi-$ unstable on R_+.

Finally, we have $\mid \Psi(t)B(t)\Psi^{-1}(t) \mid = be^{-2a(t\,+\,1)}$. Thus, B(t) satisfies the conditions of the Theorem.

Theorem 4. *In the conditions of the Example 1 or 2, if B(t) is a continuous $n\times n$ matrix function on R_+ such that*

$$L = \int_0^\infty \mid \Psi(t)B(t)\Psi^{-1}(t) \mid dt$$

is a sufficiently small number, then the linear equation (2) is $\Psi-$ unstable on R_+.

Proof: The proof is similar to that of Theorem 3. □

5 Ψ− Instability of the nonlinear equation (1)

The purpose of this section is to study the $\Psi-$ instability of trivial solution of the nonlinear Volterra integro-differential equation (1).

Theorem 5. *Suppose that:*

1.There exist supplementary projections P_1, $P_{2,}$, $P_2 \neq 0$ and a constant $K > 0$ such that the fundamental matrix Y(t) of (3) satisfies the condition

$$\int_0^t \mid \Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s) \mid ds +$$
$$+\int_t^\infty \mid \Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s) \mid ds \;\leq\; K$$

for all $t \geq 0$;

2. The function F(t,s,x) satisfies the inequality

$$\| \Psi(t)F(t,s,x(s)) \| \;\leq\; f(t,s) \,\| \Psi(s)x(s) \| \,,\; o \;\leq\; s \;\leq\; t < \infty,$$

for every continuous function $x : R_+ \longrightarrow R^n$, where f(t,s) is a continuous nonnegative function on D such that

$$\int_0^t f(t,s)ds \leq M, \text{ for all } t \;\geq\; 0,$$

where M is a positive constant;

3. $MK < 1$,

Then, the trivial solution of (1) is $\Psi-$ unstable on R_+.

Proof: Suppose the contrary. Therefore, for every $\varepsilon > 0$ and any $t_0 \geq 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution x(t) of (1) which satisfies the inequality $\| \Psi(t_0)x(t_0) \| < \delta$ exists and satisfies the inequality $\| \Psi(t)x(t) \| < \varepsilon$ for all $t \geq t_0$.

Without loss of generality, we may assume that $Y(0) = I$.

For $t_0 = 0$, we can choose $x(0) \in R^n$ such that $P_1x(0) = 0$ and $0 < \| \Psi(0)x(0)\| < \delta(\varepsilon, 0)$.

Then, $\| \Psi(t)x(t) \| < \varepsilon$ for all $t \geq 0$.

Consider the function

$$y(t) = x(t) - \int_0^t Y(t)P_1Y^{-1}(s)\int_0^s F(s,u,x(u))duds +$$

$$+ \int_t^\infty Y(t)P_2Y^{-1}(s)\int_0^s F(s,u,x(u))duds, \text{ for } t \geq 0.$$

For $v \geq t \geq 0$ we have

$$\| \int_t^v Y(t)P_2Y^{-1}(s)\int_0^s F(s,u,x(u))duds \| \leq$$

$$\leq |\Psi^{-1}(t)| \int_t^v |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \int_0^s \| \Psi(s)F(s,u,x(u)) \| duds \leq$$

$$\leq |\Psi^{-1}(t)| \int_t^v |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \int_0^s f(s,u) \| \Psi(u)x(u) \| duds \leq$$

$$\leq M\varepsilon |\Psi^{-1}(t)| \int_t^v |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| ds.$$

It follows that the integral

$$\int_t^\infty Y(t)P_2Y^{-1}(s)\int_0^s F(s,u,x(u))duds$$

is convergent.

Thus, the function y(t) exists on R_+.

Clearly, the function y(t) is continuously differentiable on R_+ and we have

$$y'(t) = x'(t) - \int_0^t Y'(t)P_1Y^{-1}(s)\int_0^s F(s,u,x(u))duds -$$

$$- Y(t)P_1Y^{-1}(t)\int_0^t F(t,u,x(u))du +$$

$$+ \int_t^\infty Y'(t)P_2Y^{-1}(s)\int_0^s F(s,u,x(u))duds -$$

$$- Y(t)P_2Y^{-1}(t)\int_0^t F(t,u,x(u))du = A(t)x(t) + \int_0^t F(t,s,x(s))ds -$$

$$- A(t)\int_0^t Y(t)P_1Y^{-1}(s)\int_0^s F(s,u,x(u))duds -$$

$$- Y(t)\,(P_1 + P_2)\,Y^{-1}(t)\int_0^t F(t,u,x(u))du +$$

$$+ A(t)\int_t^\infty Y(t)P_2Y^{-1}(s)\int_0^s F(s,u,x(u))duds = A(t)y(t), \text{ for all } t \geq 0.$$

Thus, y(t) is a solution on R_+ of the linear equation (3).
Since

$$\| \Psi(t)y(t) \| \leq \| \Psi(t)x(t) \| +$$

$$+\int_0^t | \Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s) | \int_0^s \| \Psi(s)F(s,u,x(u)) \| duds+$$

$$+\int_t^\infty | \Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s) | \int_0^s \| \Psi(s)F(s,u,x(u)) \| duds \leq$$

$$\leq \varepsilon + \int_0^t | \Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s) | \int_0^s f(s,u) \| \Psi(u)x(u) \| duds+$$

$$+\int_t^\infty | \Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s) | \int_0^s f(s,u) \| \Psi(u)x(u) \| duds \leq$$

$$\leq \varepsilon(\,1 + MK\,), \text{ for } t \geq 0,$$

it follows that the solution y(t) is $\Psi-$ bounded on R_+.
On the other hand,

$$y(t) = Y(t)Y^{-1}(0)y(0) = Y(t)[P_1y(0) + P_2y(0)] = Y(t)P_2y(0).$$

If $P_2y(0) \neq 0$, from the Lemma 2 it follows that

$$\limsup_{t\longrightarrow\infty} \| \Psi(t)y(t) \| = \infty,$$

which is contradictory.

Thus, $P_2y(0) = 0$ and then $y(t) = 0$ on R_+.

Thereafter,

$$x(t) = \int_0^t Y(t)P_1Y^{-1}(s)\int_0^s F(s,u,x(u))duds -$$

$$- \int_t^\infty Y(t)P_2Y^{-1}(s)\int_0^s F(s,u,x(u))duds, \text{ for } t \geq 0.$$

In the same manner as above, it follows that

$$\| \Psi(t)x(t) \| \leq MK \| \Psi(t)x(t) \| , \text{ for } t \geq 0,$$

which is contradictory.

Thus, the trivial solution of (1) is $\Psi-$ unstable on R_+.

The proof is complete. □

Corollary 1. *If in Theorem 5 we consider that f(t,s) satisfies one of following conditions*

a). $f(t,s) = g(t)h(s)$, where g and h are continuous functions on R_+ such that

$$g(t)\int_0^t h(s)ds \leq M, \text{ for all } t \geq 0,$$

b). $f(t,s) = k(t-s)$, where k is a continuous function on R_+ such that

$$\int_0^\infty k(u)du < +\infty,$$

then, the conclusion of the Theorem is true.

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