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A Class of Partitionable Graphs

by

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Abstract

Let us call a graph G partitionable if $\theta(G)$=$\alpha(G)$ and $\chi(G)$=$\omega(G)$ hold. We call a graph G *O-graph* if there are an optimal coloring of the set of vertices of G and an optimal coloring of $\overline{G}$, the complement of G, such that any color-class of G intersects any color-class of $\overline{G}$. The main result of this paper is (Theorem 1): A graph G with n vertices is an O-graph iff it is partitionable and n=$\alpha(G)\cdot\omega(G)$.

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1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e. finite, undirected, without loops and multiple edges) graph with vertex set V and edge set E. $\overline{G}$ designates the complement of G. If $A \subseteq V$, then $[A]$ is the subgraph of G induced by $A \subseteq V$. By $G - W$ we mean the graph $(V, E - W)$, whenever $W \subseteq E$ and by $G - X$ we mean the graph $[V - X]$, whenever $X \subseteq V$, but we will often denote is simply $G - v$ $(\forall v \in X)$ when there is no ambiguity.

A *stable* set in G is a set of mutually non-adjacent vertices, and the *stability number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum stable set.

A *clique* in G is a subset of $V(G)$ that induce a complete subgraph in G, and the *density* (or *clique number*) of G, denoted by $\omega(G)$, is the size of a largest clique in G, i.e., $\omega(G)$= $\alpha(\overline{G})$.

The *chromatic number* and the *clique covering number* of G (i.e. the chromatic number of $\overline{G}$) will be denoted respectively, by $\chi(G)$ and $\theta(G)$.

The quasi-cartesian product of the two graphs G_1=(V_1, E_1) and G_2=(V_2, E_2) is the graph $G_1 \otimes G_2$ whose vertex set is $V_1 \times V_2$ and two vertices (v_1, v_2) and $(v_1\prime, v_2\prime)$ are adjacent iff:

$$v_1 = v_1\prime \text{ and } v_2v_2\prime \in E_2;$$

$$v_1 v_1{\prime} \in E_1 \text{ and } v_2 v_2{\prime} \in E_2;$$

$$v_1 v_1{\prime} \in E_1 \text{ and } v_2 = v_2{\prime}.$$

Let us recall that in communication theory the determining of $\alpha(G^m)$, ($G^m := G \otimes G^{m-1}, m \geq 2$) has an important role [1]

It is known, [3], that for any two graphs G_1 and G_2 holds

$$\alpha(G_1 \otimes G_2) \leq \min\{\alpha(G_1)\theta(G_2), \alpha(G_2)\theta(G_1)\},$$

and, consequently, if $\alpha(G) = \theta(G)$, then $\alpha(G \otimes H) = \alpha(G) \cdot \alpha(H)$, for any graph H, and so, for such graphs we have

$$\alpha(G^m) = [\alpha(G)]^m, m \geq 1.$$

A graph G is called $\alpha -$ *partitionable* if $\alpha(G) = \theta(G)$ holds. A graph G is *perfect* if $\alpha(H)=\theta(H)$ (or, equivalently, $\chi(H)=\omega(H)$) holds for any induced subgraph H of G, i.e. every induced subgraph is $\alpha -$ *partitionable.*

The famous Strong Perfect Graph Conjecture, to structure characterization of perfect graphs, was recently proved by M.Chudnovsky, N.Robertson, P.D.Seymour, R.Thomas, [1].

Now the problem is to find other classes of $\alpha -$ *partitionable* graphs, that can be non perfect. In this paper we consider a particular class of partitionable graphs.

2 Partitonable graphs.

Definition 1. *Let us call a graph G partitionable if* $\theta(G)=\alpha(G)$ *and* $\chi(G)=\omega(G)$.

Obviously, to characterize the partitionable graphs is a very hard problem, but it is possible to find some classes of such graphs. Let us consider the following partitionable graphs.

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[1] *In 1956 C. E. Shannon [4] posed the following problem in information theory.*

Suppose we wish to send messages through in the noisy communication channels. The messages we send use letters in an alphabet $S = \{s_1, s_2, ..., s_p\}$ *and are received as letters in an alphabet* $S' = \{s'_1, s'_2, ...s'_q\}$. *Let* $p_i(j)$ *be the probability to receive the letter* $s'_j \in S'$ *when the letter* $s_i \in S$ *has been transmitted. In most cases it is necessary to transmit not only letters, but words on m* ($m \geq 1$) *letters. For channels without memory, the probability* $p_u(u')$ *to receive the word* $u' = s'_{j1} s'_{j2} ... s'_{jm}$ *when the word* $u = s_{i_1} s_{i_2} ... s_{i_m}$ *has been transmitted is equal to* $\prod_{k=1}^{m} p_{i_k}(j_k)$. *Let* U_u *denote the set of all exit words* u' *of length m for which* $p_u(u') > 0$. *Two entrance m-letters words u and v can not be confused at the exit if* $U_u \cap U_v = \emptyset$; *we will call such words unmistakable. Obviously, the set of all unmistakable m-words can be decoded at the exit without errors, and thus using in transmission only unmistakable words, they can be decoded at the reception with no error.*

The issue is the determination of the largest set of unmistakable m-lengths words. For this a graph G_m *is attached to the set of vertices consisting of all m-lengths words in which two vertices are not adjacent if they correspond to m-words, that are unmistakables. Then the maximum number of unmistakable m-lengths words is* $\alpha(G_m)$. *It is a fact, (see [3]), that* $G_m = G \otimes G \otimes ... \otimes G$ *(m times) and* $\alpha(G_m) \geq [\alpha(G)]^m$ ($G = G_1$).

Definition 2. *A graph G is called an O-graph if there are a coloring of G and a coloring of $\overline{G}$, the complement of G, such that any color-class of G intersects any color-class of $\overline{G}$.*

Theorem 1. *Let G be a graph with n vertices. Then G is an O-graph if and only if G is partitionable and $n = \alpha(G) \cdot \omega(G)$.*

Proof: Let be $\chi(G) = p$, $\theta(G) = \chi(\overline{G}) = q$, $S = (S_1, S_2, ..., S_p)$ a $p - coloring$ of G and $Q = (Q_1, Q_2, ..., Q_q)$ a $q - coloring$ of $\overline{G}$, such that $S_i \cap Q_j \neq \emptyset$, for all $i = 1, 2, ..., p$ and $j = 1, 2, ..., q$, hold. Then

$$\alpha(G) \geq |S_i| = \textstyle\sum_{j=1}^{q} |S_i \cap Q_j| = q \geq \theta(G) \geq \alpha(G) \text{ for all } i = 1, 2, ..., p.$$

It follows:

$$|S_i| = \alpha(G), i = 1, 2, ..., p,\ q = \alpha(G),\ |Q_j| = \omega(G),\ p = \omega(G), j = 1, 2, ..., q, \text{ and}$$
$$n = \textstyle\sum_{i=1}^{p} |S_i| = \alpha(G) \cdot \omega(G).$$

Because $\chi(G) = \omega(G)$ $(:= \omega)$ and $n = \alpha(G) \cdot \omega(G)$ there is an optimal coloring of G with ω stable sets $(S_1, ..., S_\omega)$ with $|S_i| = \alpha$ $(:= \alpha(G))$. Similarly, there is an optimal coloring $(Q_1, Q_2, ..., Q_\alpha)$ of $\overline{G}$ with $|Q_j| = \omega$. Obviously, $S_i \cap Q_j \neq \emptyset$, for all $i = 1, 2, ..., \omega$ and $j = 1, 2, ..., \alpha$, that is G is an O-graph. □

Corollary 1. *A graph G is an O-graph if and only if the set of vertices can be partitioned in ω stable sets each of it having α elements and in α cliques with ω vertices.*

Remark 1. *There are O-graphs which are not perfect.*

Corollary 2. *If G is an O-graph then, any color-class of any optimal coloring intersects any clique from any optimal covering with cliques of G.*

Proof: If G is an O-graph, then any optimal coloring of G has exactly ω classes of color with exactly α elements and, similarly, any optimal coloring of $\overline{G}$ has exactly α classes of color with exactly ω vertices. □

Proposition 1. *Let G be an O-graph. Then for any optimal coloring $(S_1, ..., S_\omega)$ of G, the subgraph induced by $S_i \cup S_j$ has a perfect matching for all $i, j = 1, ..., \omega$ with $i \neq j$.*

Proof: We consider $(S_1, ..., S_\omega)$ a ω-coloring of G and $(Q_1, Q_2, ..., Q_\alpha)$ a α-coloring of $\overline{G}$. According to Corollary 2, $S_i \cap Q_k \neq \emptyset$ for all $i = 1, 2, ..., \omega$ and $k = 1, 2, ..., \alpha$ and let us denote $\{x_{i_k}\} = S_i \cap Q_k$. Then for $l \neq k$, $x_{i_l} \neq x_{i_k}$ because $Q_l \cap Q_k = \emptyset$ holds. It follows $S_i = \{x_{i_1}, x_{i_2}, ..., x_{i_\alpha}\}$, for all $i = 1, 2, ..., \omega$, and, $x_{i_k} x_{j_k} \in E(G)$ for $k = 1, 2, ..., \alpha$, and $i \neq j$. Then set $\{x_{i_k} x_{j_k} | \, k = 1, 2, ..., \alpha$, and $i \neq j\}$ is a perfect matching of $[S_i \cup S_j]$. □

Proposition 2. *Let G be an O-graph of order n, size m, stability number α, density ω. Then*

$$\alpha C_{\omega}^{2} \leq m \leq \alpha^{2} C_{\omega}^{2}.$$

Proof: First we proof the following.

Let G be an O-graph with n vertices, m edges, stability number α, density ω and let $(S_1,...,S_{\omega})$ be a coloring of G. Then for any $i \neq j$ and for any two nonadjacent vertices $x \in S_i$ and $y \in S_j$ the graph $G' = G + xy$ is an O-graph, with the same parameters: n, α, ω.

Obviously $\chi(G') = \omega(G) \leq \omega(G')$, and $\theta(G') \leq \theta(G) = \alpha(G) = \alpha(G')$. It follows $\chi(G') = \omega(G') = \omega$ and $\theta(G') = \alpha(G') = \alpha$.

Because the disjoint union of α ω-cliques is an O-graph of minimum size, we have $\alpha C_{\omega}^{2} \leq m$. By above the remark the completely ω-partite graph $K_{\alpha,\ldots,\alpha}$ is O-graph of maximum size, it follows $m \leq \alpha^{2} C_{\omega}^{2}$. □

Definition 3. *Let p, q be positive integer. A graph G is called $(p,q)-$decomposable if G admits a p-coloring $(S_1,...,S_p)$ where $|S_i| = q$ for all $i = 1,...,p$.*

Corollary 3. *A graph G is O-graph if and only if G is (p,q) − decomposable and $\overline{G}$ is (q,p) − decomposable , for some p and q.*

Proof: From Corollary 1 it follows that any O-graph is both $(\omega,\alpha)-$*decomposable* and (α,ω) − *decomposable*.

If G is (p,q) − *decomposable* and (q,p) − *decomposable*, for some p and q, then:

$$p \leq \omega(G) \leq \chi(G) \leq p \text{ and,}$$
$$q \leq \alpha(G) \leq \chi(\overline{G}) \leq q.$$

It follows $\chi(G) = \omega(G)$, $\chi(\overline{G}) = \alpha(G)$, $n = \alpha(G) \cdot \omega(G)$ and, by Theorem 1, G is an O-graph. □

Definition 4. *A graph G is called a critical O-graph if G is not an O-graph, and G-x is an O-graph, for any vertex x of G.*

Definition 5. *A graph G with n vertices is called (α,ω)-partitionable (see [2]) if $n = \alpha\omega + 1$ and for every vertex v the induced subgraph $G - v$ can be partitioned into α cliques of size ω, as well as into ω stable sets of size α.*

Corollary 4. *A graph G is critically O-graph iff G is (α,ω)-partitionable.*

Proof: It follows from Definition 5 and Corollary 1. □

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