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Real rank zero and continuous fields of C^*-algebras

by
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Abstract

We give a necessary and sufficient condition for the C^*-algebra associated to an arbitrary continuous field of C^*-algebras to have real rank zero. Some applications of this result are given, including a characterization of the real rank zero property for the C^*-algebras with Hausdorff primitive spectrum.

Key Words: C^*-algebra, real rank zero, continuous field of C^*-algebras, the C^*-algebra associated to a continuous field of C^*-algebras, minimal tensor product of C^*-algebras, primitive spectrum.
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1 Introduction

A C^*-algebra A is said to have real rank zero (written $\mathrm{RR}(A) = 0$) if its unitarization $\tilde{A} := A + \mathbb{C} \cdot 1$ has the property that the set of all invertible, selfadjoint elements of $\tilde{A}$ is dense in the set of all selfadjoint elements of $\tilde{A}$ ([5]). The real rank of C^*-algebras was introduced by L.G. Brown and G.K. Pedersen ([5]). The C^*-algebras with real rank zero can be seen as a kind of noncommutative zero dimensional topological spaces. The real rank zero C^*-algebras have nice properties and they form a large class of algebras (see e.g. [5], [6]). The real rank zero property proved to be very important in Elliott's classification program for separable, amenable C^*-algebras by discrete invariants including the K-theory ([8]). Indeed, the great majority of the classification results obtained till now are for classes of amenable, separable, real rank zero C^*-algebras (see [8], [15]). Clearly, a natural and important problem is to characterize the real rank zero property for large and interesting classes of C^*-algebras. This has been done, e.g., for arbitrary C^*-algebras ([5]), for large classes of AH algebras ([1], [2], [9], [4]) and for large classes of LB algebras ([13]). In this paper we characterize the real rank zero property for the rich and important class of the C^*-algebras associated to continuous fields of C^*-algebras ([7], [10]). Let $\mathcal{A} = ((A(t))_{t\in T}, \Theta)$ be a continuous field of C^*-algebras and let A be the C^*-algebras associated to it. We prove that A has real rank zero if and only if $\{t \in T : A(t) \neq 0\}$ is zero dimensional and each fiber C^*-algebra $A(t)\,(t \in T)$ has real rank zero (Theorem 2.1). As a corollary of this result we obtain that if A and B are two nonzero C^*-algebras and A is commutative, then $\mathrm{RR}(A \otimes B) = 0$ if and only if $\mathrm{RR}(A) = \mathrm{RR}(B) = 0$ (Corollary 2.3). Note that in general (i.e. if A is not necessarily commutative), this equivalence is known not to be true ([11], [12]) (Remark 2.4). Another consequence of Theorem 2.1 is the fact that for a C^*-algebra A with Hausdorff primitive spectrum $\mathrm{Prim}(A)$, $\mathrm{RR}(A) = 0$ if and only if $\dim(\mathrm{Prim}(A)) = 0$ and $\mathrm{RR}(A/I) = 0$ for every $I \in \mathrm{Prim}(A)$ (Corollary 2.5).

This paper can be also seen as a natural continuation of our work in [14] where, among other things, we characterized the ideal property for the C^*-algebras associated to continuous fields of C^*-algebras and we also characterized the projection property for the separable C^*-algebras associated to continuous fields of C^*-algebras.

If A is a C^*-algebra, $\mathcal{P}(A)$ will denote the set of all the projections of A, i.e. $\mathcal{P}(A) = \{p \in A : p = p^* = p^2\}$, while $\mathrm{Prim}(A)$ will denote the primitive spectrum of A endowed with the Jacobson (or the hull-kernel) topology ([7]). The symbol $\otimes$ will in this paper always mean the minimal tensor product of C^*-algebras. If M is a subset of a set X, χ_M will denote the characteristic function of M, that is $\chi_M : X \to \mathbb{C}$ is defined by $\chi_M(x) = 1$ if $x \in M$ and $\chi_M(x) = 0$ if $x \in X \setminus M$ (X will be understood form the context). If T is a locally compact, Hausdorff space, $C_b(T)$ will denote the set of all the continuous and bounded functions $f : T \to \mathbb{C}$.

2 Results

The main result of our paper is the following:

Theorem 2.1. *Let $\mathcal{A} = ((A(t))_{t\in T}, \Theta)$ be a continuous field of C^*-algebras over a locally compact, Hausdorff space T. Let A be the C^*-algebra associated to $\mathcal{A}$. Then, the following are equivalent:*

(1) $\mathrm{RR}(A) = 0$;

(2) $\{t \in T : A(t) \neq 0\}$ *is totally disconnected and for every* $t \in T$, $\mathrm{RR}(A(t)) = 0$.

The proof of this theorem will use (part of) the following result:

Theorem 2.2 ([14, Theorem 2.1]) *Let $\mathcal{A} = ((A(t))_{t\in T}, \Theta)$ be a continuous field of C^*-algebras over a locally compact, Hausdorff space T. Let A be the C^*-algebra associated to $\mathcal{A}$. Then, the following are equivalent:*

(1) *A has the ideal property;*

(2) *$\{t \in T : A(t) \neq 0\}$ is totally disconnected and for every $t \in T, A(t)$ has the ideal property.*

Proof of Theorem 2.1. This proof is inspired by the proof of [14, Theorem 2.1].

(1) $\Rightarrow$ (2). Assume that $\mathrm{RR}(A) = 0$. We know that for every $t \in T, A \ni f \mapsto f(t) \in A(t)$ is a surjective $*$ - homomorphism. Then, since $\mathrm{RR}(A) = 0$ and since the property of having real rank zero passes to quotients, it follows that $\mathrm{RR}(A(t)) = 0$ for every $t \in T$. Observe also that the implication (1) $\Rightarrow$ (2) in Theorem 2.2 implies immediately that $\{t \in T : A(t) \neq 0\}$ is totally disconnected (since the fact that $\mathrm{RR}(A) = 0$ implies clearly that A has the ideal property). (Recall that a C^*-algebra is said to have the ideal property if each of its (closed, two-sided) ideals is generated (as an ideal) by its projections).

(2) $\Rightarrow$ (1). Assume that (2) is true. As it follows from the proof of [14, Theorem 2.1], the C^*-algebra associated to an arbitrary continuous field of C^*-algebras $E(t)$ $(t \in S)$ over a locally compact, Hausdorff space S is $*$ - isomorphic to the C^*-algebra associated to a continuous field of C^*-algebras $F(t)$ $(t \in S_1)$ over the locally compact, Hausdorff space $S_1 := \{t \in S : E(t) \neq 0\}$, where $F(t) := E(t)$ for every $t \in S_1$. Hence, in order to prove the implication (2) $\Rightarrow$ (1), we may assume, and we shall, that T *is totally disconnected.* Fix now $x = x^* \in A$ and $\varepsilon > 0$. We shall prove that there is an element $y = y^*$ of A with finite spectrum such that $\| x - y \| \leq \varepsilon$. Define $K := \{t \in T : \| x(t) \| \geq \varepsilon\}$. Obviously, K is a compact subset of T (by the definition of A and the fact that $x \in A$). For each fixed $t \in K$, since $\mathrm{RR}(A(t)) = 0$, by [5, Theorem 2.6] there are $n(t) \in \mathbb{N}$, $\lambda_{i,t} \in \mathbb{R}$ and pairwise orthogonal projections $p_{i,t}$ in $A(t)$, $1 \leq i \leq n(t)$ such that:

$$\left\| x(t) - \sum_{i=1}^{n(t)} \lambda_{i,t} p_{i,t} \right\| < \varepsilon. \qquad (2.1)$$

By [10, Lemma 3.1], there are an open neighborhood V_t of t and $\tilde{p}_{i,t} \in A$ such that $\tilde{p}_{i,t}(t) = p_{i,t}$ and $\tilde{p}_{1,t}(s), \tilde{p}_{2,t}(s), \ldots, \tilde{p}_{n(t),t}(s)$ are pairwise orthogonal projections in $A(s)$ for every $s \in V_t$ and $1 \leq i \leq n(t)$. Using the continuity property of the norm for continuous fields of C*-algebras, the inequality (2.1), the fact that $x, \sum_{i=1}^{n(t)} \lambda_{i,t} \tilde{p}_{i,t} \in \Theta$ and the equalities $\tilde{p}_{i,t}(t) = p_{i,t}, 1 \leq i \leq n(t)$ imply that there is an open set U_t with $U_t \subseteq V_t$ and $t \in U_t$, such that:

$$\left\| x(s) - \sum_{i=1}^{n(t)} \lambda_{i,t} \tilde{p}_{i,t}(s) \right\| < \varepsilon, s \in U_t.$$

Since T is supposed to be totally disconnected, locally compact and Hausdorff, by [3, Chapter II, §4.4, Corollary] it follows that the closed and open (clopen) subsets of T form a base for the topology of T. Therefore, we may suppose, and we shall, that U_t is clopen. Now, since K is compact, there is a finite cover $(U_k)_{1\leq k\leq n}$ of K such that each U_k is clopen and there are $m_k \in \mathbb{N}, \lambda_{i,k} \in \mathbb{R}, p_{i,k} \in A$ such that $p_{1,k}(s), p_{2,k}(s), \ldots, p_{m_k,k}(s)$ are pairwise orthogonal projections in $A(s)$ if $s \in U_k$, $1 \leq i \leq m_k$, $1 \leq k \leq n$, such that:

$$\left\| x(s) - \sum_{i=1}^{m_k} \lambda_{i,k} p_{i,k}(s) \right\| < \varepsilon, s \in U_k, 1 \leq k \leq n. \qquad (2.2)$$

Define now $W_1 := U_1$ and for every $k \in \mathbb{N}, 2 \leq k \leq n$, $W_k := (T \backslash U_1) \cap (T \backslash U_2) \cap \ldots \cap (T \backslash U_{k-1}) \cap U_k$. Note that since the $U_k^{,}s$ are clopen in T, each W_k $(1 \leq k \leq n)$ is clopen in T, $W_k \subseteq U_k$ $(1 \leq k \leq n)$ and $(W_k)_{k=1}^n$ is a partition of $U_1 \cup U_2 \cup \ldots \cup U_n$. Observe now that for each $1 \leq k \leq n, \chi_{W_k} \in C_b(T)$, because W_k is clopen in T. Using this together with [7, Proposition 10.1.9 (ii)], it follows that:

$$\chi_{W_k} p_{i,k} \in \mathcal{P}(A),\ 1 \leq i \leq m_k, 1 \leq k \leq n$$

because $\chi_{W_k}(t) p_{i,k}(t) = p_{i,k}(t) \in \mathcal{P}(A(t))$ whenever $t \in W_k$, $1 \leq i \leq m_k$, $1 \leq k \leq n$ and $\chi_{W_k}(t) p_{i,k}(t) = 0 \in \mathcal{P}(A(t))$ whenever $t \in T \backslash W_k$, $1 \leq i \leq m_k$, $1 \leq k \leq n$. Moreover, observe that:

$$\left\{ \begin{array}{l} \chi_{W_k} p_{i,k}\ (1 \leq i \leq m_k,\ 1 \leq k \leq n) \text{ are} \\ \text{pairwise orthogonal projections in } A. \end{array} \right. \qquad (2.3)$$

Indeed, consider $(k,i) \neq (l,j)$, where $1 \leq k,l \leq n, 1 \leq i \leq m_k, 1 \leq j \leq m_l$. Then, either $k \neq l$ or $k = l$ and $i \neq j$. If $k \neq l$, then $\chi_{W_k} \chi_{W_l} = 0$ and hence $(\chi_{W_k} p_{i,k}) \cdot (\chi_{W_l} p_{j,l}) = (\chi_{W_k} \chi_{W_l}) p_{i,k} p_{j,l} = 0$; if $k = l$ and $i \neq j$, then $(\chi_{W_k} p_{i,k}) \cdot (\chi_{W_l} p_{j,l}) = \chi_{W_k} p_{i,k} \cdot \chi_{W_k} p_{j,k} = \chi_{W_k} p_{i,k} p_{j,l} = 0$, since $p_{i,k}(t) p_{j,k}(t) = 0$ for every $t \in W_k \subseteq U_k$.

To end the proof of (2) $\Rightarrow$ (1) we shall prove that:

$$\left\| x - \sum_{\substack{1\leq k\leq n \\ 1\leq i\leq m_k}} \lambda_{i,k} \left(\chi_{W_k} p_{i,k} \right) \right\| \leq \varepsilon. \qquad (2.4)$$

Indeed, since $\sum_{\substack{1\le k\le n\\ 1\le i\le m_k}} \lambda_{i,k}\left(\chi_{W_k} p_{i,k}\right)$ is a linear combination with real coefficients of pairwise orthogonal projections in A (see (2.3)), it follows that

$$\sum_{\substack{1\le k\le n\\ 1\le i\le m_k}} \lambda_{i,k}\left(\chi_{W_k} p_{i,k}\right)$$

is a selfadjoint element of A with finite spectrum. This observation together with (2.4) and [5, Theorem 2.6] will imply that (1) is true. To prove that (2.4) is true, let $t \in \cup_{k=1}^{n} W_k$. Then, there is a unique $1 \le k_0 \le n$ such that $t \in W_{k_0}$ (since $W_k \cap W_l = \emptyset$ whenever $k \ne l, 1 \le k, l \le n$). This together with the inclusion $W_{k_0} \subseteq U_{k_0}$ and (2.2) imply that:

$$\left\| x(t) - \sum_{\substack{1\le k\le n\\ 1\le i\le m_k}} \lambda_{i,k}\chi_{W_k}(t) p_{i,k}(t) \right\|$$
$$= \left\| x(t) - \sum_{1\le i\le m_{k_0}} \lambda_{i,k_0} p_{i,k_0}(t) \right\| < \varepsilon.$$

If now $t \in T\backslash\left(\cup_{k=1}^{n} W_k\right) = T\backslash\left(\cup_{k=1}^{n} U_k\right) \subseteq T\backslash K$, using the definition of K one obtains:

$$\left\| x(t) - \sum_{\substack{1\le k\le n\\ 1\le i\le m_k}} \lambda_{i,k}\chi_{W_k}(t) p_{i,k}(t) \right\|$$
$$= \|x(t)\| < \varepsilon.$$

In conclusion, (2.4) is proved and the proof of (2) $\Rightarrow$ (1) is over.

Corollary 2.3. *Let A and B be nonzero C^*-algebras. Assume that A is commutative. Then, the following are equivalent:*

(1) $\mathrm{RR}(A \otimes B) = 0$;
(2) $\mathrm{RR}(A) = \mathrm{RR}(B) = 0$;
(3) $\mathrm{Prim}(A)$ *is totally disconnected and* $\mathrm{RR}(B) = 0$.

Remark 2.4. There are examples of C^*-algebras E and F such that $\mathrm{RR}(E) = \mathrm{RR}(F) = 0$ but $\mathrm{RR}(E \otimes F) \neq 0$ (see [11], [12]). Hence, if in the above corollary we drop the condition that A is commutative, the result does not hold anymore.

Corollary 2.5. *Let A be a C^*-algebra such that* Prim(A) *is Hausdorff. Then, the following are quivalent:*

(1) $\mathrm{RR}(A) = 0$;

(2) $\mathrm{Prim}(A)$ *is totally disconnected and for every* $I \in \mathrm{Prim}(A), \mathrm{RR}(A/I) = 0$.

Proof: If $A = 0$, then obviously (1)$\Leftrightarrow$(2). Assume now that $A \neq 0$. Since $\mathrm{Prim}(A)$ is Hausdorff, the proof of [10, Theorem 2.3] implies that A is $*$-isomorphic to the C^*-algebra associated to a continuous field of C^*-algebras $A(t)$ ($t \in \mathrm{Prim}(A)$) over $\mathrm{Prim}(A)$, where $A(t) := A/t$ for every $t \in \mathrm{Prim}(A)$. Note that clearly $A(t) \neq 0$ for every $t \in \mathrm{Prim}(A)$.

Now, (1)$\Leftrightarrow$ (2) by Theorem 2.1 and the above fact. □

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