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Product of paraquaternionic Kähler manifolds

by
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Abstract

In this paper, we investigate what happens with the natural product manifold of two paraquaternionic Kähler manifolds. We also construct the curvature tensor of the product manifold of two paraquaternionic space forms.

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1 Introduction

The product of two Kählerian manifolds is also a Kähler manifold([10]), but, the natural product manifold of two quaternionic Kähler manifolds does not become a quaternionic Kähler manifold([7]). In the present paper we investigate what happens in the paraquaternionic geometry.

In section 2 we present the definition and basic properties of paraquaternionic manifolds.

In section 3 we prove that the natural product manifold of two paraquaternionic Kähler manifolds is an almost paraquaternionic non Kähler manifold.

As application of the results obtained in the previous section, in the last section we construct the curvature tensor of the product manifold of two paraquaternionic space forms.

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2 Paraquaternionic manifolds

Following Garcia Rio, Matsushita, Vasquez-Lorenzo ([2]), we first recall the definitions of almost paraquaternionic structures, almost paraquaternionic manifolds, paraquaternionic Kähler manifolds, paraquaternionic sectional curvature and paraquaternionic space forms.

Definition 2.1. *Let M be a smooth manifold. An almost paraquaternionic structure on M is a rank-3 subbundle $\sigma \subset End(TM)$ such that a local basis $\{J_1, J_2, J_3\}$ exists of sections of σ satisfying:*

$$\begin{cases} J_\alpha^2 = -\epsilon_\alpha Id, \forall \alpha = \overline{1,3} \\ J_1 J_2 = -J_2 J_1 = J_3 \end{cases} \tag{1}$$

where $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$.

Libermann ([6]) calls this structure quaternionic structure of second kind.

Definition 2.2. *Let (M, g) be a semi-Riemannian manifold and σ an almost paraquaternionic structure on M. The metric g is said to be adapted to the paraquaternionic structure σ if it satisfies:*

$$g(J_\alpha X, J_\alpha Y) = \epsilon_\alpha g(X, Y), \forall \alpha = \overline{1,3} \tag{2}$$

for all vector fields X, Y on M and any local basis $\{J_1, J_2, J_3\}$ of σ.

Moreover, (M, g, σ) is said to be an almost paraquaternionic manifold.

Definition 2.3. *An almost paraquaternionic manifold (M, g, σ) is said to be paraquaternionic Kähler manifold if the Levi-Civita connection of g satisfies the following conditions:*

$$\begin{cases} \nabla_X J_1 = -\omega_3(X) J_2 + \omega_2(X) J_3 \\ \nabla_X J_2 = -\omega_3(X) J_1 + \omega_1(X) J_3 \\ \nabla_X J_3 = \quad \omega_2(X) J_1 - \omega_1(X) J_2 \end{cases} \tag{3}$$

for any vector field X on M, $\omega_1, \omega_2, \omega_3$ being local 1-forms over the open for which $\{J_1, J_2, J_3\}$ is a local basis of σ.

Definition 2.4. *Let (M, g, σ) be a paraquaternionic Kähler manifold.*

1. *A subspace π of T_pM, $p \in M$ is called non-degenerate if the restriction of g to π is no degenerate. In particular a 2-plane π in T_pM is non-degenerate if and only if it has a basis $\{X, Y\}$ satisfying:*

$$\Delta(\pi) = g(X, X) g(Y, Y) - g(X, Y)^2 \neq 0.$$

2. *If $X \in T_pM$, $p \in M$, the 4-plane $PQ(X)$ spanned by $\{X, J_1X, J_2X, J_3X\}$ is called a paraquaternionic 4-plane.*

3. *A 2-plane in T_pM,$p \in M$, spanned by $\{X, Y\}$ is called half-paraquaternionic if $PQ(X) = PQ(Y)$.*

Note that if X is a non-lightlike vector of T_pM, PQ(X) is a non-degenerate subspace.

Definition 2.5. *Let (M, g, σ) be a paraquaternionic Kähler manifold and let π be a non-degenerate 2-plane in $T_pM, p \in M$, spanned by $\{X, Y\}$. The sectional curvature $K(\pi)$ is defined by:*

$$K(\pi) = \frac{R(X, Y, X, Y)}{\Delta(\pi)}.$$

In particular, the sectional curvature for a half-paraquaternionic plane is called paraquaternionic sectional curvature.

Definition 2.6. *A paraquaternionic Kähler manifold of constant paraquaternionic sectional curvature is called a paraquaternionic space form.*

It is clear that any paraquaternionic manifold is of dimension $4n$ and any adapted metric is necessarily of neutral signature $(2n, 2n)$.

Moreover, we have the following important fundamental properties concerning paraquaternionic manifolds ([1],[2],[5],[9]):

1. Any paraquaternionic Kähler manifold is an Einstein manifold, provided that $dim M > 4$.
2. A paraquaternionic Kähler manifold with dimension $4m > 4$ and zero scalar curvature is a locally parahyperKähler manifold.
3. Any paraquaternionic space form is a locally symmetric space.
4. The curvature tensor of a paraquaternionic space form is:

$$R(X, Y)Z = \frac{c}{4}\{g(Z, Y)X - g(X, Z)Y + \sum_{\alpha=1}^{3} \epsilon_\alpha [g(Z, J_\alpha Y)J_\alpha X - g(Z, J_\alpha X)J_\alpha Y + 2g(X, J_\alpha Y)J_\alpha Z]\} \quad (4)$$

   for all vector fields X, Y, Z on M and any local basis $\{J_1, J_2, J_3\}$ of σ.

Some recent results concerning the hypersurfaces of quaternionic and paraquaternionic manifolds we find in Mangione([8]) and Ianus([4]) .

3 Product of paraquaternionic manifolds

Let M be a smooth manifold endowed with a tensor F of type (1,1) such that:

$$F^2 = Id.$$

Then (M, F) is called to be an almost product manifold with almost product structure F.

Let (M, F) be an almost product manifold and let g be a semi-Riemannian metric on M such that:

$$g(FX, FY) = g(X, Y), \forall X, Y \in \Gamma(TM).$$

Then we say that (M, F, g) is an almost product semi-Riemannian manifold.

Let (M_1, g_1, σ_1) be a paraquaternionic Kähler manifold. Then, in any coordinate neighborhood U_1 of M_1, there exists a local bases $\{J_1^{(1)}, J_2^{(1)}, J_3^{(1)}\}$ of σ_1 satisfying (1) and (2). Moreover, the Levi-Civita connection $\nabla^{(1)}$ of g_1 satisfies the following conditions:

$$\begin{cases} \nabla_X^{(1)} J_1^{(1)} = -\omega_3^{(1)}(X)J_2^{(1)} + \omega_2^{(1)}(X)J_3^{(1)} \\ \nabla_X^{(1)} J_2^{(1)} = -\omega_3^{(1)}(X)J_1^{(1)} + \omega_1^{(1)}(X)J_3^{(1)} \\ \nabla_X^{(1)} J_3^{(1)} = \quad \omega_2^{(1)}(X)J_1^{(1)} - \omega_1^{(1)}(X)J_2^{(1)} \end{cases} \tag{5}$$

for any vector field X on M_1, $\omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)}$ being local 1-forms over the open for which $\{J_1^{(1)}, J_2^{(1)}, J_3^{(1)}\}$ is a local basis of σ_1.

Let (M_2, g_2, σ_2) be an another paraquaternionic Kähler manifold. Then, in any coordinate neighborhood U_2 of M_2, there exists a local bases $\{J_1^{(2)}, J_2^{(2)}, J_3^{(2)}\}$ of σ_2 satisfying (1) and (2). Moreover, the Levi-Civita connection $\nabla^{(2)}$ of g_2 satisfies the following conditions:

$$\begin{cases} \nabla_X^{(2)} J_1^{(2)} = -\omega_3^{(2)}(X)J_2^{(2)} + \omega_2^{(2)}(X)J_3^{(2)} \\ \nabla_X^{(2)} J_2^{(2)} = -\omega_3^{(2)}(X)J_1^{(2)} + \omega_1^{(2)}(X)J_3^{(2)} \\ \nabla_X^{(2)} J_3^{(2)} = \quad \omega_2^{(2)}(X)J_1^{(2)} - \omega_1^{(2)}(X)J_2^{(2)} \end{cases} \tag{6}$$

for any vector field X on M_2, $\omega_1^{(2)}, \omega_2^{(2)}, \omega_3^{(2)}$ being local 1-forms over the open for which $\{J_1^{(2)}, J_2^{(2)}, J_3^{(2)}\}$ is a local basis of σ_2.

Let $M = M_1 \times M_2$ be the product manifold of M_1 and M_2. We define a semi-Riemannian metric g on M by:

$$g(X, Y) = g_1(\Pi_1 X, \Pi_1 Y) + g_2(\Pi_2 X, \Pi_2 Y) \tag{7}$$

for any vector fields X and Y on M, where Π_1 and Π_2 denote the projection mappings of $\Gamma(T(M_1 \times M_2))$ to $\Gamma(T(M_1))$ and $\Gamma(T(M_2))$, that is:

$$\Pi_1 : \Gamma(TM) \to \Gamma(TM_1), \Pi_2 : \Gamma(TM) \to \Gamma(TM_2).$$

Lemma 3.1. *$(M, F = \Pi_1 - \Pi_2, g)$ is an almost product semi-Riemannian manifold.*

Proof. Because we have:

$$\Pi_1^2 = \Pi_1, \Pi_2^2 = \Pi_2, \Pi_1\Pi_2 = \Pi_2\Pi_1 = 0$$

we deduce that:

$$F^2 = (\Pi_1 - \Pi_2)(\Pi_1 - \Pi_2) = \Pi_1 - \Pi_2 = F.$$

On the other hand, for any vector fields X and Y on M we have:

$$g(FX, FY) = g_1(\Pi_1 FX, \Pi_1 FY) + g_2(\Pi_2 FX, \Pi_2 FY) =$$

$$= g_1(\Pi_1(\Pi_1 - \Pi_2)X, \Pi_1(\Pi_1 - \Pi_2)Y) + g_2(\Pi_2(\Pi_1 - \Pi_2)X, \Pi_2(\Pi_1 - \Pi_2)Y) =$$
$$= g_1(\Pi_1 X, \Pi_1 Y) + g_2(\Pi_2 X, \Pi_2 Y) = g(X, Y).$$

Thus, the proof is complete.

We define the endomorphisms of tangent space of M by:

$$J_\alpha X = J_\alpha^{(1)}\Pi_1 X + J_\alpha^{(2)}\Pi_2 X, \forall \alpha = \overline{1,3}, \tag{8}$$

for any vector field X on M, where $\{J_\alpha^{(1)}\}_{\alpha=\overline{1,3}}$ and $\{J_\alpha^{(2)}\}_{\alpha=\overline{1,3}}$ are local bases of σ_1 and σ_2 respectively.

If we consider the vector bundle σ over M generated by $\{J_\alpha\}$, then we have:

Lemma 3.2. *(M, g, σ) is an almost paraquaternionic manifold.*

Proof. For any coordinate neighborhood $U_1 \times U_2$ of M, we have the local bases $\{J_1, J_2, J_3\}$ of σ. By using (8), for any vector field X on M we derive:

$$J_\alpha^2 X = J_\alpha(J_\alpha^{(1)}\Pi_1 X + J_\alpha^{(2)}\Pi_2 X) =$$
$$= J_\alpha^{(1)}\Pi_1(J_\alpha^{(1)}\Pi_1 X + J_\alpha^{(2)}\Pi_2 X) + J_\alpha^{(2)}\Pi_2(J_\alpha^{(1)}\Pi_1 X + J_\alpha^{(2)}\Pi_2 X) =$$
$$= J_\alpha^{(1)}(J_\alpha^{(1)}\Pi_1 X) + J_\alpha^{(2)}(J_\alpha^{(2)}\Pi_2 X) =$$
$$= -\epsilon_\alpha(\Pi_1 X + \Pi_2 X) = -\epsilon_\alpha X, \forall \alpha = \overline{1,3}.$$

On the other hand, we have:

$$J_1 J_2 X = J_1(J_2^{(1)}\Pi_1 X + J_2^{(2)}\Pi_2 X) =$$
$$= J_1^{(1)}\Pi_1(J_2^{(1)}\Pi_1 X + J_2^{(2)}\Pi_2 X) + J_1^{(2)}\Pi_2(J_2^{(1)}\Pi_1 X + J_2^{(2)}\Pi_2 X) =$$
$$= J_1^{(1)}J_2^{(1)}\Pi_1 X + J_1^{(2)}J_2^{(2)}\Pi_2 X =$$
$$= J_3^{(1)}\Pi_1 X + J_3^{(2)}\Pi_2 X = J_3 X$$

and similarly:

$$J_2 J_1 X = J_2(J_1^{(1)}\Pi_1 X + J_1^{(2)}\Pi_2 X) =$$
$$= J_2^{(1)}\Pi_1(J_1^{(1)}\Pi_1 X + J_1^{(2)}\Pi_2 X) + J_2^{(2)}\Pi_2(J_1^{(1)}\Pi_1 X + J_1^{(2)}\Pi_2 X) =$$
$$= J_2^{(1)}J_1^{(1)}\Pi_1 X + J_2^{(2)}J_1^{(2)}\Pi_2 X =$$
$$= -J_3^{(1)}\Pi_1 X - J_3^{(2)}\Pi_2 X = -J_3 X.$$

Thus, we conclude that σ is an almost paraquaternionic structure on M. Moreover, the metric g is adapted to the paraquaternionic structure σ, because it satisfies:

$$g(J_\alpha X, J_\alpha Y) = g_1(\Pi_1 J_\alpha X, \Pi_1 J_\alpha Y) + g_2(\Pi_2 J_\alpha X, \Pi_2 J_\alpha Y) =$$

$$= g_1(J_\alpha^{(1)}\Pi_1 X, J_\alpha^{(1)}\Pi_1 Y) + g_2(J_\alpha^{(2)}\Pi_2 X, J_\alpha^{(2)}\Pi_2 Y) =$$
$$= \epsilon_\alpha g_1(\Pi_1 X, \Pi_1 Y) + \epsilon_\alpha g_2(\Pi_2 X, \Pi_2 Y) = \epsilon_\alpha g(X, Y).$$

Thus, we deduce that (M, g, σ) is an almost paraquaternionic manifold.

Now, for any local coordinate neighborhood $U_1 \times U_2$ of M we define local 1-form $\omega_1, \omega_2, \omega_3$ by:

$$\omega_\alpha(X) = \omega_\alpha^{(1)}(\Pi_1 X) + \omega_\alpha^{(2)}(\Pi_2 X), \forall \alpha = \overline{1,3}, \tag{9}$$

for any vector field X on M.

We remark that the Levi-Civita connection ∇ on M is:

$$\nabla_X Y = \nabla^{(1)}_{\Pi_1 X}\Pi_1 Y + \nabla^{(2)}_{\Pi_2 X}\Pi_2 Y, \tag{10}$$

for any vector fields X and Y on M.

Lemma 3.3. *The Levi-Civita connection ∇ of g satisfies the following conditions:*

$$\begin{cases} \nabla_X J_1 = \frac{1}{2}[-\omega_3(X)J_2 + \omega_2(X)J_3 - \omega_3(FX)J_2F + \omega_2(FX)J_3F] \\ \nabla_X J_2 = \frac{1}{2}[-\omega_3(X)J_1 + \omega_1(X)J_3 - \omega_3(FX)J_1F + \omega_1(FX)J_3F] \\ \nabla_X J_3 = \frac{1}{2}[\ \ \omega_2(X)J_1 - \omega_1(X)J_2 + \omega_2(FX)J_1F - \omega_1(FX)J_2F] \end{cases} \tag{11}$$

for any vector field X on M.

Proof. For any vector fields X and Y on M we have:

$$(\nabla_X J_1)Y = \nabla_X J_1 Y - J_1(\nabla_X Y) =$$
$$= \nabla^{(1)}_{\Pi_1 X}\Pi_1 J_1 Y + \nabla^{(2)}_{\Pi_2 X}\Pi_2 J_1 Y - J_1\nabla^{(1)}_{\Pi_1 X}\Pi_1 Y - J_1\nabla^{(2)}_{\Pi_2 X}\Pi_2 Y =$$
$$= \nabla^{(1)}_{\Pi_1 X}J_1^{(1)}\Pi_1 Y + \nabla^{(2)}_{\Pi_2 X}J_1^{(2)}\Pi_2 Y - J_1^{(1)}\nabla^{(1)}_{\Pi_1 X}\Pi_1 Y - J_1^{(2)}\nabla^{(2)}_{\Pi_2 X}\Pi_2 Y =$$
$$= (\nabla^{(1)}_{\Pi_1 X}J_1^{(1)})(\Pi_1 Y) + (\nabla^{(2)}_{\Pi_2 X}J_1^{(2)})(\Pi_2 Y).$$

Thus, taking account of (5) and (6), we obtain:

$$(\nabla_X J_1)Y = -\omega_3^{(1)}(\Pi_1 X)J_2^{(1)}(\Pi_1 Y) + \omega_2^{(1)}(\Pi_1 X)J_3^{(1)}(\Pi_1 Y) -$$
$$-\omega_3^{(2)}(\Pi_2 X)J_2^{(2)}(\Pi_2 Y) + \omega_2^{(2)}(\Pi_2 X)J_3^{(2)}(\Pi_2 Y). \tag{12}$$

On the other hand, by using (9) we obtain:

$$-\omega_3(X)J_2 Y + \omega_2(X)J_3 Y - \omega_3(FX)J_2(FY) + \omega_2(FX)J_3(FY) =$$
$$2\{-\omega_3^{(1)}(\Pi_1 X)J_2^{(1)}(\Pi_1 Y) + \omega_2^{(1)}(\Pi_1 X)J_3^{(1)}(\Pi_1 Y) -$$
$$-\omega_3^{(2)}(\Pi_2 X)J_2^{(2)}(\Pi_2 Y) + \omega_2^{(2)}(\Pi_2 X)J_3^{(2)}(\Pi_2 Y)\}. \tag{13}$$

From (12) and (13) we deduce:

$$(\nabla_X J_1)Y = \frac{1}{2}[-\omega_3(X)J_2Y + \omega_2(X)J_3Y - \omega_3(FX)J_2(FY) + \omega_2(FX)J_3(FY)]$$

and similarly we obtain:

$$(\nabla_X J_2)Y = \frac{1}{2}[-\omega_3(X)J_1Y + \omega_1(X)J_3Y - \omega_3(FX)J_1(FY) + \omega_1(FX)J_3(FY)]$$

and

$$(\nabla_X J_3)Y = \frac{1}{2}[\ \omega_2(X)J_1Y - \omega_1(X)J_2Y + \omega_2(FX)J_1(FY) - \omega_1(FX)J_2(FY)].$$

Thus, the proof is complete.

Finally, from above lemmas, we conclude:

Proposition 3.4. *The natural product manifold of two paraquaternionic Kähler manifolds is an almost paraquaternionic non Kähler manifold.*

4 The curvature tensor of the product manifold of two paraquaternionic space forms

Now we consider the paraquaternionic space forms $M_1(c_1)$ and $M_2(c_2)$. Then, from (4) we have:

$$R_1(X,Y)Z = \frac{c_1}{4}\{g_1(Z,Y)X - g_1(X,Z)Y + \sum_{\alpha=1}^{3}\epsilon_\alpha[g_1(Z,J_\alpha^{(1)}Y)J_\alpha^{(1)}X - \\ -g_1(Z,J_\alpha^{(1)}X)J_\alpha^{(1)}Y + 2g_1(X,J_\alpha^{(1)}Y)J_\alpha^{(1)}Z]\} \quad (14)$$

for all vector fields X, Y, Z on M_1 and any local basis $\{J_1^{(1)}, J_2^{(1)}, J_3^{(1)}\}$ of σ_1 and similarly:

$$R_2(X,Y)Z = \frac{c_2}{4}\{g_2(Z,Y)X - g_2(X,Z)Y + \sum_{\alpha=1}^{3}\epsilon_\alpha[g_2(Z,J_\alpha^{(2)}Y)J_\alpha^{(2)}X - \\ -g_2(Z,J_\alpha^{(2)}X)J_\alpha^{(2)}Y + 2g_2(X,J_\alpha^{(2)}Y)J_\alpha^{(2)}Z]\} \quad (15)$$

for all vector fields X, Y, Z on M_2 and any local basis $\{J_1^{(2)}, J_2^{(2)}, J_3^{(2)}\}$ of σ_2.

Proposition 4.1. *Let $M = M_1 \times M_2$ be the natural product manifold of two paraquaternionic space forms $M_1(c_1)$ and $M_2(c_2)$. Then the curvature tensor of*

M is given by:

$$
\begin{aligned}
& R(X,Y)Z = \\
= \ & \frac{c_1+c_2}{16}\{g(Z,Y)X - g(X,Z)Y + g(Z,FY)FX - g(X,FZ)FY + \\
+ \ & \sum_{\alpha=1}^{3}\epsilon_\alpha[g(Z,J_\alpha Y)J_\alpha X - g(Z,J_\alpha X)J_\alpha Y + 2g(X,J_\alpha Y)J_\alpha Z + \\
+ \ & g(Z,FJ_\alpha Y)FJ_\alpha X - g(Z,FJ_\alpha X)FJ_\alpha Y + 2g(X,FJ_\alpha Y)FJ_\alpha Z]\} + \\
+ \ & \frac{c_1-c_2}{16}\{g(Z,FY)X - g(X,FZ)Y + g(Z,Y)FX - g(X,Z)FY + \\
+ \ & \sum_{\alpha=1}^{3}\epsilon_\alpha[g(Z,FJ_\alpha Y)J_\alpha X - g(Z,FJ_\alpha X)J_\alpha Y + 2g(X,FJ_\alpha Y)J_\alpha Z + \\
+ \ & g(Z,J_\alpha Y)FJ_\alpha X - g(Z,J_\alpha X)FJ_\alpha Y + \\
+ \ & 2g(X,J_\alpha Y)FJ_\alpha Z]
\end{aligned}
\tag{16}
$$

for all vector fields X, Y, Z on M, where F is the almost product structure on M as in lemma 4.1 and J_α is given by (8).

Proof. A straightforward computation leads to the following expression for the curvature tensor of M:

$$
\begin{aligned}
R(X,Y)Z = \ & \frac{c_1}{4}\{g_1(\Pi_1 Z,\Pi_1 Y)\Pi_1 X - g_1(\Pi_1 X,\Pi_1 Z)\Pi_1 Y + \\
& + \sum_{\alpha=1}^{3}\epsilon_\alpha[g_1(\Pi_1 Z, J_\alpha^{(1)}\Pi_1 Y)J_\alpha^{(1)}\Pi_1 X - g_1(\Pi_1 Z, J_\alpha^{(1)}\Pi_1 X)J_\alpha^{(1)}\Pi_1 Y + \\
& + 2g_1(\Pi_1 X, J_\alpha^{(1)}\Pi_1 Y)J_\alpha^{(1)}\Pi_1 Z]\} + \frac{c_2}{4}\{g_2(\Pi_2 Z,\Pi_2 Y)\Pi_2 X - \\
& - g_2(\Pi_2 X,\Pi_2 Z)\Pi_2 Y + \sum_{\alpha=1}^{3}\epsilon_\alpha[g_2(\Pi_2 Z, J_\alpha^{(2)}\Pi_2 Y)J_\alpha^{(2)}\Pi_2 X - \\
& - g_2(\Pi_2 Z, J_\alpha^{(2)}\Pi_2 X)J_\alpha^{(2)}\Pi_2 Y + 2g_2(\Pi_2 X, J_\alpha^{(2)}\Pi_2 Y)J_\alpha^{(2)}\Pi_2 Z]\}
\end{aligned}
\tag{17}
$$

On the other hand, we obtain:

$$
\begin{aligned}
& \frac{c_1+c_2}{16}[g(Z,Y)X - g(X,Z)Y + g(Z,FY)FX - g(X,FZ)FY] + \\
& + \frac{c_1-c_2}{16}[g(Z,FY)X - g(X,FZ)Y + g(Z,Y)FX - g(X,Z)FY] = \\
& = \frac{c_1}{4}[g_1(\Pi_1 Z,\Pi_1 Y)\Pi_1 X - g_1(\Pi_1 X,\Pi_1 Z)\Pi_1 Y] + \\
& + \frac{c_2}{4}[g_2(\Pi_2 Z,\Pi_2 Y)\Pi_2 X - g_2(\Pi_2 X,\Pi_2 Z)\Pi_2 Y],
\end{aligned}
\tag{18}
$$

$$\frac{c_1+c_2}{16}\sum_{\alpha=1}^{3}\epsilon_\alpha[g(Z,J_\alpha Y)J_\alpha X+g(Z,FJ_\alpha Y)FJ_\alpha X]+$$
$$+\frac{c_1-c_2}{16}\sum_{\alpha=1}^{3}\epsilon_\alpha[g(Z,FJ_\alpha Y)J_\alpha X+g(Z,J_\alpha Y)FJ_\alpha X]=$$
$$=\frac{c_1}{4}\sum_{\alpha=1}^{3}\epsilon_\alpha g_1(\Pi_1 Z,J_\alpha^{(1)}\Pi_1 Y)J_\alpha^{(1)}\Pi_1 X+$$
$$+\frac{c_2}{4}\sum_{\alpha=1}^{3}\epsilon_\alpha g_2(\Pi_2 Z,J_\alpha^{(2)}\Pi_2 Y)J_\alpha^{(2)}\Pi_2 X, \tag{19}$$

$$\frac{c_1+c_2}{16}\sum_{\alpha=1}^{3}\epsilon_\alpha[-g(Z,J_\alpha X)J_\alpha Y-g(Z,FJ_\alpha X)FJ_\alpha Y]+$$
$$+\frac{c_1-c_2}{16}\sum_{\alpha=1}^{3}\epsilon_\alpha[-g(Z,FJ_\alpha X)J_\alpha Y-g(Z,J_\alpha X)FJ_\alpha Y]=$$
$$=-\frac{c_1}{4}\sum_{\alpha=1}^{3}\epsilon_\alpha g_1(\Pi_1 Z,J_\alpha^{(1)}\Pi_1 X)J_\alpha^{(1)}\Pi_1 Y-$$
$$-\frac{c_2}{4}\sum_{\alpha=1}^{3}\epsilon_\alpha g_2(\Pi_2 Z,J_\alpha^{(2)}\Pi_2 X)J_\alpha^{(2)}\Pi_2 Y \tag{20}$$

and

$$\frac{c_1+c_2}{16}\sum_{\alpha=1}^{3}\epsilon_\alpha[2g(X,J_\alpha Y)J_\alpha Z+2g(X,FJ_\alpha Y)FJ_\alpha Z]+$$
$$+\frac{c_1-c_2}{16}\sum_{\alpha=1}^{3}\epsilon_\alpha[2g(X,FJ_\alpha Y)J_\alpha Z+2g(X,J_\alpha Y)FJ_\alpha Z]=$$
$$=\frac{c_1}{2}\sum_{\alpha=1}^{3}\epsilon_\alpha g_1(\Pi_1 X,J_\alpha^{(1)}\Pi_1 Y)J_\alpha^{(1)}\Pi_1 Z+$$
$$+\frac{c_2}{2}\sum_{\alpha=1}^{3}\epsilon_\alpha g_2(\Pi_2 X,J_\alpha^{(2)}\Pi_2 Y)J_\alpha^{(2)}\Pi_2 Z. \tag{21}$$

Finally, taking account of (17),(18),(19),(20) and (21) the proof is complete.

Corollary 4.2. *Let $M = M_1 \times M_2$ be the natural product manifold of two paraquaternionic space forms $M_1^{4n_1}(c_1)$ and $M_2^{4n_2}(c_2)$. Then the Ricci tensor* *of M is given by:*

$$Ric(X,Y) = \frac{c_1 + c_2}{16}[(4n+16)g(X,Y) + g(FX,Y)\sum_{i=1}^{4n}\epsilon_i g(FE_i, E_i)] +$$
$$+\frac{c_1 - c_2}{16}[(4n+16)g(FX,Y) + g(X,Y)\sum_{i=1}^{4n}\epsilon_i g(FE_i, E_i)] \quad (22)$$

for any $X, Y \in \Gamma(TM)$, where $4n = dimM$, $\{E_i\}_{i=\overline{1,4n}}$ is a local pseudo-orthonormal basis for the vector fields on M and $\epsilon_i = g(E_i, E_i), \forall i = \overline{1,4n}$.

Proof. The statement follows direct from (16).

Corollary 4.3. *Let $M = M_1 \times M_2$ be the natural product manifold of two paraquaternionic space forms $M_1(c)$ and $M_2(c)$. If $dimM_1 = dimM_2$, then M is an Einstein manifold.*

Proof. Because $c_1 = c_2 = c$, from (22) we derive:

$$Ric(X,Y) = \frac{c}{8}[(4n+16)g(X,Y) + g(FX,Y)\sum_{i=1}^{4n}\epsilon_i g(FE_i, E_i)]. \quad (23)$$

On the other hand, because dim M_1=dim M_2 it follows easy that we have:

$$\sum_{i=1}^{4n}\epsilon_i g(FE_i, E_i) = 0. \quad (24)$$

By using (23) and (24) we deduce:

$$Ric(X,Y) = \frac{c(n+4)}{2}g(X,Y),$$

for any $X, Y \in \Gamma(TM)$ and hence M is an Einstein manifold.

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