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On the Henstock and McShane integrability

by
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Abstract

It is known that any Lebesgue integrable function is McShane integrable, and in fact we know only McShane's proof (*see* [7] , $p.90-94$). In the first part of this paper we provide another proof of this result, and in the second part, we shall study the Lebesgue measurability of the Henstock integrable functions.We shall show that any function which is Henstock integrable is Lebesgue measurable, and thus we shall prove in a different way from the former proof from Y.Kubota in [3] the Lebesgue measurability of the McShane integrable functions. We also present six corollaries of the others two results, some of them being already known.

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1 Definitions and Notations.

We denote by R the real line, by $I=[a,b]$ a compact interval on the line and by $|E|$ the Lebesgue measure of a set E. For the exterior measure of the set E we use the notation $|E|^*$, and the characteristic function of E will be denoted by λ_E. Let $\mathcal{I}$ be the collection of all closed intervals that are contained in I. Any collections $\mathcal{T}$ of pairs $(\Delta_k,\xi_k)\in\mathcal{I}\times I$, $k=\overline{1,n}$ is called *a partition* of the interval I, if the intervals Δ_i,Δ_j, are non-overlapping for $i\neq j$ and $\bigcup_{k=1}^{n}\Delta_k=I$.

Let $\delta:I\to(0,\infty)$ be a positive function defined on I.

A partition $\mathcal{T}$ of I is called $Henstock\ \delta-fine$, if every pair $(\Delta,\xi)\in\mathcal{T}$ satisfies

$$\xi\in\Delta\subset(\xi-\delta(\xi),\xi+\delta(\xi)).$$

A partition $\mathcal{T}$ of I is called $McShane\ \delta-fine$, if every pair $(\Delta,\xi)\in\mathcal{T}$ satisfies

$$\Delta\subset(\xi-\delta(\xi),\xi+\delta(\xi)).$$

Definition 1. *A function $f : I \to R$ is called Henstock integrable on I or $\mathcal{H}-integrable$ on I if there exists $\alpha \in R$ with the following property : for each $\varepsilon \succ 0$, there exists a positive function $\delta : I \to (0, \infty)$ such that*

$$\left| \sum_{\mathcal{T}} f(\xi_k) |\Delta_k| - \alpha \right| \prec \varepsilon,$$

whenever $\mathcal{T}$ is a Henstock δ - fine partition of I. We shall put $\alpha = (\mathcal{H}) \int_I f$ and by $\mathcal{H}(I)$ or $\mathcal{H}(a, b)$ we shall denote the class of all $\mathcal{H}-integrable$ functions on I.

Definition 2. *A function $f : I \to R$ is called McShane integrable on I or $\mathcal{M}-integrable$ on I if there exists $\beta \in R$ with the following property : for each $\varepsilon \succ 0$, there exists a positive function $\delta : I \to (0, \infty)$ such that*

$$\left| \sum_{\mathcal{T}} f(\xi_k) |\Delta_k| - \beta \right| \prec \varepsilon,$$

whenever $\mathcal{T}$ is a McShane δ - fine partition of I. We shall put $\beta = (\mathcal{M}) \int_I f$ and by $\mathcal{M}(I)$ or $\mathcal{M}(a, b)$ we shall denote the class of all $\mathcal{M}-integrable$ functions on I.

Obviously, if f is $\mathcal{M}-integrable$, then f is $\mathcal{H}-integrable$ and

$$(\mathcal{M}) \int_I f = (\mathcal{H}) \int_I f.$$

If $f : I \to R$ is Lebesgue integrable we shall denote its integral by $(\mathcal{L}) \int_I f$ and by $\mathcal{L}(I)$ or $\mathcal{L}(a, b)$ we shall denote the class of all $\mathcal{L}-integrable$ functions on I.A function $f : I \to R$ is called an *elementary function* if it is of the type

$$f = \sum_{i=1}^{m} \alpha_i \lambda_{E_i}$$

where $m \in N^*, \alpha_i \in R, \forall\ i = \overline{1, m}$, and E_i, $i = \overline{1, m}$ are Lebesgue measurable sets.

2 Preliminaries

Theorem 1. $(Theorem\ 3.3\ of\ [6])$. *Let $f_n : I \to R$ be a sequence of functions of $\mathcal{M}(I)$. We suppose for each $x \in [a, b]$*

$$f_1(x) \le f_2(x) \le f_3(x) \le ......f_n(x) \le f_{n+1}(x) \le .....$$

and $\lim_{n\to\infty} f_n(x) = f(x)$ *is finite. If the sequence* $\left((\mathcal{M})\int_I f_n\right)_n$ *is convergent to* $\alpha \in R$, *then*

$$f \in \mathcal{M}(I) \text{ and } \alpha = (\mathcal{M})\int_I f.$$

The following will be obvious.

Corollary 1. . *Let* $E \subset I$ *be a set of type* $\bigcup_{n=1}^{\infty}(a_n, b_n)$, *where* $(a_n, b_n) \bigcap (a_m, b_m) = \phi$ *for* $m \neq n$. *Then*

$$\lambda_E \in \mathcal{M}(I) \text{ and } (\mathcal{M})\int_I \lambda_E = |E|.$$

Theorem 2. (*Henstock's lemma of* [5]) . *Let* $f \in \mathcal{H}(I)$ *be a function and given* $\varepsilon \succ 0$ *let* $\delta : I \to (0,\infty)$ *be a positive function such that for each partition* $\mathcal{T}$ *Henstock* $\delta-$ *fine of* I, *we have*

$$\left|\sum_{\mathcal{T}} f(\xi_k)|\Delta_k| - (\mathcal{H})\int_I f\right| \prec \varepsilon.$$

Then for each $\mathcal{P} \subseteq \mathcal{T}$ *we have*

$$\left|\sum_{\mathcal{P}}\left(f(\xi_k)|\Delta_k| - (\mathcal{H})\int_{\Delta_k} f\right)\right| \leq \varepsilon,$$

$$\sum_{\mathcal{P}}\left|f(\xi_k)|\Delta_k| - (\mathcal{H})\int_{\Delta_k} f\right| \leq 2\varepsilon.$$

It is shown without difficulty

Theorem 3. . *Let* $(f_n)_{n\geq 1} \subset \mathcal{M}(I)$ *be a sequence of real functions with the following property:* $\lim_{n\to\infty} f_n(x) = f(x)$ *uniformly on* I. *Then*

$$f \in \mathcal{M}(I) \text{ and } (\mathcal{M})\int_I f = \lim_{n\to\infty}(\mathcal{M})\int_I f_n.$$

Lemma 1. . *Let* $E \subset I$ *be a Lebesgue measurable set having* $|E| = 0$. *Then*

$$\lambda_E \in \mathcal{M}(I) \text{ and } (\mathcal{M})\int_I \lambda_E = 0.$$

Proof: Let $\varepsilon \succ 0$ be a positive number.There exists a countable set J and a family of disjoint open intervals $(a_j, b_j)_{j \in J}$ such that

$$E \subset \bigcup_{j \in J} (a_j, b_j), \quad \sum_{j \in J} (b_j - a_j) \prec \varepsilon.$$

Now, we define $\delta : I \to (0, \infty)$ as follows: for $x \in E$, there exists a unique $j \in J$ such that $x \in (a_j, b_j)$ and we shall put $\delta(x) = \min(x - a_j, b_j - x)$, and when $x \notin E$ we take $\delta(x) = 1$. For a partition $\mathcal{T} = \left\{ (\Delta_k, \xi_k) \in \mathcal{I} \times I, k = \overline{1, n} \right\}$ of I, McShane $\delta-$ fine, we have $\Delta_k \subset (\xi_k - \delta(\xi_k), \xi_k + \delta(\xi_k))$. Then

$$\sum_{\mathcal{T}} \lambda_E(\xi_k) |\Delta_k| = \sum_{\xi_k \in E} |\Delta_k|.$$

But $|\Delta_k| \prec 2\delta(\xi_k) = 2\min(\xi_k - a_j, b_j - \xi_k)$ for $\xi_k \in (a_j, b_j)$. Since

$$\min(\xi_k - a_j, b_j - \xi_k) \leq \frac{b_j - a_j}{2}$$

we have

$$|\Delta_k| \prec b_j - a_j.$$

Then

$$\sum_{\xi_k \in E} |\Delta_k| \prec \sum_{j \in J} (b_j - a_j) \prec \varepsilon$$

and the statement is proved. □

Lemma 2. . *Let $E \subset I$ be a Lebesgue measurable set. Then*

$$\lambda_E \in \mathcal{M}(I) \ \ and \ \ (\mathcal{M}) \int_I \lambda_E = |E|.$$

Proof: For each $n \in N^*$ there exists $E_n = \bigcup_{k=1}^{\infty} (a_k^n, b_k^n)$, $(a_i^n, b_i^n) \bigcap (a_j^n, b_j^n) = \phi$ for $i \neq j$, such that

$$E \subseteq E_n \ and \ |E_n| \prec |E| + \frac{1}{n}. \tag{1}$$

Without restricting the generality we can suppose $E_n \supseteq E_{n+1}$, for all $n \geq 1$. Let $A = \bigcap_{n=1}^{\infty} E_n$. Obviously $E \subseteq A$ and

$$\forall \ x \in I, \ \lambda_1(x) \geq \lambda_2(x) \geq \lambda_3(x) \geq \ldots.\lambda_n(x) \geq \lambda_{n+1}(x) \geq \ldots.\lambda_A(x)$$

where $\lambda_n(x) = \lambda_{E_n}(x)$, $x \in I$. Then $u_n(x) = \lambda_1(x) - \lambda_n(x)$, $n \geq 1$, $x \in I$ satisfies

$$u_1(x) \leq u_2(x) \leq u_3(x) \leq \ldots..u_n(x) \leq u_{n+1}(x) \leq \ldots..$$

and

$$\lim_{n\to\infty} u_n(x) = \lambda_1(x) - \lambda_A(x).$$

According to corollary 1 $(u_n)_{n\geq 1} \subset \mathcal{M}(I)$ and satisfies the conditions of the theorem 1, therefore $\lambda_1 - \lambda_A \in \mathcal{M}(I)$. Then $\lambda_A \in \mathcal{M}(I)$ and in addition

$$(\mathcal{M})\int_I \lambda_A = \lim_{n\to\infty} (\mathcal{M})\int_I \lambda_n = \lim_{n\to\infty} |E_n|. \tag{2}$$

But from (1) and $E \subset A$ we get

$$\lim_{n\to\infty} |E_n| = |E| = |A|. \tag{3}$$

Then $|A - E| = 0$ and according to lemma 1,

$$\lambda_{A-E} \in \mathcal{M}(I) \text{ and } (\mathcal{M})\int_I \lambda_{A-E} = 0.$$

From $\lambda_E = \lambda_A - \lambda_{A-E}$ we get $\lambda_E \in \mathcal{M}(I)$. Then from (2) and (3)

$$(\mathcal{M})\int_I \lambda_E = |E|.$$

Thus the proof is finished. □

Corollary 2. . *Let $f : I \to R$ be a bounded function of $\mathcal{L}(I)$. Then*

$$f \in \mathcal{M}(I) \text{ and } (\mathcal{M})\int_I f = (\mathcal{L})\int_I f.$$

Proof: According to lemma 2 any elementary function belongs to $\mathcal{M}(I)$.But any bounded application of $\mathcal{L}(I)$ is the uniform limit of a sequence of elementary functions and the statement is deduced immediately from theorem 3. □

Corollary 3. . *Let $f : I \to [0,\infty)$, $f \in \mathcal{L}(I)$. Then*

$$f \in \mathcal{M}(I) \text{ and } (\mathcal{M})\int_I f = (\mathcal{L})\int_I f.$$

Proof: Because f is the simple limit of an increasing sequence of bounded applications of $\mathcal{L}(I)$ then the corollary 2 and theorem 1 prove the assertion. □

3 Main results.

Theorem 4. . *(see* [7], *p.90 − 94). Let $f : I \to R$ such that $f \in \mathcal{L}(I)$. Then*

$$f \in \mathcal{M}(I) \ \ and \ \ (\mathcal{M}) \int_I f = (\mathcal{L}) \int_I f.$$

Proof: From $f \in \mathcal{L}(I)$ we have $f^+, \ f^- \in \mathcal{L}(I)$ and consequently from Corollary 3,

$$f^+, \ f^- \in \mathcal{M}(I), \ \ (\mathcal{M}) \int_I f^+ = (\mathcal{L}) \int_I f^+, \ \ (\mathcal{M}) \int_I f^- = (\mathcal{L}) \int_I f^-.$$

Therefore the fact is proven. □

Theorem 5. . *If $f \in \mathcal{H}(I)$, then the function $F : I \to R$, $F(x) = (\mathcal{H}) \int_a^x f(t)\, dt$, $a \prec x \leq b$, $F(a) = 0$ is continuous and $F'(x) = f(x)$ a.e.*

Proof: (i). First we shall show that F is continuous on $[a, b]$ .Given $\varepsilon \succ 0$, there is a positive function $\delta(x)$ such that for any partition $\mathcal{T} = \left\{(\Delta_k, \xi_k) \in \mathcal{I} \times I, k = \overline{1, n}\right\}$ of I, Henstock $\delta-$fine, we have

$$\left| \sum_{\mathcal{T}} f(\xi_k) |\Delta_k| - (\mathcal{H}) \int_I f \right| \prec \varepsilon.$$

Let $a \leq x_0 \prec b$ and $0 \prec x - x_0 \prec \delta(x_0)$. Since $([x_0, x], \ x_0)$ is $\delta(x_0)$ −fine, by theorem 2 we have

$$\left| f(x_0)(x - x_0) - (\mathcal{H}) \int_{x_0}^{x} f(t)\, dt \right| \leq \varepsilon.$$

If we put $\delta'(x_0) = \min(\varepsilon, \delta(x_0))$ we have

$$|F(x) - F(x_0)| \leq \varepsilon + |f(x_0)|(x - x_0) \prec \varepsilon(1 + |f(x_0)|) \ \ for \ 0 \prec x - x_0 \prec \delta'(x_0).$$

This implies the right continuity of F at x_0. Similarly, we can show that F is left continuous at x_0 for $a \prec x_0 \leq b$.

(ii). Now we shall prove that $F'(x) = f(x)$ *a.e.*

Let

$$A = \left\{ x \in (a, b) \ \ : \ \limsup_{z \to x} \left| \frac{F(z) - F(x)}{z - x} \ - \ f(x) \right| \succ 0 \right\}$$

and for every $n \geq 1$ we put

$$A_n = \left\{ x \in (a,b) \ : \ \limsup_{z \to x} \left| \frac{F(z) - F(x)}{z - x} \ - \ f(x) \right| \succ \frac{1}{n} \right\}.$$

Obviously

$$A = \bigcup_{n=1}^{\infty} A_n \ .$$

We fix $n \geq 1$. We shall prove $|A_n|^* = 0$.

Indeed, given $\varepsilon \succ 0$ there exists a positive function $\delta : I \to (0, \infty)$ such that for every partition $\mathcal{T} = \left\{ (\Delta_k, \xi_k) \in \mathcal{I} \times I, k = \overline{1,m} \right\}$ of I, Henstock $\delta-$fine, we have

$$\left| \sum_{\mathcal{T}} f(\xi_k) |\Delta_k| - (\mathcal{H}) \int_I f \right| \prec \frac{\varepsilon}{8n}. \tag{1}$$

Let $x \in A_n$. For all $p \in N^*$ there exists $x_p \in (a,b)$ such that

$$|F(x_p) - F(x) - f(x)(x_p - x)| \succ \frac{1}{n} |x_p - x| \tag{2}$$

$$|x_p - x| \prec \min \left(\delta(x) \, , \ \frac{1}{p} \right) \tag{3}$$

If we denote by $x_p^{'}$ the symetric of x_p relative to x, we get $x_p^{'} \in (a,b)$.

Now let $U(x)$ be the family of all closed intervals with end points $x_p, x_p^{'}, p \in \ N^*$.

Then $U(A_n) = \bigcup_{x \in A_n} U(x)$ is a Vitali cover of A_n. Using the Vitali covering lemma there exists a finite family $I_1, I_2, I_3, ... I_r$ of disjoint elements of $U(A_n)$ such that

$$|A_n|^* \prec \sum_{k=1}^{r} |I_k| + \frac{\varepsilon}{2}. \tag{4}$$

If x_k is the middle of I_k , $k = \overline{1,r}$, then x_k divides I_k in two intervals I_k^1, I_k^2 with the same lenght.According to (2) at least one of the two intervals I_k^1, I_k^2 denoted by Δ_k and with the end points x_k, y_k satisfies

$$|F(\Delta_k) - f(x_k)|\Delta_k|| \succ \frac{1}{n} |\Delta_k| \tag{5}$$

where by $F(\Delta_k)$ we denote $F(x_k) - F(y_k)$ if $\Delta_k = [y_k, x_k]$ or $F(y_k) - F(x_k)$ if $\Delta_k = [x_k, y_k]$. Since

$$|\Delta_k| = \frac{|I_k|}{2}$$

we get from (5)

$$|F(\Delta_k) - f(x_k)|\Delta_k|| \succ \frac{1}{2n} |I_k| . \tag{6}$$

Then from (1), (4), (6) and theorem 2 we get

$$|A_n|^* \prec \varepsilon.$$

Because $\varepsilon \succ 0$ was chosen arbitrarily we get

$$|A_n|^* = 0.$$

From

$$A = \bigcup_{n=1}^{\infty} A_n$$

we get immediately the proof. □

Corollary 4. . *A function $f : I \to R$ which is $\mathcal{H}$ -integrable on I is necessarily Lebesgue measurable.*

Corollary 5. . (Pr*oposition* 5 *of* [6]) . *If $f \in \mathcal{M}(I)$, then its indefinite integral*

$$F(x) = (\mathcal{M}) \int_a^x f(t)\, dt, \;\; a \prec x \leq b, \;\; F(a) = 0$$

is continuous and $F'(x) = f(x)$ a.e.

The proof is immediate according to theorem 5 and the fact that $\mathcal{M}$ -integrability implies $\mathcal{H}$ -integrability and the equality

$$(\mathcal{M}) \int_a^x f(t)\, dt = (\mathcal{H}) \int_a^x f(t)\, dt, \;\; \forall\, x \in I.$$

Corollary 6. . (*Corollary* 3 *of* [3]) . *A function which is $\mathcal{M}$ -integrable on I is necessarily Lebesgue measurable.*

Theorem 6. *If $f : I \to [0, \infty)$ is $\mathcal{H}$ -integrable on I, then $f \in \mathcal{L}(I)$ (and $f \in \mathcal{M}(I)$) and*

$$(\mathcal{L}) \int_I f = (\mathcal{H}) \int_I f.$$

Proof: According to corollary 4, $f : I \to [0, \infty)$ is necessarily Lebesgue measurable. Then, there exists an increasing sequence of functions $(f_n)_{n \geq 1}$ such that $\forall$ $n \in N^*$ f_n is bounded and

$$\lim_{n \to \infty} f_n(x) = f(x), \;\; \forall\, x \in I.$$

Then for all $\forall\ n \in N^*\ f_n \in \mathcal{L}(I)$ and by corollary 3 $\forall\ n \in N^*\ f_n \in \mathcal{M}(I)$. In addition we have

$$(\mathcal{M}) \int_I f_n = (\mathcal{L}) \int_I f_n.$$

Thus, the theorem 1 and a classical result of Lebesgue integrability finishes the proof. □

Theorem 7. . *Let $f : I \to R$. The following assertions are equivalent:*
i) f is Lebesgue integrable on I
ii) f is Lebesque measurable and $|f|$ is $\mathcal{H}$ -integrable on I.

Proof: Let $f : I \to R$ be a Lebesgue integrable.Then $|f| : I \to [0, \infty)$ is Lebesgue integrable and according to corollary 3, $|f| \in \mathcal{M}(I) \subset \mathcal{H}(I)$.

Reciprocally if $|f| : I \to [0, \infty)$ is $\mathcal{H}$ -integrable, then by theorem 6, $|f| \in \mathcal{L}(I)$ so $f \in \mathcal{L}(I)$. □

Theorem 8. (*Theorem* 2 *of* [3] *or Theorem* 1 *of* [8]). *Let $f : I \to R$. The following assertions are equivalent:*
i) f is Lebesgue integrable on I
ii) f is McShane integrable on I.

Proof: Let $f : I \to R$, $\mathcal{M}$ -integrable.Then f is Lebesgue measurable and $|f|$ $\mathcal{M}$ -integrable (*corollary* 1 *of* [3]). According to theorem 7, the statement is closed. □

As a conclusion we can say that any McShane integrable function is Lebesgue integrable and a function is Lebesgue integrable if and only if it is Lebesgue measurable and $|f|$ is Henstock integrable.

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