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The maximum principle for discrete delay inclusions with end point constraints

by
Aurelian Cernea and Carmina Georgescu

Abstract

We study an optimization problem given by a discrete delay inclusion with end point constraints. Necessary optimality conditions in the form of maximum principle for this problem are obtained.

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1 Introduction

Consider the problem

$$\text{minimize} \quad g(x_N) \qquad (1.1)$$

over the solutions of the discrete inclusion

$$x(t+1) \in F_t(x(t), x(t-k)),\ t = 0, 1, ..., N-1, \quad x(l) = x_0(l),\ l = -k, ..., 0, \qquad (1.2)$$

with end point constraints of the form

$$x(N) \in X_N, \qquad (1.3)$$

where $k, N \in \mathbb{N}$, $F_t(.,.) : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, $t = 0, ..., N-1$, $x(.) = (x(-k), ..., x(0), ..., x(N)) \in \mathbb{R}^{(N+k+1)n}$, $X_N \subset \mathbb{R}^n$, $x_0(l) \in \mathbb{R}^n$, $l = -k, ..., 0$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given.

The aim of this paper is to obtain necessary optimality conditions for a solution $\overline{x}(.) = (\overline{x}(-k), ..., \overline{x}(0), ..., \overline{x}(N))$ to the problem (1.1)-(1.3) in terms of the variational inclusion associated to the problem (1.2) and in terms of the cone of interior directions (Dubovitskij-Miljutin tangent cone) to the set X_N at $x(N)$.

Optimal control problems for systems described by discrete inclusions have been studied by many authors ([2], [5], [8], [10], [11], [12] etc.). In the framework of discrete delay inclusions sufficient conditions for local controllability are obtained in [8], [4] and necessary optimality conditions are obtained in [4] using the notion of derived cone.

The general idea of our approach is to use a result of nonsmooth analysis which states that the intersection of the quasitangent (intermediate) cone to a given set with the cone of interior directions to another given set at a common point is contained in the quasitangent cone of the intersection of the two sets. This idea has been, already used in [6], [3] and [5] to obtain necessary optimality conditions for optimal control problems given by differential inclusions, hyperbolic differential inclusions and discrete inclusions. Finally, a last step uses the Minchenko and Sirotko duality results in [8], that characterize the positive dual of the solution set of the variational inclusion associated to (1.2) in terms of the adjoint inclusion.

The paper is organized as follows: in Section 2 we present the notations and definitions to be used in the sequel while in Section 3 we present our main results.

2 Preliminaries

In what follows we are concerned with the discrete delay inclusion

$$x(t+1) \in F_t(x(t), x(t-k)), \quad x(l) = x_0(l), \qquad (2.1)$$

where $t = 0, 1, ..., N-1$, $l = -k, ..., 0$, and k, N are positive integers,

$$F_t(.,.) : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$$

has nonempty closed convex values, $t = 0, 1, ..., N-1$ and $x_0(l) \in \mathbb{R}^n$, $l = -k, ..., -1, 0$.

A sequence $x(-k), ..., x(N)$ that satisfies inclusion (2.1) is called a *trajectory* of the inclusion (2.1) and it is denoted by $x(.) = (x(-k), ..., x(N))$.

Denote by S_F the solution set of inclusion (2.1), i.e.

$$S_F := \{x(.) = (x(-k), ..., x(N)); \quad x(.) \text{ is a trajectory of } (2.1)\}.$$

and by $R_F^N := \{x_N; \quad x \in S_F\}$ the reachable set of inclusion (2.1).

We consider $\overline{x}(.) = (\overline{x}(-k), ..., \overline{x}(N)) \in S_F$ a trajectory of (2.1).

Since the reachable set R_F^N is, generally, neither a differentiable manifold, nor a convex set, its infinitesimal properties may be characterized only by tangent cones in a generalized sense, extending the classical concepts of tangent cones in Differential Geometry and Convex Analysis, respectively.

From the multitude of the intrinsic tangent cones in the literature(e.g. [1]), the contingent, the quasitangent and Clarke's tangent cones, defined, respectively, by

$$K_xX = \{v \in \mathbb{R}^n; \quad \exists\, s_m \to 0+,\ x_m \in X : \tfrac{x_m - x}{s_m} \to v\}$$
$$Q_xX = \{v \in \mathbb{R}^n; \quad \exists\, c(.) : [0, s_0) \to X,\ c(0) = x,\ c'(0) = v\}$$
$$C_xX = \{v \in \mathbb{R}^n; \quad \forall\, (x_m, s_m) \to (x, 0+),\ x_m \in X,\ \exists y_m \in X : \tfrac{y_m - x_m}{s_m} \to v\}$$

seem to be among the most oftenly used in the study of different problems involving nonsmooth sets and mappings.

We recall that, in contrast with K_xX, Q_xX, the cone C_xX is convex and one has $C_xX \subset Q_xX \subset K_xX$.

Another important tangent cone is the *cone of interior directions* (Dubovitskij-Miljutin tangent cone) defined by

$$I_xX := \{v \in \mathbb{R}^n; \quad \exists\, s_0,\, r > 0 : \, x + sB(v,r) \subset X \quad \forall\, s \in [0, s_0)\},$$

$$B(v,r) := \{w \in \mathbb{R}^n; \quad ||w - v|| < r\}, \quad \overline{B}(v,r) := \mathrm{cl}B(v,r)$$

From the properties of the quasitangent cones we recall only the following (e.g. [1]):

$$Q_xX_1 \cap I_xX_2 \subset Q_x(X_1 \cap X_2). \tag{2.2}$$

We recall that two cones $C_1, C_2 \subset \mathbb{R}^n$ are said to be *separable* if there exists $q \in \mathbb{R}^n \backslash \{0\}$ such that:

$$< q, v > \leq 0 \leq < q, w > \quad \forall v \in C_1, w \in C_2.$$

We denote by C^+ the positive dual cone of $C \subset \mathbb{R}^n$

$$C^+ = \{q \in \mathbb{R}^n; \quad < q, v > \geq 0, \quad \forall\, v \in C\}.$$

The negative dual cone of $C \subset \mathbb{R}^n$ is $C^- = -C^+$.

For a mapping $g(.) : X \subset \mathbb{R}^n \to \mathbb{R}$ which is not differentiable, the classical (Fréchet) derivative is replaced by some generalized directional derivatives. We recall only the upper right-contingent derivative, defined by:

$$\overline{D}_K g(x; v) = \limsup_{(\theta, w) \to (0+, v)} \frac{g(x + \theta w) - g(x)}{\theta}, \; v \in K_xX$$

and in the case when $g(.)$ is locally-Lipschitz at $x \in int(X)$ by Clarke's generalized directional derivative, defined by:

$$D_C g(x; v) = \limsup_{(y, \theta) \to (x, 0+)} \frac{g(y + \theta v) - g(y)}{\theta}, \; v \in \mathbb{R}^n.$$

The results in the next sections will be expresed, in the case where $g(.)$ is locally-Lipschitz at x, in terms of the Clarke generalized gradient, defined by:

$$\partial_C g(x) = \{q \in \mathbb{R}^n; \quad < q, v > \leq \underline{D}_C g(x; v) \quad \forall\, v \in \mathbb{R}^n\}.$$

Corresponding to each type of tangent cone, say τ_xX one may introduce (e.g. [1]) a *set-valued directional derivative* of a multifunction $G(.) : X \subset \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ (in particular of a single-valued mapping) at a point $(x, y) \in \mathrm{Graph}(G)$ as follows

$$\tau_y G(x; v) = \{w \in \mathbb{R}^n; \quad (v, w) \in \tau_{(x,y)}\mathrm{Graph}(G)\}, \quad v \in \tau_xX.$$

We recall that a set-valued map, $A(.) : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be a *convex* (respectively, closed convex) *process* if $\text{Graph}(A(.)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is a convex (respectively, closed convex) cone.

For the basic properties of convex processes we refer to [1], but we shall use here only the above definition.

In what follows, we shall assume the following hypothesis.

Hipotheses 2.1 *i) The values of $F_t(.,.)$ are nonempty compact convex $\forall t \in \{0, ..., N-1\}$.*

ii) There exists $l(t) > 0$ such that $F_t(.,.)$ is Lipschitz with the Lipschitz constant $l(t)$, $\forall t \in \{0, ..., N-1\}$.

iii) There exists $A_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, $t = 0, 1, ..., N-1$ a family of closed convex processes such that

$$A_t(u, v) \subset Q_{\overline{x}(t+1)} F_t((\overline{x}(t), \overline{x}(t-k)); (u, v)) \quad \forall (u, v) \in \mathbb{R}^n \times \mathbb{R}^n,$$

$\forall t \in \{0, 1, ..., N-1\}$.

To the problem (2.1) we associate the linearized problem

$$w(t+1) \in A_t(w(t), w(t-k)), \quad w(l) = 0, \ , \quad (2.3)$$

where $t = 0, 1, ..., N-1$ and $l = -k, ..., 0$.

Denote by S_A the solution set of inclusion (2.3) and by R_A^N the reachable set of inclusion (2.3).

We recall that if $A : \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^n)$ is a set-valued map then the adjoint of A is the multifunction $A^* : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^m)$ defined by

$$A^*(p) = \{q \in \mathbb{R}^n; \ < q, v > \leq < p, v' > \quad \forall (v, v') \in graph A(.)\}.$$

The next lemma, due to Minchenko and Sirotko, characterizes the positive dual of the reachable set R_A^N of the problem (2.3).

Lemma 2.2 ([8]) *Assume that Hipotheses 2.1 are satisfied.*
Then, one has

$$\begin{aligned} & (R_A^N)^+ = \\ = \ & \{\eta \in \mathbb{R}^n; \ \exists \quad q(t), p(t) \quad t = 0, ..., N \quad \text{such that} \\ & \quad \eta = p(N), \quad (p(t), q(t)) \in A_t^*(p(t+1)) + (q(t+k), 0), \ \ t = N-1, ..., 0 \\ & \text{and} \quad q(t) = 0 \text{ for } t \geq N\}. \end{aligned}$$

3 The main result

We prove first an approximation of the reachable set R_F^N at $\overline{x}_N$.

Theorem 3.1 *Assume that Hipotheses 2.1 are satisfied. Then*

$$R_A^N \subset Q_{\overline{x}(N)} R_F^N.$$

Proof. Let $w \in R_A^N$ and $s_m \to 0+$. It follows that there exists $w(.) = (w(-k), ..., w(N))$ solution to (2.3) such that $w = w(N)$.

In particular, $w(-k) = ... = w(0) = 0$. On the other hand, $w(1) \in A_0(0,0) \in Q_{\overline{x}(1)} F_0((\overline{x}(0), \overline{x}(-k)); (0,0))$ and by the definition of the quasitangent derivative of F_0 we have that there exist $(\tilde{w}_1^m, \tilde{w}_0^m, \tilde{w}_{-k}^m) \to (w(1), 0, 0)$ such that

$$\overline{x}(1) + s_m \tilde{w}_1^m \in F_0(\overline{x}(0) + s_m \tilde{w}_0^m, \overline{x}(-k) + s_m \tilde{w}_{-k}^m) \quad \forall m \in \mathbb{N}.$$

Using the lipschitzianity of the set-valued map $F_0(.,.)$ one may write

$$\overline{x}(1) + s_m \tilde{w}_1^m \in F_0(\overline{x}(0), \overline{x}(-k)) + s_m l(0)(||\tilde{w}_0^m|| + ||\tilde{w}_{-k}^m||)B(0,1) \quad \forall m \in \mathbb{N}.$$

Thus, there exists $b_m^1 \in B(0,1)$ such that

$$\overline{x}(1) + s_m[\tilde{w}_1^m - l(0)(||\tilde{w}_0^m|| + ||\tilde{w}_{-k}^m||)b_m^1] \in F_1(\overline{x}_0 + s_k w_0^k) \quad \forall k \in \mathbb{N}$$

and if we define $w_1^m := \tilde{w}_1^m - l(0)(||\tilde{w}_0^m|| + ||\tilde{w}_{-k}^m||)b_m^1$ we have $w_1^m \to w_1$ and

$$\overline{x}(1) + s_m w_1^m \in F_0(\overline{x}(0), \overline{x}(-k)) \quad \forall k \in \mathbb{N}.$$

By repeating this construction for $t = 1, ..., N-1$ we find that there exists $w_t^k \in \mathbb{R}^n$ such that $w_t^k \to w(t)$ $t = 1, ..., N-1$ and

$$\overline{x}(t) + s_m w_{t+1}^m \in F_t(\overline{x}(t) + s_m w_t^m, \overline{x}(t-k) + s_m w_{t-k}^m) \quad \forall m \in \mathbb{N}, t = 0, 1, ..., N-1.$$

In particular, for $s_m \to 0+$ there exists $w_N^m \to w(N)$ such that $\overline{x}(N) + s_m w_N^m \in R_F^N$, i.e. $w = w(N) \in Q_{\overline{x}(N)} R_F^N$ and the proof is complete. □

Consider now the problem (1.1)-(1.3). The main result of this paper is the following theorem.

Theorem 3.2 *Let $X_N \subset \mathbb{R}^n$ be a closed set, let $\overline{x}(.) = (\overline{x}(-k), ..., \overline{x}(N)) \in S_F$ be an optimal solution for problem (1.1)-(1.3) such that Hipotheses 2.1 are satisfied and let $g(.) : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function.*

Then for any convex cone $C_1 \subset I_{\overline{x}(N)} X_N$ there exist $\lambda \in \{0,1\}$ and

$$(p(0), p(1), ..., p(N)) \in \mathbb{R}^{(N+1)n}, \ (q(0), q(1), ..., q(N)) \in \mathbb{R}^{(N+1)n},$$

such that

$$(p(t) - q(t+k), q(t)) \in A_t^*(p(t+1)), \ , \quad q(t) = 0 \qquad (3.1)$$

for $t \geq N$, $t = N-1, ..., 0$.

$$p(N) \in \lambda \partial_C g(\overline{x}(N)) - C_1^+, \qquad (3.2)$$

$$< -p(t+1), \overline{x}(t+1) >= \max\{< -p(t+1), v >; \quad v \in F_t(\overline{x}(t), \overline{x}(t-k))\}, \quad (3.3)$$

$t = 0, ..., N-1$,

$$\lambda + ||p(0)|| + ... + ||p(N)|| > 0. \quad (3.4)$$

Proof. We have $g(\overline{x}(N)) = \min\{g(x) : x \in X_N \cap R_F^N\}$ and from definitions it follows

$$D_C g(\overline{x}(N); v) \geq \overline{D}_K g(\overline{x}(N); v) \geq 0 \quad \forall v \in K_{\overline{x}(N)}(X_N \cap R_F^N). \quad (3.5)$$

For all $t = 0, ..., N-1$ and $x, y \in \mathbb{R}^n$ we define

$$\tilde{A}_t(x, y) = cl(A_t(x, y) + \bigcup_{h>0} \frac{1}{h}(F_t(\overline{x}(t), \overline{x}(t-k)) - \overline{x}(t+1))).$$

Then, by Proposition 3.5 in [7], $\{\tilde{A}_t(.,.)\}_{t=0,...,N-1}$ is a family of closed convex processes satisfying Hipotheses 2.1, $A_t \subset \tilde{A}_t$ and, moreover

$$\tilde{A}_t^*(p) = \quad (3.6)$$

$$\begin{cases} A_t^*(p) & \text{if} \quad < -p, \overline{x}(t+1) >= \max\{< -p, v >; v \in F_t(\overline{x}(t), \overline{x}(t-k))\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

According to Theorem 3.1 one has

$$R_{\tilde{A}}^N \subset Q_{\overline{x}(N)} R_F^N. \quad (3.7)$$

From (2.2) and (3.7) we obtain

$$R_{\tilde{A}}^N \cap I_{\overline{x}(N)} X_N \subset Q_{\overline{x}(N)}(X_N \cap R_F^N).$$

So, from (3.5) one has

$$D_C g(\overline{x}(N); v) \geq 0 \quad \forall v \in R_{\tilde{A}}^N \cap C_1. \quad (3.8)$$

Obviously, since $\tilde{A}_t, t = 0, ..., N-1$ are convex processes it follows that $R_{\tilde{A}}^N$ is a convex cone. We have two cases.

In the case when $R_{\tilde{A}}^N$ and C_1 are separable, there exists $\eta \in \mathbb{R}^n \backslash \{0\}$ such that

$$< \eta, v > \leq 0 \leq < \eta, w > \quad \forall v \in C_1, w \in R_{\tilde{A}}^N,$$

hence $\eta \in -C_1^+$ and $\eta \in (R_{\tilde{A}}^N)^+$. According to Lemma 2.2 and (3.6) there exist $(p(0), p(1), ..., p(N)) \in \mathbb{R}^{(N+1)n}$ and $(q(0), q(1), ..., q(N)) \in \mathbb{R}^{(N+1)n}$ such that (3.1) and (3.3) hold true. Therefore, if we take $\lambda = 0$ then (3.1)-(3.4) are verified.

In the case when $R_{\tilde{A}}^N$ and C_1 are not separable we have $(R_{\tilde{A}}^N \cap C_1)^+ = (R_{\tilde{A}}^N)^+ + C_1^+$.

From a simple separation result (e.g. Lemma 5.1 in [9]), from the definition of the Clarke's generalized gradient and from (3.6) we obtain the existence of

$\eta \in \partial_C g(\overline{x}_N)) \cap ((R^N_{\tilde{A}})^+ + C_1^+)$. Hence there exist $\eta_1 \in (R^N_{\tilde{A}})^+, \eta_2 \in C_1^+$ such that $\eta = \eta_1 + \eta_2$. As in the first case, using Lemma 2.2 we deduce the existence of $(p(0), p(1), ..., p(N)) \in \mathbb{R}^{(N+1)n}$ and $(q(0), q(1), ..., q(N)) \in \mathbb{R}^{(N+1)n}$ such that (3.1) holds true. As in the first case, from (3.6) we obtain (3.3). We take in this case $\lambda = 1$ and (3.4) is also verified. □

Remark 4.4 If in Theorem 3.2 $k = 0$ (i.e, no delay is present) then Theorem 3.2 yields Theorem 3.2 in [5].

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Faculty of Mathematics and Informatics,
University of Bucharest,
Academiei 14,
010014 Bucharest, Romania.
E-mail: acernea@math.math.unibuc.ro

Department of Applied Mathematics,
Spl. Independenţei 31,
Bucharest, Romania.