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On a property of conics

by
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To Professor Ion D. Ion on the occasion of his 70th Birthday

Abstract

In the present paper we explain a problem, using some general facts about linear systems of conics.

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1 Introduction

The starting point of this paper is the following well known problem (see e.g. [LN], p.20):

Problem 1.1. *Let* $f,\ g : \mathbf{R} \to \mathbb{R}$,

$$f(x) = 2x^2 + 2,\ g(x) = x^2 + 2x + 1.$$

If h is another polynomial function of degree 2, such that

$$g(x) \leq h(x) \leq f(x),\ \textit{for all}\ x \in \mathbb{R},$$

then there exists λ, $0 \leq \lambda \leq 1$ *such that*

$$h = \lambda f + (1 - \lambda) g.$$

This problem can be solved, for example, using inequalities, as in the following.

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Proof: Let $h(x) = ax^2 + bx + c$; then

$$(a-1)x^2 + (b-2)x + (c-1) \geq 0, \ \textit{for all } x \in \mathbb{R}$$

and

$$(2-a)x^2 - bx + (2-c) \geq 0, \ \textit{for all } x \in \mathbb{R}.$$

Therefore, $1 \leq a \leq 2$ and

$$(b-2)^2 \leq 4(a-1)(c-1),$$

$$b^2 \leq 4(2-a)(2-c).$$

Let $\lambda = a-1, \ \mu = c-1$, where λ and $\mu \in [0;1]$. Using the inequality between the arithmetical and the geometrical means, we obtain that

$$2 \leq |b| + |2-b| \leq 2\sqrt{\lambda\mu} + 2\sqrt{(1-\lambda)(1-\mu)} \leq 2.$$

Therefore, $\lambda = \mu$, $b = 2\lambda$ and we are done. □

The aim of this paper is to explain the above problem, toward algebraic geometry results.

2 Linear systems of conics

In this section, we use standard notations as for instance those in [H] or [Ş]. We prefer to include some known results about linear systems of conics in order to make the paper self-contained.

Let $|2H|$ be the complete linear system of conics on $\mathbf{P}^2_{\mathbf{C}}$ and let $P_1, P_2, \ldots, P_r$ be points of $\mathbf{P}^2$. We denote by

$$\mathcal{L} = |2H - P_1 - P_2 - \ldots - P_r|$$

the sub-linear system of $|2H|$ consisting in those conics C passing through the given points. We say that $P_1, P_2, \ldots P_r$ are the **assigned base points** of $\mathcal{L}$.

This definition makes sense even if some of the assigned base points are equal: if, for example, $P_1 = P_2$, then $C \in \mathcal{L}$ if it passes through P_1 with multiplicity of at least 2.

We can consider even that some of the given points are infinitely near points of $\mathbf{P}^2$. Such a point determines a direction through one of the ordinary assigned points, and we impose to the conics in $\mathcal{L}$ to be tangent at this direction.

Let $\hat{\mathbf{P}}^2$ be the surface obtained by blowing up the assigned base points on $\mathbf{P}^2$ and let $f : \hat{\mathbf{P}}^2 \to \mathbf{P}^2$ be the corresponding morphism. We consider the linear system

$$\mathcal{L}_1 = |f^*(2H) - E_1 - \ldots - E_r|$$

on $\hat{\mathbf{P}}^2$, where E_i are the exceptional curves (or their total transforms, if some of the given points are infinitely near points). We say that a base point of $\mathcal{L}_1$ is an **unassigned base point** of $\mathcal{L}$.

Proposition 2.1. (See, e.g. [H], p. 396). *Let $\mathcal{L}$ be a linear system of curves on $\mathbf{P}^2$. If $\mathcal{L}$ has no unassigned base points, then for any point Q,*

$$dim|\mathcal{L} - Q| = dim(\mathcal{L}) - 1.$$

Proof: Since $\mathcal{L}$ has no unassigned base points, Q impose a new condition to curves in $\mathcal{L}$, independent to all the old conditions. Then the vector space corresponding to $|\mathcal{L} - Q|$ is a hyperplane in the vector space corresponding to $\mathcal{L}$, and the dimension drops exactly by one. □

Proposition 2.2. *Let*

$$\mathcal{L} = |2H - P_1 - P_2 - \ldots - P_r|,$$

$$\mathcal{F} = |H - P_1 - P_2 - \ldots - P_s|$$

be two linear systems of conics on $\mathbf{P}^2$. If $\mathcal{L}$ has no unassigned base points, and if $s \leq r$, that remains true for $\mathcal{F}$.

Proof: Consider the blow-up $\hat{\mathbf{P}}^2$ of $\mathbf{P}^2$ in $P_1, P_2, \ldots, P_r$ and the linear systems

$$\hat{\mathcal{L}} = |f^*(2H) - E_1 - E_2 - \ldots - E_r|, \ \hat{\mathcal{F}} = |f^*(2H) - E_1 - E_2 - \ldots - E_s|.$$

Since $\hat{\mathcal{L}}$ is a subsystem of $\hat{\mathcal{F}}$, we have that

$$Bs(\hat{\mathcal{F}}) \subseteq Bs(\hat{\mathcal{L}}) = \emptyset.$$

Then $\hat{\mathcal{F}}$ has no unassigned base points. □

Proposition 2.3. (See, e.g. [H], p. 396) *Let $P_1, P_2, \ldots, P_r$ be (ordinary or infinitely near) distinct points of $\mathbf{P}^2$.*

1. *For $k \geq 3$, $|2H - kP_1| = \emptyset$.*
2. *$\mathcal{L}_1 = |2H - 2P_1|$ has no unassigned points and $dim(\mathcal{L}_1) = 2$.*
3. *$dim(|2H - 2P_1 - P_2|) = 1$.*
4. *Suppose that P_1, P_2, P_3, P_4 are ordinary distinct points, any three of them no collinear. Then*

   $$\mathcal{L}_2 = |2H - P_1 - P_2 - P_3 - P_4|$$

   has no unassigned base points.

5. *Let P_1, P_2, P_3 be ordinary points on $\mathbf{P}^2$, and let P_4 be infinitely near P_1. Suppose that P_1, P_2, P_3 are no collinear, and that neither P_2, nor P_3 does belong to the line of $\mathbf{P}^2$ corresponding with P_4. Then*

$$\mathcal{L}_3 = |2H - P_1 - P_2 - P_3 - P_4|$$

has no unassigned base points.

Proof: *1.* If C would be a conic in $|2H - kP_1|$, then the multiplicity of P_1 on C would be at least k, which is impossible.
2. A conic C is in $\mathcal{L}_1$ if and only if C is a union of two lines passing through P_1. Since $|H - P_1|$ has no unassigned base points, we conclude that $\mathcal{L}_1 = |2H - 2P_1|$ has no unassigned base points, and that $dim(\mathcal{L}_1) = 2$.
3. We can apply Proposition 2.1 to the linear system $|2H - 2P_1|$.
4. We will prove that, for any (ordinary or infinitely near) point Q, there exists a conic in $\mathcal{L}_2$, not passing through Q; therefore, $\mathcal{L}_2$ will have no unassigned base points.
Let $l_{i,j}$ denote the line P_iP_j. Consider the conics

$$C = l_{1,2} \cup l_{3,4}$$

and

$$D = l_{1,3} \cup l_{2,4}.$$

Since

$$C \cap D = \{P_1, P_2, P_3, P_4\}$$

we conclude in the case when Q is different from the assigned base points. On the other hand, if, for example, $Q = P_1$, this point could not be an unassigned base point, because C passes through P_1 with a multiplicity exactly one.
5. Let g be the blow-up morphism from $X = \hat{\mathbf{P}}^2(P_1, P_2, P_3)$ to $\mathbf{P}^2$, and let h be the blow-up morphism from $\hat{X}(P_4)$ to X. We consider the morphism $f = h \circ g$ and the linear system

$$\hat{\mathcal{L}}_3 = |f^*(2H) - E_1 - E_4 - E_3 - E_4|,$$

where E_1, E_2, E_3 are the total transforms by h of the exceptional curves of g, and E_4 is the exceptional curve of h.
Let $l_{1,4}$ be the line in $\mathbf{P}^2$ passing through P_1 and corresponding to direction P_4. The linear system $\mathcal{L}_3$ contains the conics

$$C = l_{1,4} \cup l_{2,3} \text{ and } D = l_{1,2} \cup l_{1,3}.$$

(Observe that the proper transform $\tilde{D}$ of D, i.e.

$$\tilde{D} = g^*(D) - E_1 - E_2 - E_3 = \tilde{l}_{1,2} \cup \tilde{l}_{1,3} \cup E_1$$

is a curve on X, passing through P_4.)
On X, the proper transforms $\tilde{C}$ and $\tilde{D}$ does not intersect, and the same is true on $\tilde{X}$. Therefore, $\hat{\mathcal{L}}_3$ has no base points. □

Corollary 2.1. *Let P_1 and P_2 be ordinary points on $\mathbf{P}^2$, and let Q_1, Q_2 be infinitely near points to P_1, P_2. Suppose that P_1 does no belongs to the line of $\mathbf{P}^2$ corresponding to Q_2, and that P_2 does not belongs to the line corresponding to Q_1. Then*

$$dim(|2H - P_1 - P_2 - Q_1 - Q_2|) = 1.$$

3 An explanation for Problem 1.1

In this section we give an explanation for the Problem 1.1, using the results from above. We use the notations from Section 1.

We consider the complex projective plane $\mathbf{P}^2_{\mathbf{C}}$ with coordinates $(x : y : z)$. Let C, C_1 and C_2 be the projective conics, defined in the affine open set

$$U = (z \neq 0) \subset \mathbf{P}^2_{\mathbf{C}}$$

by the equations

$$y = h(x),\ y = f(x), \text{respective } y = g(x),$$

i.e.

$$C = V(yz - ax^2 - bxz - cz^2),$$
$$C_1 = V(yz - 2x^2 - 2z^2),$$
$$C_2 = V(yz - x^2 - 2xz - z^2).$$

Observe that C_1 and C_2 have a common tangent line in $P_1 = (0 : 1 : 0)$. In fact, this is a common property for all the conics, described as the projective closure of an affine parabola, graph of a 2nd degree function. Therefore, C have the same tangent line in P_1 as the above curves.

Since

$$C_1 \cap C_2 = \{(0 : 1 : 0), (1 : 4 : 1)\},$$

C_1 and C_2 have another common tangent line in $P_2 = (1 : 4 : 1)$.

If we denote by Q_1 and Q_2 the infinitely near points corresponding to the tangent lines from above, we obtain that C_1 and C_2 are members of the linear system

$$\mathcal{L} = |2H - P_1 - P_2 - Q_1 - Q_2|.$$

On the other hand, from the condition

$$f(x) \leq h(x) \leq g(x), \text{for all } x \in \mathbb{R},$$

we obtain that the curve C passes through P_2. The same condition give that

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1} \frac{h(x) - h(1)}{x - 1},$$

i.e.

$$f'(1) = g'(1) = h'(1).$$

Therefore, C have the same tangent line in P_2 as C_1 and C_2, and we obtain that $C \in \mathcal{L}$.

From the Corollary 2.1, we know that the (projective) dimension of $\mathcal{L}$ is 1. Therefore, $\mathcal{L}$ corresponds to a vector sub-space in $H^0(\mathbf{P}^2, \mathcal{O}(2))$, of dimension 2. Any two different conics in $\mathcal{L}$, in particular C_1 and C_2 determine a base of this vector space. Therefore, there exists λ and μ in $\mathbb{C}$ such that

$$C = \lambda C_1 + \mu C_2.$$

Express this equality in terms of equations, in the affine open set $U = (z \neq 0)$ to obtain that

$$y - h(x) = \lambda(y - g(x)) + \mu(y - f(x)),$$

i.e.

$$h(x) = \lambda g(x) + \mu f(x).$$

A simpler re-interpretation of the problem gives $\lambda, \mu \in \mathbb{R}$ and $\lambda + \mu = 1$.

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