Bull. Math. Soc. Sc. Math. Roumanie
Tome 49(97) No. 1, 2006, 63–68

On some properties of squarefree and squareful numbers

by
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To Professor Ion D. Ion on the occasion of his 70th Birthday

Abstract

A number is called squareful if it contains at least a square in its prime factorization.We study the integers that can be written as sums of two squarefree numbers, as sums of two squareful numbers and as sums of a squarefree and a squareful number. We also show that between n^2 and $(n+1)^2$ there are more than n squarefree numbers.

Key Words: Squarefree numbers, squareful numbers, additive representations.
2000 Mathematics Subject Classification: Primary: 11A67, Secondary 11N25, 11N80.

1 Introduction

We recall that an integer is called squareful (sometimes also nonsquarefree) if it contains at least a square in its prime factorization.
Throughout the paper we will denote by:

- $(a_k)_{k\geq 1}$ the increasing sequence of squarefree numbers ($a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, ...$)
- $(n_k)_{k\geq 1}$ the increasing sequence of squareful numbers ($n_1 = 4, n_2 = 8, n_3 = 9, n_4 = 12, ...$)
- $(p_k)_{k\geq 1}$ the increasing sequence of prime numbers
- $Q(x)$ the number of squarefree integers not exceeding x

The conjecture of Goldbach is one of the most popular unsolved problems in mathematics because of its very simple-looking statement which makes at first sight rather incredible the lack of success of all the attempts to solve it so far.
The statement of the conjecture is that every even number bigger than 2 can be

written as a sum of two primes.
The most significant results on this topic are the theorem of Chen which states that every large enough even number can be written as a sum of a prime and a product of at most two primes [1], and the theorem of Vinogradov according to which every large enough odd integer can be written as a sum of three primes [2]
We will study in Section 2 the much simpler problem of the representation of integer numbers as sums of two squarefree integers, of two squareful integers and also as sums of a squarefree and a squareful integer.
Another unsolved problem concerning the distribution of primes is the one stating that between every two consecutive squares there is at least a prime number. We will prove in Section 3 that between every two consecutive squares we find several squarefree numbers.
We will use the following estimates:

$$\left|Q(x) - \frac{6}{\pi^2}x\right| < 0.5\sqrt{x} \text{ for all } x \geq 8. \tag{1}$$

(issued from [3]), and

$$\left|Q(x) - \frac{6}{\pi^2}x\right| < 0.036438\sqrt{x} \text{ for all } x \geq 82005. \tag{2}$$

(issued from [4]).

2 Representation results

We begin with the squarefree case:

Theorem 1. *Every integer number $n \geq 2$ can be written as a sum of two squarefree integers.*

Proof: Let us suppose there exist positive integers $n \geq 31$ which cannot be written as sums of two squarefree numbers.
Let $a_1, a_2, \ldots, a_k < n \leq a_{k+1}$; we then have

$$k \geq Q(n) - 1 \tag{3}$$

For each $i = \overline{1,k}$ we have $1 \leq n - a_i < n$. Besides, for all $i, j = \overline{1,k}$ we may write $n - a_i \neq a_j$. Since

$$\{a_1, a_2, \ldots, a_k, n - a_1, n - a_2, \ldots, n - a_k\} \subset \{1, 2, \ldots, n-1\},$$

relation (3) leads to $n - 1 \geq 2k \geq 2(Q(n) - 1)$, so

$$n \geq 2Q(n) - 1. \tag{4}$$

Relation (1) implies

$$Q(n) > \frac{6}{\pi^2}n - \frac{1}{2}\sqrt{n} \qquad \text{for all} \quad n \geq 8. \tag{5}$$

Relations (4) and (5) give

$$n > \frac{12}{\pi^2}n - \sqrt{n} - 1,$$

which implies $\sqrt{n} < 5.48$, so $n < 30.04$, a contradiction.
For $2 \leq n \leq 30$ the statement of the theorem follows by direct calculation. □

Remark 2. *A more technical approach for the proof of Theorem 1 would have been the use of Schnirelmann sequences. This approach, however, would have required (bearing in mind that we have to put a zero term to the Schnirelmann sequences before adding them) the weakened statement "Every integer number $n \geq 2$ can be written as a sum of at most two squarefree integers".*

The problem of the representation of integers as sums of two squareful numbers is much simpler. We have:

Proposition 3. *Let $n \geq 24$ be an integer. Then n can be written as a sum of two squareful integers.*

Proof: 24=4+20; 25=9+16; 26=8+18; 27=9+18. For $n \geq 28$ we have the following cases:

- if $n = 4k$, then $n = 4 + 4(k-1)$
- if $n = 4k+1$, then $n = 9 + 4(k-2)$
- if $n = 4k+2$, then $n = 18 + 4(k-4)$
- if $n = 4k+3$, then $n = 27 + 4(k-6)$

□

Remark 4. *The only positive integers which cannot be written as a sum of two squareful terms are 1,2,3,4,5,6,7,9,10,11,14,15,19 and 23.*

We now turn to the integers which can be written as a sum of a squarefree and a squareful number. We begin by a technical result:

Lemma 5. *Let x be positive, p a prime number with $p^2 \leq x$, $\alpha \in \{1, 2, \ldots, p^2 - 1\}$, and $n = \left[\frac{x-\alpha}{p^2}\right]$. Then the number of squarefree numbers in the sequence $\alpha, \alpha + p^2, \alpha + 2p^2, \ldots, \alpha + np^2$ is greater than*

$$\left(2 - \frac{\pi^2}{6}\right)\left(\frac{x}{p^2} - 1\right) - \sqrt{x}.$$

Proof: Let $\mathcal{M} = \{\alpha, \alpha + p^2, \alpha + 2p^2, \ldots, \alpha + np^2\}$. Then $\mathcal{M}$ contains at most $\left[\frac{n+1}{k^2}\right] + 1$ multiples of k^2 for each integer $k \in \{2, \ldots, [\sqrt{x}]\}$ (because, if $(k,p) = 1$, these multiples of k^2 will appear at a rate of one at every k^2 terms of the progression $\mathcal{M}$, while if $p|k$ the progression $\mathcal{M}$ will contain no multiples of k^2 at all). Consequently, the number $L_{\mathcal{M}}(x)$ of squarefree integers in $\mathcal{M}$ will be at least

$$n + 1 - (n+1)\sum_{k=2}^{[\sqrt{x}]} \frac{1}{k^2} - \sqrt{x} + 1.$$

We derive the inequality

$$L_{\mathcal{M}}(x) > (n+1)\left(2 - \frac{\pi^2}{6}\right) - \sqrt{x} + 1 > \left(\left[\frac{x-\alpha}{p^2}\right] + 1\right)\left(2 - \frac{\pi^2}{6}\right) - \sqrt{x}; \quad (6)$$

if we use the conditions on α, (6) leads to the relation

$$L_{\mathcal{M}}(x) > \left(2 - \frac{\pi^2}{6}\right)\left(\frac{x}{p^2} - 1\right) - \sqrt{x}.$$

□

Theorem 6. *Every integer number $n \geq 8$ can be written as a sum of a squarefree and a squareful term.*

Proof: Let $n \geq 16 \cdot 13^4$ and $k \geq 6$ such as $n \in [16p_k^4, 16p_{k+1}^4)$, p_m being the m^{th} prime. Consider also $j \in \{1, 2, \ldots, k\}$ and $\alpha \in \{1, 2, \ldots, p_j^2 - 1\}$. If we denote $2 - \pi^2/6$ by β, since $\beta > 1/3$, $4\beta - 1 > 1/3$, and $4p_j - 1 > 3$, we obtain $(4p_j - 1)[(4\beta - 1)p_j + \beta] \geq p_j + 1$, so

$$\left(\frac{\sqrt{n}}{p_j} - 1\right)\left[\beta\left(\frac{\sqrt{n}}{p_j} + 1\right) - p_j\right] \geq p_j + 1,$$

leading to

$$\beta\left(\frac{n}{p_j^2} - 1\right) - p_j\left(\frac{\sqrt{n}}{p_j} - 1\right) \geq p_j + 1,$$

or

$$\beta\left(\frac{n}{p_j^2} - 1\right) - \sqrt{n} \geq 1. \quad (7)$$

Since, according to Lemma 5, the progression $\mathcal{M}_j = \{\alpha, \alpha + p_j^2, \alpha + 2p_j^2, \ldots, \alpha + n_j p_j^2\}$ (with $n_j = \left[\frac{n-\alpha}{p_j^2}\right]$) contains more than $\beta\left(\frac{n}{p_j^2} - 1\right) - \sqrt{n}$ squarefree terms, we derive from (7) the presence of at least one squarefree term in $\mathcal{M}_j$.

Let us now suppose that n cannot be written as the sum of a squarefree and a squareful number. Then $n - \lambda p_j^2$ would be squareful for each $j \leq k$ and $\lambda \in$

$\{0, 1, \ldots, n_j\}$. This means all the terms of the progression $\mathcal{M}_j$ (α being the residue of n modp_j^2, while $n_j = \left[\frac{n-\alpha}{p_j^2}\right]$) are squareful. According to relation (7) and its consequence, this means $\alpha = 0$, so $p_j^2 | n$.
Therefore, n has to be divisible by $p_1^2 p_2^2 \cdots p_k^2$.
On the other hand, since $k \geq 6$,

$$n \geq p_1^2 p_2^2 \cdots p_k^2 \geq 4 \cdot 9 \cdot 25 \cdot 49 \cdots p_{k-1}^2 \cdot p_k^2. \tag{8}$$

Now, the Bertrand-Chebychev Theorem implies $p_k > p_{k+1}/2$ and $p_{k-1} > p_{k+1}/4$. Together with (8), these relations imply

$$n > 4 \cdot 9 \cdot 25 \cdot 49 \cdot \frac{p_{k+1}^2}{16} \cdot \frac{p_{k+1}^2}{4} > 16 p_{k+1}^4,$$

inequality which comes in contradiction with the choice of k.
The statement of the theorem is thus true for $n \geq 16 \cdot 13^4$.
Computer checking for the values $n < 16 \cdot 13^4 = 456976$ completes the proof. □

Remark 7. *The only positive integers which cannot be written as a sum of a squarefree and a squareful term are 1,2,4, and 8.*

3 Squarefree integers between consecutive squares

We prove in this section

Theorem 8. *For every positive integer n, the interval $(n^2, (n+1)^2)$ contains more than n squarefree integers.*

Proof: Let us denote $a = 0.036438$. According to relation (2), for $n > 287 = [\sqrt{82005}]$ we have the inequality

$$Q((n+1)^2) - Q(n^2) > \frac{6}{\pi^2}(n+1)^2 - a(n+1) - \frac{6}{\pi^2}n^2 - an = n\left(\frac{12}{\pi^2} - 2a\right) + \frac{6}{\pi^2} - a. \tag{9}$$

Since $12/\pi^2 - 2a - 1 > 0.01 > 0$, the relation

$$n\left(\frac{12}{\pi^2} - 2a\right) + \frac{6}{\pi^2} - a > n \tag{10}$$

holds true for every positive integer n; moreover, for $n > 287$ we may even write

$$n\left(\frac{12}{\pi^2} - 2a\right) + \frac{6}{\pi^2} - a > n + 1. \tag{11}$$

Relations (9) and (11) prove the claim for $n > 287$.
Computer checking of the statement for $n \leq 287$ completes the proof. □

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Received: 5.12.2005

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