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Some interpretations of hypergroups

by
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To Professor Ion D. Ion on the occasion of his 70th Birthday

Abstract

First we look at some definitions, interpreting hypergroups as sets with a ternary special relation. Then we establish the relationships between canonical hypergroups defined usually (starting with Marty, Krasner, Eaton) and those defined by J. R. McMullen and J. F. Price by using some mappings.

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1 Introduction

We consider here the basic definitions of some types of hypergroups. We show that the canonical hypergroups can be defined as McMullen hypergroups over a commutative unitary ring. In the third section, we shall see that the hypergroups could be obtained by considering a ternary relation with some properties on a nonempty set H.

First we recall some necessary definitions.

Definition 1.1. *A* hypergroup *is an ordered pair* $(H, \circ)$,*where* H *is a nonempty set and* "$\circ$" *is a hyperoperation, i.e.* $\circ : H^2 \to \mathcal{P}(H) \setminus \{\emptyset\}$, *such that the following conditions are fulfilled:*

(*i*) $(x \circ y) \circ z = x \circ (y \circ z)$, *for all* $x, y, z \in H$,

(*ii*) $x \circ H = H \circ x$, *for all* $x \in H$.

Here, we denote $A \circ B := \bigcup \{a \circ ba \in A,\, b \in B\}$, and $\{x\} \circ H = x \circ H$.

If H contains an element e such that $x \circ e = x$, for all $x \in H$ (such an element is called a *right identity* and, because of condition $|e \circ x| = 1$, for all $x \in H$, e is called a *right scalar*) and, from $x \circ y \cap x \circ z \neq \emptyset$ for $x, y, z \in H$, we have $e \circ y = e \circ z$, then the hypergroup is *of type C* (on the right side, but since we

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use only such hypergroups, we shall forget about the word "right"), after Yves Sureau [Su, 1991], or it is a *weak cogroup,* after S. D. Comer [Co, 1984], [Co, 1991].

A weak cogroup is a *right cogroup* if $|y \circ x| = |z \circ x|$, for all $x, y, z \in H$ (after J. Eaton [Ea, 1940]).

Definition 1.2. *A* canonical hypergroup *is a hypergroup* $(H, \circ)$ *with a left scalar identity, satisfying also the conditions:*

(i) $x \circ y = y \circ x$, *for all* $x, y \in H$.

(ii) *For all* $x \in H$, *there is a unique* $x' \in H$, *such that* $e \in x \circ x'$.

(iii) *If* $z \in x \circ y$, *for* $x, y, z \in H$, *then* $y \in x' \circ z$.

Canonical hypergroups, defined by J.Mittas in 1972, remain ones of the most visited algebraic hyperstructures since they are pretty good generalizations for abelian groups and they give a framework for hyperstructures replacing rings, modules, fields or used in geometry (join spaces).

Another interesting hypergroups are the $D-$*hypergroups ; these* are hypergroups isomorphic to a hypergroup of left cosets of a group $(G, \cdot)$ with respect to a subgroup S, G/S, with hyperoperation:

$$xS \circ yS := \{zS/z \in xSy\}.$$

Each $D-$hypergroup is a cogroup ([Ea, 1940]), but there are cogroups which are not $D-$hypergroups (Y. Utumi, [Ut, 1949]). However, if $(H, \circ)$ has a structure of $D-$hypergroup and there exists an equivalence relation ρ on H, verifying the conditions

(i) $e^\rho = \{e\}$;

(ii) $(x \circ y^\rho)^\rho = x^\rho \circ y^\rho$;

(iii) $e \circ x \subseteq x^\rho$, for all $x, y \in H$,

then we may define a structure of right cogroup by using the hyperoperation:

$$x * y := x \circ y^\rho, \text{ for } x, y \in H.$$

(Here $x^\rho = \{y \in H/y\rho x\}$.)

These cogroups are called *cogroups of Utumi type.* L. Haddad and Y. Sureau [Ha, Su, 1990] have shown that there are cogroups which are not of Utumi type.

On a $C-$hypergroup $(H, \circ)$, one considers those permutations $\sigma \in S(H)$, (-here $S(H)$ is the symmetric group of H,- for which

$$\sigma(x \circ y) = \sigma(x) \circ y, \text{ for all } x, y \in H.$$

These permutations form a subgroup $M(H)$ of $S(H)$. For a subgroup G of $M(H)$ and $x \in H$, we denote $G(x) = \{\sigma(x) \,|\sigma \in G\}$.

Putting together results of M. Krasner, Y. Utumi, L. Haddad and Y. Sureau, we get the complete characterization of a $D-$hypergroup, as a $C-$hypergroup for which there is a subgroup of $M(H)$, G, with the properties:

(i) $G(e) = H$;

(ii) if $T = \{\sigma \in G \,|\sigma(e) = e\}$, then $T(x) = e \circ x$, for all $x \in H$.

Also, we have the characterization of the $C-$hypergroups which are of Utumi type as those having a subgroup G of $M(H)$, with the property $G(e) = H$.

2 McMullen hypergroups

We rewrite the definition of a hypergroup given by J. R. McMullen [Mc, 1979].

Definition 2.1. *An* abelian McMullen hypergroup *is a nonempty set* H *together with three maps:*

$m : H^3 \to \mathbf{N}$, $d : H \to \mathbf{N}^*$, $- : H \to H$ $(a \mapsto \bar{a})$, *- called, respectively, multiplicity, degree and conjugation, - and a distinguished element* e *in* H*, such that the axioms (H.1) - (H.8) are fulfiled:*

(H.1) For all $a, b \in H$*, the subset* $\{x \in H \,|m(a,b,x) \neq 0\}$ *is nonempty and finite.*

(H.2) (Associativity) For all $a, b, c, d \in H$,

$$\sum_{x \in H} m(a,b,x)\, m(x,c,d) = \sum_{y \in H} m(b,c,y)\, m(a,y,d).$$

(H.3) (Identity) For all $a, b \in H$, $m(e,a,b) = m(a,e,b) = \delta_{ab}$, *where* $\delta_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b. \end{cases}$

(H.4) (Reversibility) For all $a, b \in H$, $m\left(e, \bar{b}, c\right) \neq 0$ *if and only if* $a = b$ *and* $m\left(a, \bar{a}, e\right) = m\left(\bar{a}, a, e\right)$.

(H.5) (Conjugation) For all $a, b, c \in H$, $m\left(\bar{a}, \bar{b}, \bar{c}\right) = m(b,a,c,)$ *and* $d\left(\bar{a}\right) = d(a)$.

(H.6) (Degree) For all $a, b \in H$, $d(a) \cdot d(b) = \sum_{x \in H} m(a,b,x)\, d(x)$.

(H.7) (Division) For all $a \in H$, $m\left(a, \bar{a}, e\right)$ *divides* $(d(a))^2$.

(H.8) (Commutativity) For all $a, b, c \in H$, $m(a,b,c) = m(b,a,c)$.

Proposition 2.2. *Any canonical hypergroup with* $|x \circ y| < +\infty$ *is an abelian McMullen hypergroup and conversely, each abelian McMullen hypergroup is a canonical hypergroup.*

Proof: Indeed, in a canonical hypergroup $(H, \circ, e)$ with $|x \circ y| < +\infty$, we define:

$$m(a,b,c) = \begin{cases} 1, \text{ if } c \in a \circ b, \\ 0, \text{ if } c \notin a \circ b, \end{cases}$$

$d(x) = 1$, for all $x \in H$, $\bar{a} = a'$, and we verify (H.l) - (H.8).

Conversely, in an abelian McMullen hypergroup, we define a canonical hypergroup structure, by:

$$a \circ b := \{x \in H \,|m(a,b,x) \neq 0\}.$$

□

Let us note that we may take, instead of **N**, any nonempty subset C of a commutative ring with identity 1, which is closed under multiplication and addition.

In his paper [Mc, 1979], Mc Mullen has distinguished the subset of elements:

$$G = \{a \in H \,|d(a) = 1\},$$

which is also characterized by one of the conditions:

$$\left\{a \in H \,\middle|\, m\left(a, \bar{a}, b\right) = \delta_{eb}, \text{ for all } b \in H\right\}$$

or

$$\{a \in H \,|\forall b \in H, \exists c \in H \text{ unique, such that } m(a, b, x) = \delta_{cx}, \forall x \in H\}.$$

Then $(G, \circ)$ is a group and its elements are called *grouplike* elements.

Proposition 2.3. *Let $(H, m, d, -, e)$ be an abelian McMullen hypergroup over R, with R a commutative unitary ring. Then we may define the hypergroup ring (and algebra) RH, by considering the elements of the form $\sum\limits_{a \in H}{}^{*} \alpha_a a$, where $\alpha_a \in R$ and only a finite number of α_a are different from 0.*

Proof: The addition is, as usual, a componentwise addition of two formal sums, the scalar multiplication is the usual one and the multiplication is:

$$\sum_{a \in H}{}^{*} \alpha_a a \cdot \sum_{b \in H}{}^{*} \beta_b b := \sum_{a \in H, b \in H, c \in H} \sum \sum \alpha_a \beta_b m(a, b, c)\, c.$$

We consider the map $H \to RH$, given by $a \to \sum\limits_{b \in H} \delta_{ab} b$,which embeds H in RH. □

In the case of a field k, McMullen and Price [Mc, Pr, 1977] have proved the following result (we restate it a little bit different manner).

Proposition 2.4. *Let H be a finite McMullen hypergroup with coefficients in a field k. The hypergroup algebra kH is a* double algebra *over k with the mappings u, Δ, d, S defined below only on spanning elements and linearly extended:*

$u : k \to kH$, $u(1) = e$ (considered in kH);

$\Delta : H \to kH \otimes kH$, $\Delta(a) = d(a)^{-1} a \otimes a$;

$S : H \to H$, $S(a) = \bar{a}$.

We do not prove it here, but let us remark that while a group leads, with the same considerations, to a Hopf algebra, in the case of a canonical hypergroup we get only a double algebra.

In the case kH, the set of grouplike elements is $\left\{d(a)^{-1} \,|a \in H\right\}$.

Now, if A is a double algebra over the field k and H is the set of its grouplike elements, then H can be endowed canonically with a hypergroup structure with A as the hypergroup algebra over k.

Remark 2.5. *In [Mc, Pr, 1977], some constructions of finite Mc Mullen hypergroups are given, which can be added to those given in [St, 1995] or [St,1997].*

3 Hypergroups and ternary relations.

Let $(H, \circ)$ be a hypergroup. Then we may consider the ternary relation ρ on H associated to the hyperoperation:

$$(a, b, c) \in \rho \text{ if and only if } c \in a \circ b.$$

The relation satisfies the following conditions:

(3. 1) For all $a, b \in H$, there exists at least one element $c \in H$, such that $(a, b, c) \in \rho$.

(3. 2) If for $a, b, c, z \in H$ there is $x \in H$, such that $(a, b, x), (x, c, z) \in \rho$, then there exists $y \in H$, such that $(a, y, z), (b, c, y) \in \rho$ and conversely.

(3. 3) For all $a, b \in H$, there exists $x, y \in H$, such that $(a, x, b) \in \rho$ and $(y, a, b) \in \rho$.

Conversely, if (H, ρ),- ρ being a ternary relation on H, $-$ satisfies (3. 1) - (3. 3), then, by taking the hyperoperation:

(3. 4) $a \circ b := \{x \in H \,|(a, b, x) \in \rho\}$, $(H, \circ)$ is a hypergroup.

The commutativity of (3. 4) is transled into a condition for ρ:

(3. 5) $(a, b, x) \in \rho$ if and only if $(b, a, x) \in \rho$, for all $a, b, x \in H$.

The two-sided identity e is characterized by:

(3. 6) For all $a \in H$, $(e, a, a) \in \rho$ and $(a, e, a) \in \rho$.

The inverse of an element is given by:

(3. 7) For all $a \in H$, there exists $a' \in H$, for which $(a, a', e) \in \rho$ and $(a', a, e) \in \rho$.

An element $s \in H$ is a *scalar* in H if:

$$(a, s, b), (a, s, c) \in \rho \text{ implies } b = c$$

and

$$(s, a, b), (s, a, c) \in \rho \text{ implies } b = c,$$

for any $a \in H$.

It would be nice if somebody can find connections between this definition and the hypergroups associated to ordered structures.

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