Bull. Math. Soc. Sc. Math. Roumanie
Tome 49(97) No. 1, 2006, 13–24

Residuated lattice of fractions relative to a $\wedge$-closed system

by
Dumitru Buşneag and Dana Piciu

To Professor Ion D. Ion on the occasion of his 70th Birthday

Abstract

The aim of this paper is to introduce (taking as a guide-line the case of rings, see [12]) the notion of residuated lattice of fractions relative to a $\wedge$-closed system. For the case of Hilbert algebras, MV and pseudo MV-algebras, BL and pseudo BL-algebras see [5], [6], [7], [8] and [18].

With this paper we initiate a study for the localization of residuated lattices.

Key Words: Residuated lattice, residuated lattice of fractions, $\wedge$-closed system.
2000 Mathematics Subject Classification: Primary: 03G10, Secondary 06B20, 08A72, 03G25.

1 Introduction

The origin of residuated lattices is in Mathematical Logic without contraction. They have been investigated by Krull ([15]), Dilworth ([9]), Ward and Dilworth ([21]), Ward ([20]), Balbes and Dwinger ([1]) and Pavelka ([17]).

In [11], Idziak prove that the class of residuated lattices is equational. These lattices have been known under many names: *BCK- latices* in [10], *full BCK-algebras* in [15], *FL_{ew}- algebras* in [16], and *integral, residuated, commutative l-monoids* in [3].

Definition 1. A *residuated lattice* ([2], [19]) is an algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) equipped with an order $\leq$ satisfying the following:

(LR_1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,

(LR_2) $(A, \odot, 1)$ is a commutative ordered monoid,

(LR_3) $\odot$ and $\rightarrow$ form an adjoint pair, i.e. $c \leq a \rightarrow b$ iff $a \odot c \leq b$ for all $a, b, c \in A$.

The relations between the pair of operations $\odot$ and $\to$ expressed by Definition 1 (LR_3), is a particular case of the *law of residuation* ([2]). Namely, let A and B two posets, and $f : A \to B$ a mapping. Then f is called *residuated* if there is a map $g : B \to A$, such that for any $a \in A$ and $b \in B$, we have $f(a) \leq b$ iff $b \leq g(a)$ (this is also expressed by saying that the pair (f, g) is a *residuated pair*).

Now setting A a residuated lattice, $B = A$, and defining, for any $a \in A$, two mappings $f_a, g_a : A \to A$, $f_a(x) = x \odot a$ and $g_a(x) = a \to x$, for any $x \in A$, we see that $x \odot a = f_a(x) \leq y$ iff $x \leq g_a(y) = a \to y$ for every $x, y \in A$, that is, for every $a \in A$, (f_a, g_a) is a pair of residuation.

The symbols $\Rightarrow$ and $\Leftrightarrow$ are used for logical implication and logical equivalence.

Proposition 1. ([*11*]) *The class* $\mathcal{RL}$ *of residuated lattices is equational.*

Example 1. Let p be a fixed natural number and $A = [0, 1]$ the real unit interval. If for $x, y \in A$, we define $x \odot y = 1 - \min\{1, [(1-x)^p + (1-y)^p]^{1/p}\}$ and $x \to y = \sup\{z \in [0, 1] : x \odot z \leq y\}$, then $(A, \max, \min, \odot, \to, 0, 1)$ is a residuated lattice.

Example 2. If we preserve the notation from Example 1, and we define for $x, y \in A$, $x \odot y = (\max\{0, x^p + y^p - 1\})^{1/p}$ and $x \to y = \min\{1, (1 - x^p + y^p)^{1/p}\}$, then $(A, \max, \min, \odot, \to, 0, 1)$ become a residuated lattice called *generalized Lukasiewicz structure.* For $p = 1$ we obtain the notion of *Lukasiewicz structure* $(x \odot y = \max\{0, x + y - 1\}, x \to y = \min\{1, 1 - x + y\})$.

Example 3. If on $A = [0, 1]$, for $x, y \in A$ we define $x \odot y = \min\{x, y\}$ and $x \to y = 1$ if $x \leq y$ and y otherwise, then $(A, \max, \min, \odot, \to, 0, 1)$ is a residuated lattice (called Gödel structure).

Example 4. If consider on $A = [0, 1]$, $\odot$ to be the usual multiplication of real numbers and for $x, y \in A, x \to y = 1$ if $x \leq y$ and y/x otherwise, then $(A, \max, \min, \odot, \to, 0, 1)$ is a residuated lattice (called *Products structure* or *Gaines structure*).

Example 5. If $(A, \vee, \wedge, ', 0, 1)$ is a Boolean algebra, then if we define for $x, y \in A, x \odot y = x \wedge y$ and $x \to y = x' \vee y$, then $(A, \vee, \wedge, \odot, \to, 0, 1)$ become a residuated lattice.

Definition 2. ([19]) A residuated lattice $(A, \wedge, \vee, \odot, \to, 0, 1)$ is called *BL-algebra*, if the following two identities hold in A :

(BL_1) $x \odot (x \to y) = x \wedge y$;

(BL_2) $(x \to y) \vee (y \to x) = 1$.

Remark 1. Lukasiewicz structure, Gödel structure and Product structure are $BL-$ algebras. Not every residuated lattice, however, is a BL-algebra (see [19], p.16).

Remark 2. If in a $BL-$ algebra A, $x^{**} = x$ for all $x \in A$, and for $x, y \in A$ we denote $x \oplus y = (x^* \odot y^*)^*$ then we obtain an algebra $(A, \oplus, ^*, 0)$ of type $(2, 1, 0)$ called MV- *algebras* (see [19]).

Remark 3. ([19]) A residuated lattice $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is an MV-algebra iff it satisfies an additional condition: $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, for any $x, y \in A$.

Example 6. ([13])We give an example of a residuated lattice, which is not a BL-algebra. Let $A = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$, but a, b are incomparable. A become a residuated lattice relative to the following operations:

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

,

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

.

The condition $x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$, for all $x, y \in A$ is not verified, since $c = a \vee b \neq [(a \rightarrow b) \rightarrow b] \wedge [(b \rightarrow a) \rightarrow a] = (b \rightarrow b) \wedge (a \rightarrow a) = 1$, hence A is not a BL-algebra.

In what follows by A we denote a residuated lattice; for $x \in A$ and a natural number n, we define $x^* = x \rightarrow 0, (x^*)^* = x^{**}$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geq 1$.

Theorem 1. ([*14*], [*19*]) *Let $x, x_1, x_2, y, y_1, y_2, z \in A$. Then we have the following rules of calculus:*

$(lr - c_1)$ $1 \rightarrow x = x, x \rightarrow x = 1, y \leq x \rightarrow y, x \rightarrow 1 = 1, 0 \rightarrow x = 1$;

$(lr - c_2)$ $x \odot y \leq x, y$, *hence* $x \odot y \leq x \wedge y$ *and* $x \odot 0 = 0$;

$(lr - c_3)$ $x \odot y \leq x \rightarrow y$;

$(lr - c_4)$ $x \leq y$ *iff* $x \rightarrow y = 1$;

$(lr - c_5)$ $x \rightarrow y = y \rightarrow x = 1 \Leftrightarrow x = y$;

$(lr - c_6)$ $x \odot (x \rightarrow y) \leq y, x \leq (x \rightarrow y) \rightarrow y, ((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$;

$(lr - c_7)$ $x \odot (y \rightarrow z) \leq y \rightarrow (x \odot z) \leq (x \odot y) \rightarrow (x \odot z)$;

$(lr - c_8)$ $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$;

$(lr - c_9)$ $x \leq y$ *implies* $x \odot z \leq y \odot z$;

$(lr - c_{10})$ $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$;

$(lr - c_{11})$ $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$;

$(lr-c_{12})$ $x \leq y$ *implies* $z \to x \leq z \to y, y \to z \leq x \to z$ *and* $y^* \leq x^*$;

$(lr-c_{13})$ $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;

$(lr-c_{14})$ $x_1 \to y_1 \leq (y_2 \to x_2) \to [(y_1 \to y_2) \to (x_1 \to x_2)]$;

$(lr-c_{15})$ $x \odot x^* = 0$ *and* $x \odot y = 0$ *iff* $x \leq y^*$;

$(lr-c_{16})$ $x \leq x^{**}, x^{**} \leq x^* \to x$;

$(lr-c_{17})$ $1^* = 0$, $0^* = 1$;

$(lr-c_{18})$ $x \to y \leq y^* \to x^*$;

$(lr-c_{19})$ $x^{***} = x^*, (x \odot y)^* = x \to y^* = y \to x^* = x^{**} \to y^*$.

Theorem 2. ([14], [19]) *If* A *is a complete residuated lattice,* $x \in A$ *and* $(y_i)_{i \in I}$ *a family of elements of* A*, then :*

$(lr-c_{20})$ $x \odot (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \odot y_i)$;

$(lr-c_{21})$ $x \odot (\bigwedge_{i \in I} y_i) \leq \bigwedge_{i \in I} (x \odot y_i)$;

$(lr-c_{22})$ $x \to (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} (x \to y_i)$;

$(lr-c_{23})$ $(\bigvee_{i \in I} y_i) \to x = \bigwedge_{i \in I} (y_i \to x)$;

$(lr-c_{24})$ $\bigvee_{i \in I} (y_i \to x) \leq (\bigwedge_{i \in I} y_i) \to x$;

$(lr-c_{25})$ $\bigvee_{i \in I} (x \to y_i) \leq x \to (\bigvee_{i \in I} y_i)$;

$(lr-c_{26})$ $(\bigvee_{i \in I} y_i)^* = \bigwedge_{i \in I} y_i^*$;

$(lr-c_{27})$ $(\bigwedge_{i \in I} y_i)^* \geq \bigvee_{i \in I} y_i^*$.

Corollary 1. *If* $x, x', y, y', z \in A$ *then:*

$(lr-c_{28})$ $x \vee y = 1$ *implies* $x \odot y = x \wedge y$;

$(lr-c_{29})$ $x \to (y \to z) \geq (x \to y) \to (x \to z)$;

$(lr-c_{30})$ $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$*, hence* $x^m \vee y^n \geq (x \vee y)^{mn}$*, for any* m, n *natural numbers;*

$(lr-c_{31})$ $(x \to y) \odot (x' \to y') \leq (x \vee x') \to (y \vee y')$;

$(lr-c_{32})$ $(x \to y) \odot (x' \to y') \leq (x \wedge x') \to (y \wedge y')$.

Proof: $(lr-c_{28})$ Suppose $x\vee y=1$. Clearly $x\odot y\leq x$ and $x\odot y\leq y$. Let now $t\in A$ such that $t\leq x$ and $t\leq y$. By $lr-c_7$ we have $t\to(x\odot y)\geq x\odot(t\to y)=x\odot 1=x$ and $t\to(x\odot y)\geq y\odot(t\to x)=y\odot 1=y$, so $t\to(x\odot y)\geq x\vee y=1$, hence $t\to(x\odot y)=1\Leftrightarrow t\leq x\odot y$, that is, $x\odot y=x\wedge y$.

$(lr-c_{29})$ We have by $lr-c_{13}:\ x\to(y\to z)=(x\odot y)\to z$ and $(x\to y)\to(x\to z)=[x\odot(x\to y)]\to z$. But $x\odot y\leq x\odot(x\to y)$, so we obtain $(x\odot y)\to z\geq[x\odot(x\to y)]\to z\Leftrightarrow x\to(y\to z)\geq(x\to y)\to(x\to z)$.

$(lr-c_{30})$ By $lr-c_{20}$ we deduce $(x\vee y)\odot(x\vee z)=x^2\vee(x\odot y)\vee(x\odot z)\vee(y\odot z)\leq$ $\leq x\vee(x\odot y)\vee(x\odot z)\vee(y\odot z)=x\vee(y\odot z)$.

$(lr-c_{31})$ From the inequalities:

$x\odot(x\to y)\odot(x^{'}\to y^{'})\leq x\odot(x\to y)\leq x\wedge y\leq y\vee y^{'}$ and

$x'\odot(x\to y)\odot(x^{'}\to y^{'})\leq x'\odot(x'\to y')\leq x'\wedge y'\leq y\vee y^{'}$ we deduce that

$(x\to y)\odot(x^{'}\to y^{'})\leq x\to(y\vee y^{'})$ and $(x\to y)\odot(x^{'}\to y^{'})\leq x'\to(y\vee y^{'})$.

So, $(x\to y)\odot(x^{'}\to y^{'})\leq[x\to(y\vee y^{'})]\wedge[x'\to(y\vee y^{'})]\overset{lr-c_{23}}{=}(x\vee x^{'})\to(y\vee y^{'})$.

$(lr-c_{32})$ From the inequalities:

$(x\wedge x')\odot(x\to y)\odot(x^{'}\to y^{'})\leq x\odot(x\to y)\overset{lr-c_6}{\leq}y$ and

$(x\wedge x')\odot(x\to y)\odot(x^{'}\to y^{'})\leq x'\odot(x'\to y')\overset{lr-c_6}{\leq}y'$ we deduce that

$(x\to y)\odot(x^{'}\to y^{'})\leq(x\wedge x')\to y$ and $(x\to y)\odot(x^{'}\to y^{'})\leq(x\wedge x')\to y'$.

So, $(x\to y)\odot(x^{'}\to y^{'})\leq[(x\wedge x')\to y]\wedge[(x\wedge x')\to y']\overset{lr-c_{22}}{=}(x\wedge x^{'})\to(y\wedge y^{'})$. □

2 Boolean center of a residuated lattice

Let $(L,\vee,\wedge,0,1)$ be a bounded lattice. Recall that an element $a\in L$ is called *complemented* if there is an element $b\in L$ such that $a\vee b=1$ and $a\wedge b=0$; if such element b exists it is called a *complement* of a. We will denote $b=a'$ and the set of all complemented elements in L by $B(L)$. Complements are generally not unique, unless the lattice is distributive.

In residuated lattices however, although the underlying lattices need not be distributive, the complements are unique.

Lemma 1. *([14]) Suppose that $a\in A$ have a complement $b\in A$. Then, the following hold:*

(i) If c is another complement of a in A, then $c=b$;

(ii) $a'=b$ and $b'=a$;

(iii) $a^2=a$.

Let $B(A)$ the set of all complemented elements of A.

Lemma 2. *If $e\in B(A)$, then $e'=e^*$ and $e^{**}=e$.*

Proof: If $e \in B(A)$, and $a = e'$, then $e \vee a = 1$ and $e \wedge a = 0$. Since $e \odot a \leq e \wedge a = 0$, then $e \odot a = 0$, hence $a \leq e \to 0 = e^*$. On the another hand, $e^* = 1 \odot e^* = (e \vee a) \odot e^* \overset{lr-c_{20}}{=} (e \odot e^*) \vee (a \odot e^*) = 0 \vee (a \odot e^*) = a \odot e^*$, hence $e^* \leq a$, that is, $e^* = a$. The equality $e^{**} = e$ follows from Lemma 1, (ii). □

Remark 4. ([14]) If $e, f \in B(A)$, then $e \wedge f, e \vee f \in B(A)$. Moreover, $(e \vee f)' = e' \wedge f'$ and $(e \wedge f)' = e' \vee f'$. So, $e \to f = e' \vee f \in B(A)$ and

$(lr - c_{33})$ $e \odot x = e \wedge x$, for every $x \in A$.

Corollary 2. *([14]) The set $B(A)$ is the universe of a Boolean subalgebra of A, called the Boolean center of A.*

Proposition 2. *For $e \in A$ the following are equivalent:*

(i) $e \in B(A)$,

(ii) $e \vee e^* = 1$.

Proof: $(i) \Rightarrow (ii)$. If $e \in B(A)$, by Lemma 2, $e \vee e' = e \vee e^* = 1$.

$(ii) \Rightarrow (i)$. Suppose that $e \vee e^* = 1$. We have: $0 = 1^* = (e \vee e^*)^* \overset{lr-c_{26}}{=} e^* \wedge e^{**} \geq e^* \wedge e$, (by $lr - c_{16}$), hence $e^* \wedge e = 0$, that is, $e \in B(A)$. □

Proposition 3. *For $e \in A$ we consider the following assertions:*

(1) $e \in B(A)$,

(2) $e^2 = e$ *and* $e = e^{**}$,

(3) $e^2 = e$ *and* $e^* \to e = e$,

(4) $(e \to x) \to e = e$, *for every* $x \in A$,

(5) $e \wedge e^* = 0$. *Then:*

(i) $(1) \Rightarrow (2), (3), (4)$ *and* (5),

(ii) $(2) \not\Rightarrow (1), (3) \not\Rightarrow (1), (4) \not\Rightarrow (1), (5) \not\Rightarrow (1)$.

Proof: (i). $(1) \Rightarrow (2)$. Follows from Lemma 1 (iii), and Lemma 2.

$(1) \Rightarrow (3)$. If $e \in B(A)$, then $e \vee e^* = 1$. Since $1 = e \vee e^* \leq [(e \to e^*) \to e^*] \wedge [(e^* \to e) \to e]$ (by $lr - c_6$ and $lr - c_1$), we deduce that $(e \to e^*) \to e^* = (e^* \to e) \to e = 1$, hence $e \to e^* \leq e^*$ and $e^* \to e \leq e$ (by $lr - c_4$), that is, $e \to e^* = e^*$ and $e^* \to e = e$ (by $lr - c_1$).

$(1) \Rightarrow (4)$. If $x \in A$, then from $0 \leq x$ we deduce $e^* \leq e \to x$ hence $(e \to x) \to e \leq e^* \to e = e$, by $(1) \Rightarrow (3)$. Since $e \leq (e \to x) \to e$ we obtain $(e \to x) \to e = e$.

$(1) \Rightarrow (5)$. Follows from Proposition 2 (since by Lemma 2, $e' = e^*$).

(ii). Consider the residuated lattice $A = \{0, a, b, c, 1\}$ from the Example 6; it is easy to verify that $B(A) = \{0, 1\}$.

(2) $\not\Rightarrow$ (1). We have $a^2 = a, a^* = b, b^* = a$, hence $a^{**} = b^* = a$, but $a \notin B(A)$.

(3) $\not\Rightarrow$ (1). We have $a^2 = a$ and $a^* \to a = b \to a = a$, but $a \notin B(A)$.

(4) $\not\Rightarrow$ (1). It is easy to verify that $(a \to x) \to a = a$ for every $x \in A$, but $a \notin B(A)$.

(5) $\not\Rightarrow$ (1). We have $a \wedge a^* = a \wedge b = 0$, but $a \vee a^* = a \vee b = c \neq 1$, hence $a \notin B(A)$. $\square$

Remark 5. ([7]) If A is a $BL-$ algebra, then all assertions from the above proposition are equivalent.

Lemma 3. *If* $e, f \in B(A)$ *and* $x, y \in A$, *then:*

$(lr - c_{34})$ $x \odot (x \to e) = e \wedge x, e \odot (e \to x) = e \wedge x;$

$(lr - c_{35})$ $e \vee (x \odot y) = (e \vee x) \odot (e \vee y);$

$(lr - c_{36})$ $e \wedge (x \odot y) = (e \wedge x) \odot (e \wedge y);$

$(lr - c_{37})$ $e \odot (x \to y) = e \odot [(e \odot x) \to (e \odot y)];$

$(lr - c_{38})$ $x \odot (e \to f) = x \odot [(x \odot e) \to (x \odot f)];$

$(lr - c_{39})$ $e \to (x \to y) = (e \to x) \to (e \to y).$

Proof: $(lr - c_{34})$. Since $e \leq x \to e$, then $x \odot e \leq x \odot (x \to e)$, hence $x \wedge e \leq x \odot (x \to e)$. From $x \odot (x \to e) \leq x, e$ we deduce the another inequality $x \odot (x \to e) \leq x \wedge e$, so $x \odot (x \to e) = e \wedge x$.

Analogous for the sequend equality.

$(lr - c_{35})$. We have

$$(e \vee x) \odot (e \vee y) \overset{lr-c_{20}}{=} [(e \vee x) \odot e] \vee [(e \vee x) \odot y] = [(e \vee x) \odot e] \vee [(e \odot y) \vee (x \odot y)]$$

$$= [(e \vee x) \wedge e] \vee [(e \odot y) \vee (x \odot y)] = e \vee (e \odot y) \vee (x \odot y) = e \vee (x \odot y).$$

$(lr - c_{36})$. As above,

$$(e \wedge x) \odot (e \wedge y) = (e \odot x) \odot (e \odot y) = (e \odot e) \odot (x \odot y) = e \odot (x \odot y) = e \wedge (x \odot y).$$

$(lr - c_{37})$. By $lr - c_8$ we have $x \to y \leq (e \odot x) \to (e \odot y)$, hence $e \odot (x \to y) \leq e \odot [(e \odot x) \to (e \odot y)]$. Conversely, $(e \odot x) \odot [(e \odot x) \to (e \odot y)] \leq e \odot y \leq y$ so $e \odot [(e \odot x) \to (e \odot y)] \leq x \to y$. Hence $e \odot [(e \odot x) \to (e \odot y)] \leq e \odot (x \to y)$.

$(lr - c_{38})$. We have $x \odot [(x \odot e) \to (x \odot f)] = x \odot [(x \odot e) \to (x \wedge f)] \overset{lr-c_{22}}{=} x \odot [(x \odot e \to x) \wedge (x \odot e \to f)] = x \odot [1 \wedge (x \odot e \to f)] = x \odot (x \odot e \to f) \overset{lr-c_{13}}{=}$ $\overset{lr-c_{13}}{=} x \odot [x \to (e \to f)] \overset{lr-c_{34}}{=} x \wedge (e \to f) = x \odot (e \to f)$, since $e \to f \in B(A)$, see Remark 4.

$(lr - c_{39})$. Follows from $lr - c_{13}$ and $lr - c_{34}$ since $e \wedge x = e \odot x$. $\square$

Definition 3. ([4])Let A and B be residuated lattices. A mapping $f : A \to B$ is a *morphism of residuated lattices* if f is morphism of bounded lattices and for every $x, y \in A : f(x \odot y) = f(x) \odot f(y)$ and $f(x \to y) = f(x) \to f(y)$.

3 Residuated lattice of fractions relative to a $\wedge-$ closed system

In this section, taking as a guide-line the case of rings (see [12]) we introduce for a residuated lattice A the notion of *residuated lattice of fractions relative to a* $\wedge$*-closed system* S. In particular if A is an MV-algebra (pseudo MV-algebra), BL-algebra, (pseudo BL-algebra) we obtain the results from [6], [7], [8] and [18] (see Remarks 9 and 10).

Definition 4. A nonempty subset $S \subseteq A$ is called $\wedge-$*closed system* in A if $1 \in S$ and $x, y \in S$ implies $x \wedge y \in S$.

If $\mathcal{P}$ is a prime ideal of the underlying lattice $L(A) = (A, \wedge, \vee)$ (that is, $\mathcal{P} \neq A$ and if $x, y \in A$ such that $x \wedge y \in \mathcal{P}$, then $x \in \mathcal{P}$ or $y \in \mathcal{P}$), then $S = A \backslash \mathcal{P}$ is a $\wedge-$closed system.

We denote by $S(A)$ the set of all $\wedge-$closed system of A (clearly $\{1\}, A \in S(A)$).

For $S \in S(A)$, on A we consider the relation θ_S defined by $(x, y) \in \theta_S$ iff there is $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$.

Lemma 4. *The relation* θ_S *is a congruence on* A.

Proof: The reflexivity (since $1 \in S \cap B(A)$) and the symmetry of θ_S are immediately. To prove the transitivity of θ_S, let $(x, y), (y, z) \in \theta_S$. Thus there are $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $y \wedge f = z \wedge f$. If denote $g = e \wedge f \in S \cap B(A)$, then $g \wedge x = (e \wedge f) \wedge x = (e \wedge x) \wedge f = (y \wedge e) \wedge f = (y \wedge f) \wedge e = (z \wedge f) \wedge e = z \wedge (f \wedge e) = z \wedge g$, hence $(x, z) \in \theta_S$.

To prove the compatibility of θ_S with the operations $\wedge, \vee, \odot$ and $\rightarrow$, let $x, y, z, t \in A$ such that $(x, y) \in \theta_S$ and $(z, t) \in \theta_S$. Thus there are $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $z \wedge f = t \wedge f$; we denote $g = e \wedge f \in S \cap B(A)$, see Remark 4.

We obtain:

$$(x \wedge z) \wedge g = (x \wedge z) \wedge (e \wedge f) = (x \wedge e) \wedge (z \wedge f) = (y \wedge e) \wedge (t \wedge f) = (y \wedge t) \wedge g,$$

hence $(x \wedge z, y \wedge t) \in \theta_S$ and

$$(x \vee z) \wedge g \overset{lr-c_{35}}{=} (x \vee z) \odot g \overset{lr-c_{20}}{=} (x \odot g) \vee (z \odot g) \overset{lr-c_{35}}{=} [(e \wedge f) \wedge x] \vee [(e \wedge f) \wedge z] =$$

$$= [(e \wedge x) \wedge f] \vee [e \wedge (f \wedge z)] = [(e \wedge y) \wedge f] \vee [e \wedge (f \wedge t)] =$$

$$= [(e \wedge f) \wedge y] \vee [(e \wedge f) \wedge t] \overset{lr-c_{35}}{=} (y \odot g) \vee (t \odot g) \overset{lr-c_{20}}{=} (y \vee t) \odot g \overset{lr-c_{30}}{=} (y \vee t) \wedge g.$$

hence $(x \vee z, y \vee t) \in \theta_S$.

By $lr - c_{35}$ we obtain:

$$(x \odot z) \wedge g = (x \odot z) \odot g = (x \odot e) \odot (z \odot f) = (x \wedge e) \odot (z \wedge f) = (y \wedge e) \odot (t \wedge f) =$$

$$= (y \odot e) \odot (t \odot f) = (y \odot t) \odot g = (y \odot t) \wedge g,$$

hence $(x \odot z, y \odot t) \in \theta_S$ and by $lr - c_{39}$:

$$(x \to z) \wedge g = (x \to z) \odot g = g \odot [(g \odot x) \to (g \odot z)] = g \odot [(g \wedge x) \to (g \wedge z)] =$$

$$= g \odot [(g \wedge y) \to (g \wedge t)] = g \odot [(g \odot y) \to (g \odot t)] = (y \to t) \odot g = (y \to t) \wedge g,$$

hence $(x \to z, y \to t) \in \theta_S$. □

For $x \in A$ we denote by x/S the equivalence class of x relative to θ_S and by $A[S] = A/\theta_S$. By $p_S : A \to A[S]$ we denote the canonical mapping defined by $p_S(x) = x/S$, for every $x \in A$. Clearly, $A[S]$ become a residuated lattice, where $\mathbf{0} = 0/S$, $\mathbf{1} = 1/S$ and for every $x, y \in A, x/S \wedge y/S = (x \wedge y)/S, x/S \vee y/S = (x \vee y)/S, x/S \odot y/S = (x \odot y)/S, x/S \to y/S = (x \to y)/S$. So, p_S is an onto morphism of residuated lattices.

Remark 6. Since for every $s \in S \cap B(A)$, $s \wedge s = s \wedge 1$ we deduce that $s/S = 1/S = \mathbf{1}$, hence $p_S(S \cap B(A)) = \{\mathbf{1}\}$.

Remark 7. If $S = \{1\}$ or S is such that $1 \in S$ and $S \cap (B(A) \backslash \{1\}) = \varnothing$, then for $x, y \in A, (x, y) \in \theta_S \iff x \wedge 1 = y \wedge 1 \iff x = y$, hence in this case $A[S] = A$.

Remark 8. If S is an $\wedge-$closed system such that $0 \in S$ (for example $S = A$ or $S = B(A)$), then for every $x, y \in A$, $(x, y) \in \theta_S$ (since $x \wedge 0 = y \wedge 0$ and $0 \in S \cap B(A)$), hence in this case $A[S] = \mathbf{0}$.

Proposition 4. *If $a \in A$, then $a/S \in B(A[S])$ iff there is $e \in S \cap B(A)$ such that $a \vee a^* \geq e$. So, if $e \in B(A)$, then $e/S \in B(A[S])$.*

Proof: For $a \in A$, we have by Proposition 2, $a/S \in B(A[S]) \Leftrightarrow a/S \vee (a/S)^* = 1 \Leftrightarrow (a \vee a^*)/S = 1/S$ iff there is $e \in S \cap B(A)$ such that $(a \vee a^*) \wedge e = 1 \wedge e = e \Leftrightarrow a \vee a^* \geq e$. If $e \in B(A)$, since $1 \in S \cap B(A)$ and $1 = e \vee e^* \geq 1$, we deduce that $e/S \in B(A[S])$. □

Theorem 3. *If A' is a residuated lattice and $f : A \to A'$ is an morphism of residuated lattices such that $f(S \cap B(A)) = \{\mathbf{1}\}$, then there is an unique morphism of residuated lattices $f' : A[S] \to A'$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{p_S} & A[S] \\ \searrow_{f} & & \swarrow_{f'} \\ & A' & \end{array}$$

is commutative (i.e. $f' \circ p_S = f$).

Proof: If $x, y \in A$ and $p_S(x) = p_S(y)$, then $(x, y) \in \theta_S$, hence there is $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$. Since f is morphism of residuated lattices, we obtain that $f(x \wedge e) = f(y \wedge e) \Leftrightarrow f(x) \wedge f(e) = f(y) \wedge f(e) \Leftrightarrow f(x) \wedge \mathbf{1} = f(y) \wedge \mathbf{1} \Leftrightarrow f(x) = f(y)$.

From this remark, we deduce that the mapping $f' : A[S] \to A'$ defined for $x \in A$ by $f'(x/S) = f(x)$ is correct defined. Clearly, f' is a morphism of residuated lattices. The unicity of f' follows from the fact that p_S is an onto mapping. □

Definition 5. Theorem 3 allows us to call $A[S]$ the *residuated lattice of fractions relative to the* $\wedge-$ *closed system* S.

Remark 9. If the residuated lattice A is a $BL-$ algebra (see Definition 2), then $x/S \wedge y/S = (x \wedge y)/S = (x \odot (x \to y))/S = x/S \odot (x/S \to y/S)$ and $(x/S \to y/S) \vee (y/S \to x/S) = ((x \to y) \vee (y \to x))/S = 1/S = \mathbf{1}$, hence $A[S]$ is a $BL-$ algebra. In this case, $A[S]$ is the BL-algebra of fractions relative to the $\wedge-$closed system S, and we obtain the results from [7]. Analogous if A is a pseudo $BL-$ algebra, so we obtain the result from [8].

Remark 10. If the residuated lattice A is a $BL-$ algebra and this is an $MV-$ algebra (i.e. $x^{**} = x$, for all $x \in A$), then $(x/S)^{**} = x^{**}/S = x/S$, hence $A[S]$ is an $MV-$ algebra. So, $A[S]$ is the MV-algebra of fractions relative to the $\wedge-$closed system S, and we obtain the results from [6]. Analogous if A is a pseudo $MV-$ algebra, so we obtain the result from [18].

Example 7. We consider $MV-$algebra $A = \{0, a, b, c, d, 1\}$ from [13]. The $\wedge-$closed systems of A which do not contain 0 are:

$$S = \{1\}, \{a, 1\}, \{b, 1\}, \{c, 1\}, \{d, 1\}, \{a, c, 1\}, \{b, c, 1\} \text{ and } \{b, c, d, 1\}.$$

In the cases $S = \{1\}, \{b, 1\}, \{c, 1\}, \{b, c, 1\}$, $A[S] = A$ (because $S \cap B(A) = \{1\}$, hence θ_S is the identity; see Remark 7). In the cases $S = \{a, 1\}, \{a, c, 1\}$ we obtain $0/S = b/S = d/S = \{0, b, d\}, 1/S = a/S = c/S = \{a, c, 1\}$so $A[S] \approx L_2$, and for $S = \{d, 1\}, \{b, d, 1\}, \{b, c, d, 1\}$ we obtain $0/S = a/S = \{0, a\}, b/S = c/S = \{b, c\}, d/S = 1/S = \{1, d\}$. In this case $A[S]$ is not a Boolean algebra because $b/S \oplus b/S = (b \oplus b)/S = d/S \neq b/S$.

Suppose now that $\mathcal{P}$ is a prime ideal of the underlying lattice $L(A)$. Then $\mathcal{P} \neq A$ and $S = A \backslash \mathcal{P}$ is a $\wedge-$closed system in A;we denote $A[S]$ by $A_{\mathcal{P}}$ and $I_{\mathcal{P}} = \{x/S : x \in \mathcal{P}\}$.

Lemma 5. *If* $x \in A$ *such that* $x/S \in I_{\mathcal{P}}$, *then* $x \in \mathcal{P}$.

Proof: If $x/S \in I_{\mathcal{P}}$, then $x/S = y/S$ with $y \in \mathcal{P} \Rightarrow$ there is $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e \leq y \Rightarrow x \wedge e \in \mathcal{P} \Rightarrow x \in \mathcal{P}$ (since $\mathcal{P}$ is prime and $e \in S = A \backslash \mathcal{P}$, hence $e \notin \mathcal{P}$). □

Proposition 5. *The set* $I_{\mathcal{P}}$ *is a proper prime ideal of the underlying lattice* $L(A_{\mathcal{P}})$.

Proof: If $x, y \in \mathcal{P}$, then $x/S \vee y/S = (x \vee y)/S \in A_{\mathcal{P}}$ (since $x \vee y \in \mathcal{P}$). Consider now $x \in \mathcal{P}$ and $y \in A$ such that $y/S \leq x/S$. Then $y/S \to x/S = 1/S \Leftrightarrow (y \to x)/S = 1/S \Leftrightarrow$ there is $e \in S \cap B(A)$ such that $e \wedge (y \to x) = e \wedge 1 = e$, hence $e \leq y \to x \Leftrightarrow e \odot y \leq x \Leftrightarrow e \wedge y \leq x$. Then $e \wedge y \in \mathcal{P}$, hence $y \in \mathcal{P}$, so $y/S \in I_{\mathcal{P}}$, that is, $I_{\mathcal{P}}$ is an ideal of $A_{\mathcal{P}}$.

If by contrary, $I_{\mathcal{P}} = A_{\mathcal{P}}$, then $1/S \in I_{\mathcal{P}}$, hence $1 \in \mathcal{P}$ (by Lemma 5) $\Leftrightarrow$ $\mathcal{P} = A$, a contradiction.

To prove that $I_{\mathcal{P}}$ is prime, let $x, y \in A$ such that $x/S \wedge y/S \in I_{\mathcal{P}}$. Then $(x \wedge y)/S \in I_{\mathcal{P}} \Rightarrow x \wedge y \in \mathcal{P}$, by Lemma 5 $\Rightarrow x \in \mathcal{P}$ or $y \in \mathcal{P} \Rightarrow x/S \in I_{\mathcal{P}}$ or $y/S \in I_{\mathcal{P}}$, hence $I_{\mathcal{P}}$ is a proper prime ideal in lattice $L(A_{\mathcal{P}})$. □

Remark 11. Following the model of commutative rings, the process of passing from A to $A_{\mathcal{P}}$ is called *localization* at $\mathcal{P}$ (taking as a guide-line the case of rings, see [12]).

References

[1] R. Balbes, Ph. Dwinger, *Distributive Lattices*, University of Missouri Press, 1974.

[2] T. S. Blyth, M. F. Janovitz, *Residuation Theory*, Pergamon Press, 1972.

[3] W. J. Blok, D. Pigozzi, Algebraizable Logics, Memoirs of the American Mathematical Society, No. 396, Amer. Math. Soc, Providence, 1989.

[4] S. Buris, H. P. Sankappanavar, *A Course in Universal Algebra*, Graduate Texts in Mathematics, No. 78, Springer, 1981.

[5] D. Buşneag, Hilbert algebra of fractions relative to an $\underline{\vee}$-closed system, Analele Universităţii din Craiova, Seria Matematică-Fizică-Chimie, vol. XVI, (1998), 34-38.

[6] D. Buşneag, D. Piciu, MV-algebra of fractions relative to an $\wedge$-closed system, Analele Universităţii din Craiova, Seria Matematică-Informatică, vol. XXX, (2003), 1-6.

[7] D. Buşneag, D. Piciu, BL-algebra of fractions relative to an $\wedge$-closed system, Analele Ştiinţifice ale Universităţii Ovidius, Constanţa, Seria Matematică , vol. XI, fascicola 1 (2003), 39-48.

[8] D. Buşneag, D. Piciu, Pseudo BL-algebra of fractions relative to an $\wedge$-closed system, Analele Ştiinţifice ale Universităţii A.I. Cuza din Iaşi, Tomul L, s. I. a f. 2, Matematică, (2004), 459-472.

[9] R. P. Dilworth, Non-commutative residuated lattices, Transactions of the American Mathematical Society, 46 (1939), 426-444.

[10] U. Höhle, Commutative residuated monoids, in: U. Höhle, P. Klement (eds), Non-classical Logics and Their Aplications to Fuzzy Subsets, Kluwer Academic Publishers, 1995.

[11] P. M. Idziak, Lattice operations in BCK-algebras, Mathematica Japonica, 29(1984), 839-846.

[12] I. D. ION, N. RADU, *Algebra* , (in romanian), Ed. Didactică şi Pedagogică, Bucureşti, 1991.

[13] A. IORGULESCU, Classes of BCK algebras-Part III, Preprint Series of The Institute of Mathematics of the Romanian Academy, nr.3 (2004), 1-37.

[14] T. KOWALSKI, H. ONO, *Residuated lattices: an algebraic glimpse at logic without contraction*, 2001.

[15] W. KRULL, Axiomatische Begründung der allgemeinen Ideal theorie, Sitzungsberichte der physikalisch medizinischen Societäd der Erlangen, 56 (1924), 47-63.

[16] M. OKADA, K. TERUI, The finite model property for various fragments of intuitionistic linear logic, Journal of Symbolic Logic, 64 (1999), 790-802.

[17] J. PAVELKA, On fuzzy logic II. Enriched residuated lattices and semantics of propositional calculi, Zeitschrift für mathematische Logik und Grundlagen der Mathematik, 25 (1979), 119-134.

[18] D. PICIU, Pseudo MV-algebra of fractions relative to an $\wedge$-closed system, Analele Universităţii din Craiova, Seria Matematică-Informatică, vol. XXX, (2003), 7-13.

[19] E. TURUNEN, *Mathematics Behind Fuzzy Logic*, Physica-Verlag, 1999.

[20] M. WARD, Residuated distributive lattices, Duke Mathematical Journal, 6 (1940), 641-651.

[21] M. WARD, R. P. DILWORTH, Residuated lattices, Transactions of the American Mathematical Society, 45 (1939), 335-354.

Received: 5.12.2005

Faculty of Mathematics and Computer Science,
University of Craiova,
13, Al.I. Cuza st., 200585,
Craiova, Romania
E-mail: `busneag@central.ucv.ro`
E-mail: `danap@central.ucv.ro`