Bull. Math. Soc. Sc. Math. Roumanie
Tome 49(97) No. 1, 2006, 55–60

Ultraproducts of transitive rings of linear transformations

by
ION. D. ION AND CONSTANTIN NIŢĂ

Abstract

If F is a nonprincipal ultrafilter on an infinite set I, then for any family $\{M_i\}_{i\in I}$ of modules $M_i \in R_i$-mod we have a natural immersion $\varphi : \left(\prod_{i\in I} End_{R_i}(M_i)\right)_F \to End_{R_F}(M_F)$ where $R_F = \left(\prod_{i\in I} R_i\right)_F$ and $M_F = \left(\prod_{i\in I} M_i\right)_F$ (theorem 1.1). Generally, φ is not an isomorphism as we can see in Examples 1.2 and 1.3. An ultraproduct of 2−transitive rings of linear transformations is a 2−transitive ring (theorem 2.2). As a consequence we obtain a classical result which says that the immersion φ in theorem 1.1 is an isomorphism in case each M_i is a simple faithful module (corollary 2.5). Finally we prove a result with applications in PI-theory: an ultraproduct of closed primitive rings is a closed primitive ring.

Key Words: Ultraproducts, rings of linear transformations, m-transitive rings, closed primitive rings.
2000 Mathematics Subject Classification: Primary: 03C20, Secondary: 03C60, 16S50, 15A04, 16D60.

1 Ultraproducts of rings of endomorphisms

Let F be a nonprincipal ultrafilter on an infinite set I. We denote by $R = \prod_{i\in I} R_i$ the direct product of the rings R_i, $i \in I$ and by $M = \prod_{i\in I} M_i$ the direct product of the modules $M_i \in R_i-$mod, $i \in I$. Clearly $M \in R-$mod and for an element $x \in M$, $x = (x_i)$ we write $Z(x) = \{i \in I \mid x_i = 0\}$.

If $Z_F(M) = \{x \in M \mid Z(x) \in F\}$ then $Z_F(M)$ is an $R-$submodule of M. Also, $Z_F(R) = \{a \in R \mid Z(a) \in F\}$ is a two-sided ideal of R. The quotient module $M_F = M/Z_F(M) = \{x_F \mid x_F = x + Z_F(M), x \in M\}$ is called the **ultraproduct** of the family $\{M_i\}_{i\in I}$ of modules. The quotient ring $R_F = R/Z_F(R) = \{a_F \mid a_F = a+Z_F(R), a \in R\}$ is called the **ultraproduct** of the family of rings $\{R_i\}_{i\in I}$.

Obviously $Z_F(R)M \subseteq Z_F(M)$ and $RZ_F(M) \subseteq Z_F(M)$. It follows that the application

$$R_F \times M_F \to M_F,\ (a_F, x_F) \mapsto a_F x_F = (ax)_F$$

is correctly defined and so M_F becomes an R_F-module.

If

$$M = \prod_{i\in I} M_i,\ N = \prod_{i\in I} N_i,$$

where $M_i, N_i \in R_i-$mod, then for every $f \in Hom_R(M,N)$ there exists $f_i \in Hom_R(M_i,N_i)$, $i \in I$ uniquely determined such that $f(x) = (f_i(x_i))$, $x \in M$, $x = (x_i)$. In this case we put $f = (f_i)$.

The map

$$f^* : M_F \to N_F, f^*(x_F) = (f_i(x_i))_F, x = (x_i) \in M$$

is correctly defined and $f^* \in Hom_{R_F}(M_F, N_F)$.

Clearly, f^* is the unique R_F-morphism from M_F to N_F such that the following diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow \varphi_M & & \downarrow \varphi_N \\ M_F & \xrightarrow{f^*} & N_F \end{array}$$

is commutative, where φ_M and φ_N are the canonical maps.

Theorem 1.1 *Let $(M_i)_{i\in I}$ be a family of modules $M_i \in R_i-$mod, $i \in I$ and $M = \prod_{i\in I} M_i$. If $E_i = End_{R_i}(M_i)$, $i \in I$ and $E = \prod_{i\in I} E_i$, then the map*

$$\rho : E \to End_{R_F}(M_F),\ \rho(f) = f^*,\ f = (f_i)$$

is a ring morphism and $ker\rho = Z_F(E)$.

In particular, the map

$$\tilde{\rho} : E_F \to End_R(M_F),\ \tilde{\rho}(f_F) = f^*$$

is an injective ring morphism.

Proof: Clearly $(f+g)^* = f^* + g^*$, $(f \circ g)^* = f^* \circ g^*$ and $(1_{M_i})^* = 1_{M_F}$ for any $f, g \in E$. So, ρ is a ring morphism.

Suppose that $f = (f_i) \in Z_F(E)$. Then $\{i \mid f_i = 0\} \in F$. It follows that $\{i \mid f_i(x_i) = 0\} \in F$ for any $x = (x_i) \in M$, so

$$f^*(x_F) = (f_i(x_i))_F = 0,$$

hence $Z_F(E) \subseteq Ker\rho$.

If $f \in E \setminus Z_F(E)$, $f = (f_i)$, then $A = \{i \mid f_i \neq 0\} \in F$. Choose $x^0 = (x_i^0) \in M$ such that $f(x_i^0) \neq 0$ for any $i \in A$. Then $f^*(x_F^0) = (f_i(x_i^0))_F \neq 0$, so $f^* \neq 0$. It follows that $Z_F(E) = ker\rho$. □

Remark. *Under adequate conditions on M_i, $i \in I$, $\tilde{\rho}$ is an isomorphism. For example, if the modules M_i, $i \in I$ are free with R_i-bases of bounded cardinality, then $\tilde{\rho}$ is an isomorphism ([2]). Also, $\tilde{\rho}$ is an isomorphism if each M_i, $i \in I$ is a simple faithful module. Now, we present two examples when ρ is not surjective: one in the category Ens of sets and the other one in the category of vector spaces.*

Example 1.2. For $n \in \mathbf{N}^*$ let $B_n = \{x_{n1}, x_{n2}, ..., x_{nn}\}$ be a set with n elements. Suppose that $B_m \bigcap B_n = \emptyset$ for any $m \neq n$. For each $k \in \mathbf{N}^*$, choose $x^{(k)} \in B = \prod_{n \geq 1} B_n$, $x^{(k)} = (x_n^{(k)})$, where

$$x_n^{(k)} = \begin{cases} x_{nn} \ , \ n \leq k \\ x_{nk} \ , \ n > k. \end{cases}$$

Let F be a nonprincipal ultrafilter on $\mathbf{N}^*$ and $S = \{x_F^{(1)}, x_F^{(2)}, ..., x_F^{(k)}, ...\}$. If $x = (x_{nn}) \in B$ then $x_F \in B_F$ and $x_F \notin S$, so S is a propre subset B_F.

The map $\xi : B_F \to B_F$, defined via $\xi(x_F^{(k)}) = x_F^{(2k)}$ for every $k \in \mathbf{N}^*$ and $\xi(x_F) = x_F$ for any $x_F \in B_F \setminus S$ is injective but not surjective.

Suppose that for $\xi \in End_{Ens}(B_F)$ there exists $f = (f_n) \in \prod_{n \geq 1} End_{Ens}(B_n) = E$ such that $f^* = \xi$. Then f^* is an injective map. It follows that $A = \{n \in \mathbf{N}^* \mid f_n \text{ an injective map}\} \in F$. Indeed, if $A \notin F$ then $C = \mathbf{N}^* \setminus A \in F$. Choose $u, v \in B$, $u = (u_n)$, $v = (v_n)$ such that $u_n \neq v_n$ and $f_n(u_n) = f_n(v_n)$ for all $n \in C$. We have $u_F \neq v_F$, an $f^*(u_F) = f^*(v_F)$, contradiction. So, $A \in F$ and f_n is bijective for any $n \in A$. It follows that $\xi = f^*$ is bijective, contradiction. Hence the map $\rho : E \to End_{Ens}(B_F)$, given by $\rho(f) = f^*$, is not surjective.

Example 1.3. For $n \in \mathbf{N}^*$ we choose an $n-$dimensional vector space V_n over a field K_n with a K_n-basis $B_n = (x_{n1}, x_{n2}, ..., x_{nn})$.

If F is a nonprincipal ultrafilter on $\mathbf{N}^*$, $V = \prod_{n \geq 1} V_n$ and $K = \prod_{n \geq 1} K_n$, then V_F is a vector space over the field K_F. As in Example 1.2., we consider the subset $S = \{x_F^{(1)}, x_F^{(2)}, ..., x_F^{(1)}, ...\}$ of V_F. The elements of S are linearly independent vectors over K_F.Indeed, suppose that

$$a_F^{(1)} x_F^{(1)} + a_F^{(2)} x_F^{(2)} + ... + a_F^{(q)} x_F^{(q)} = 0,$$

with $q \in \mathbf{N}^*$, $a_F^{(k)} \in K_F$, $a^{(k)} = (a_n^{(k)})$, $a_n^k \in K_n$.

Because the ultrafilter F on $\mathbf{N}^*$ contains the co-finite subsets of $\mathbf{N}^*$ the set

$$A = \left\{ n \mid \sum_{k=1}^{q} a_n^{(k)} x_{nk} = 0 \right\} \cap \{q, q+1, ...\}$$

belongs to F. Since the elements of B_n are linearly independent vectors over K_n we have $a_n^{(k)} = 0$ for any $n \in A$ and $1 \leq k \leq q$. It follows that $a_F^{(k)} = 0$, $1 \leq k \leq q$, so the elements of S are linearly independent vectors over K_F. Let B be a K_F-basis of V_F such that $S \subseteq B$. There exists $\xi \in End_{K_F}(V_F)$ such that $\xi(x_F^{(k)}) = x_F^{(2k)}$ for any $k \in \mathbf{N}^*$ and $\xi(x_F) = x_F$ for any $x_F \in B \setminus S$.

Clearly ξ is an injective but not surjective map.

If $f = (f_n) \in \prod_{n \geq 1} End_{K_n}(V_n)$ is such that $f^* = \xi$, then the set $C = \{n \mid f_n$ is injective$\}$ belongs to F. Because $dim_{K_n} V_n < \infty$, f_n is a bijective map for $n \in C$, so $\xi = f^*$ is bijective, contradiction. Hence the map $\rho : \prod_{n \geq 1} End_{K_n}(V_n) \to End_{K_n}(V_n)$, $\rho(f) = f^*$ is not surjective.

2 Ultraproducts of transitive rings of linear transformations

If D is a division ring and V a right vector space over D, then a subring R of $End(V_D)$ is called a **ring of linear transformations** on V_D. In this case, we can view V as a left $R-$module via $rv = r(v) \in V$ for any $r \in R$ and $v \in V$. Because $(rv)d = r(vd)$ for any $r \in R$, $v \in V$ and $d \in D$, V has a bimodule structure. If $m \in \mathbf{N}^*$, $m \leq dim(V_D)$, we say that R is $m-$**transitive** on V if for any set of $n \leq m$ linearly independent vectors $v_1, v_2, ..., v_n$ and any other set of n vectors $v'_1, v'_2, ..., v'_n$ in V, there exists $r \in R$ such that $r(v_i) = v'_i$ for all i. Clearly if R acts $1-$transitive on V, then ${}_RV$ is simple an hence $ann_R V = 0$, R is a (left) primitive ring.

If $d \in D$ and $\theta_d : V \to V$, $\theta_d(v) = vd$, then $\theta_d \in End({}_RV)$. The map $D \to End({}_RV)$, $d \mapsto \theta_d$ is an injective ring morphism, so $D \subseteq End({}_RV)$.

This inclusion may be proper. Indeed, if $D = \mathbf{R}, V = \mathbf{R}^2$ and $f \in End(V_R)$ is such that $f(e_1) = e_2$, $f(e_2) = -e_1$ where (e_1, e_2) is the standard basis, then for the subring R of $End({}_RV)$ generated by f we have $End({}_RV) \simeq \mathbf{C} \supset \mathbf{R}$.

We have the following well-known result:

Lemma 2.1. ([1]) *For a ring R of linear transformations on V_D, $dim(V_D) \geq 2$, the following assertions are equivalent:*

(1) R is $2-$transitive.

(2) ${}_RV$ is a simple faithful module and $End({}_RV) \simeq D$

(3) R is a $n-$transitive for any $n \in \mathbf{N}^*$, $n \leq dim(V_D)$.

Theorem 2.2. *Suppose that $\{D_i\}_{i \in I}$ is a family of division rings. For any $i \in I$ let R_i be a $2-$transitive ring of linear transformations of a right vector space*

V_{D_i}, $dim(V_{D_i}) \geq 2$. If $V = \prod_{i\in I} V_i$, $R = \prod_{i\in I} R_i$ and $D = \prod_{i\in I} D_i$, then for every nonprincipal ultrafilter F in I, R_F is a $2-$transitive ring of linear transformations of the right vector space V_F over the division ring D_F.

Proof: We have $dim_{D_F}(V_F) \geq 2$. Indeed, for any $i \in I$ let W_i be a $2-$dimensional subspace of V_i and put $W = \prod_{i\in I} W_i$. Then W_F is a D_F-subspace of V_F and $dim_{D_F}(W_F) = 2$ ([2]). It follows, that $dim_{D_F}(D_F) \geq 2$.

Assume that u_F, v_F, u'_F, v'_F are vectors in V_F such that $ind_{D_F}(u_F, v_F)$ which means, that the vectors u_F and v_F are linearly independent over D_F. It suffice to show that there exists an element $a_F \in R_F$ such that $a_F u_F = u'_F$ and $a_F v_F = v'_F$. We may assume that $u_i \neq 0$ and $v_i \neq 0$ for any $i \in I$. Consider the set

$$A = \{i \in I \mid ind_{D_i}(u_i, v_i)\}.$$

If $A \notin F$, then $B = I \setminus A \in F$. For any $i \in B$ there exists $\alpha_i, \beta_i \in D_i \setminus \{0\}$ such that $\alpha_i u_i + \beta_i v_i = 0$, so $v_i = \gamma_i u_i$, with $\gamma_i \in D_i$. Since $B \in F$, we have $v_F = \gamma_F u_F$ with $\gamma_F \in D_F$, where $\gamma = (\gamma_i)$ with γ_i arbitrary in D_i if $i \in A$, contradiction.

So $A \in F$. For any $i \in A$ there exists $a_i \in D_i$ such that $a_i u_i = u'_i$ and $a_i v_i = v'_i$. Put $a = (a_i) \in R$ with a_i arbitrarily taken in R_i if $i \notin A$. Then $a_F u_F = u'_F$ and $a_F v_F = v'_F$. It follows that R_F is a $2-$transitive ring of linear transformations on V_F. □

Lemma 2.3. *Let $(R_i)_{i\in I}$ be a family of unitary rings. If for any $i \in I$, M_i is a left R_i-module, then*

$$ann_{R_F}(M_F) = \left(\prod_{i\in I} ann_{R_i}(M_i)\right)_F$$

for any nonprincipal ultrafilter on I.

Proof: If $a_F \notin \left(\prod_{i\in I} ann_{R_i}(M_i)\right)_F$, then $\{i \mid a_i \in ann_{R_i}(M_i)\} \in F$. We have $a_F x_F = (ax)_F = 0$ for any $x_F \in M_F$, hence $a_F \in ann_{R_F}(M_F)$.

If $a_F \notin ann_{R_F}(M_F)$ there exists $x^0_F \in M_F$, $x^0 = (x^0_i)$ such that $a_F x^0_F \neq 0$. Then $\{i \mid a_i x^0_i \neq 0\} \in F$, hence $a_F \notin \left(\prod_{i\in I} ann_{R_i}(M_i)\right)_F$. □

Corollary 2.4. *Let $(R_i)_{i\in I}$ be a family of unitary rings. If for any $i \in I$, M_i is a left faithful module, then M_i is a left faithful R_F-module.*

With the aid of Lemma 2.1., theorem 2.2. and the Jacobson-Chevalley density theorem we can prove:

Corollary 2.5. ([1]) *Let $(R_i)_{i\in I}$ be a family of left primitive rings. For any $i \in I$ let M_i be a left simple faithful left $R-$module. Then for any nonprincipal ultrafilter F on I we have:*

(1) R_F is left primitive and M_F is a simple faithful R_F-module.

(2) $End_{R_F}(M_F) \simeq \left(\prod_{i\in I} End_{R_i}(M_i)\right)_F$.

We say that a primitive ring R is **closed** if there exists a simple faithful left $R-$module such that $End_R(M) \simeq Z(R)$, where $Z(R)$ is the center of R.

If M is a left simple faithful module and K is a maximal subfield of the division ring $D = End_R(M)$, then $R \subseteq End_K(M)$, $K \subseteq End_K(M)$, $M$is a simple faithful $RK-$ module, and $Z(RK) = K \simeq End_{RK}(M)$. It follows that RK is closed primitive ring and $RK \supset R$. ([4])

Theorem 2.6. *An ultraproduct of closed primitive rings is a closed primitive ring.*

Proof: Let $(R_i)_{i\in I}$ a family of closed primitive rings. For each $i \in I$ let M_i be a simple faithful R_i-module such that $End_{R_i}(M_i) \simeq Z(R_i)$.

If F is a nonprincipal ultrafilter on I, then M_F is a simple faithful R_F-module and

$$End_{R_F} \simeq \left(\prod_{i\in I} End_{R_i}(M_i)\right)_F \simeq \left(\prod_{i\in I} Z(R_i)\right)_F \simeq Z(R_F).$$

□

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Received: 15.12.2005

Faculty of Mathematics, University of Bucharest,
Str. Academiei 14, Bucharest,
RO-010014, Romania
E-mail: idi@al.math.unibuc.ro
E-mail: cnita@al.math.unibuc.ro