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An infinite product of formal power series

by
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To Professor Ion D. Ion on the occasion of his 70th Birthday

Abstract

We extend the construction given by the first author which produces SFT domains with non-SFT power series extension. We also extend a technique initiated by Kang and Park, to produce integral domains D with $D[[X]][K]$ being infinite-dimensional, where K is the quotient field of D.

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Let D be a domain with Krull dimension n. It is a classical result that $n+1 \leq \dim D[X] \leq 2n+1$ (e.g., [6, Corollary 30.3]). The analogue problem for formal power series is much more difficult and still far from being completely resolved.

In 1973, J. Arnold showed that if $\dim R[[X]] < \infty$ then $\dim R < \infty$ and R is of strong-finite-type (SFT), a near Noetherian property. We recall that an ideal I of a ring R is SFT if there exist a finitely generated ideal $B \subseteq I$ and a natural number N such that $z^n \in J$ for each $z \in I$. A ring R is SFT if every (prime) ideal of R is SFT. As it turns out, the converse of thus statement is not true.

Indeed, the first author of the present paper constructed in [4] an example of a one-dimensional quasi-local SFT domain W with quotient field K such that $W[[X]]$ is not SFT and such that $W[[X]][K]$ is infinite-dimensional. Recently, there has been much work done in an attempt to understand $\dim R[[X]]$ in terms of $\dim R$ (and, more generally, the relationship between Spec $R[[X]]$ and Spec R).

In [5], Kang and Park obtained some strinking results concerning Spec $V[[X]]$ where V is a nondiscrete valuation domain. As a tool, a notion of "infinite product" in power series rings was developed.

In this short note we obtain some generalizations of the results of [4] and [5] by looking at some properties of this infinite product. We show that the Kang-Park product has a natural topological interpretation. We prove that if B is an

Archimedean domain which does not satisfy the ascending chain condition for the principal ideals, then $A[[X]]$ is not an SFT domain for each subring A of B with $A : B \neq 0$ (Theorem 1).

We also prove that if D is a characteristic zero quasi-local domain having a strictly ascending chain of nonzero principal ideals with certain additional properties, then $D[[X]]$ is infinite-dimensional (Theorem 3).

Finally, we prove the if D is domain which has a sequence of discrete valuations $(v_n)_{n\geq 1}$ and a sequence of elements $(g_i)_{i\geq 1}$ such that $v_n(g_i) = n^i$ for $i, n \geq 1$, then D has an infinite strictly descending chain of prime ideals (Proposition 4).

Throughout this note, all rings are commutative with identity, usually integral domains. Our terminology and notation will follow that of [6].

We begin by recalling the definition of the infinite product of formal power series as given by Kang and Park in [5]. Let D be an integral domain, $D[[X]]$ the power series ring with coefficients in D and let $(f_n)_{n\geq 1}$ be a sequence of powers series. Assume that $(f_n)_{n\geq 1}$ satisfies the following two conditions:

(a) $(f_n)_{n\geq 1}$ is *echelon*, that is, $\lim\limits_{n\to\infty} \operatorname{ord}(f_n - f_n(0)) = \infty$,

(b) there exists a nonzero element $a \in \bigcap\limits_{n\geq 1} f_1(0)\cdots f_n(0)D$.

In particular, every f_n has nonzero constant term. Now, consider the sequence of power series $(g_n)_{n\geq 1}$, $g_n \in D[[X]]$, given by

$$g_n = \frac{a}{f_1(0)\cdots f_n(0)} f_1 \cdots f_n.$$

For every $n \geq 2$,

$$g_n - g_{n-1} = \frac{a}{f_1(0)\cdots f_n(0)} f_1 \cdots f_{n-1}(f_n - f_n(0)).$$

Since $(f_n)_{n\geq 1}$ is echelon, it follows that $(g_n)_{n\geq 1}$ is a Cauchy sequence in the (X)-adic topology of $D[[X]]$, so it has a unique limit in $D[[X]]$. Set

$$(\prod_{n=1}^{\infty} f_n;\ a) = \lim_{n\to\infty} g_n.$$

Since $g_n(0) = a$ for each $n \geq 1$, it follows that $(\prod_{n=1}^{\infty} f_n;\ a)$ has constant term a. We shall call $(\prod_{n=1}^{\infty} f_n;\ a)$ *the product of the series* $(f_n)_{n\geq 1}$ *with constant term* a. Note also that the coefficients of $(\prod_{n=1}^{\infty} f_n;\ a)$ belong to the ideal $\bigcup\limits_{n\geq 1} (a/f_1(0)\cdots f_n(0))D$.

The product $(\prod_{n=1}^{\infty} f_n;\ a)$, first developed by Kang and Park [5], has a nice multiplicative behavior. Indeed,

$$(\prod_{n=1}^{\infty} f_n;\ a) = \lim_{n\to\infty} \frac{a}{f_1(0)\cdots f_n(0)} f_1\cdots f_n =$$

$$= f_1 \lim_{n\to\infty} \frac{a/f_1(0)}{f_2(0)\cdots f_n(0)} f_2\cdots f_n = f_1(\prod_{n=2}^{\infty} f_n;\ \frac{a}{f_1(0)})$$

so, iterating, we get for each $m \geq 1$

$$(\prod_{n=1}^{\infty} f_n;\ a) = f_1\cdots f_m(\prod_{n=m+1}^{\infty} f_n;\ \frac{a}{f_1(0)\cdots f_m(0)}).$$

Let R be a commutative ring. In [2], J. Arnold showed that if $R[[X]]$ has finite (Krull) dimension, then R has finite dimension and R is a SFT ring. Recall that R is said to be a SFT ring if for each ideal I of R, there exist $n \geq 1$ and a finitely generated ideal $J \subseteq I$ such that $x^n \in J$ for each $x \in I$.

Subsequently, there has been a great amount of work to see if the following two conjectures are true

(1) a finite-dimensional SFT ring R has necessarily $R[[X]]$ of finite dimension,

(2) whether R being SFT implies $R[[X]]$ being SFT.

The first author of the present paper denied both giving a single counterexample. Indeed, using the infinite product of power series presented above, he constructed an example of a one-dimensional quasi-local SFT domain W whose power series ring $W[[X]]$ is not SFT and such that $W[[X]][K]$ is infinite-dimensional, where K is the quotient field of W (see [4]).

We show that the proof given in [4] works in a more general situation. Recall that a domain D is called an *Archimedean domain* if $\bigcap_{n\geq 1} a^n D = 0$ for every nonunit a of D.

Theorem 1. *Let B be an Archimedean domain which does not satisfy the ascending chain condition for the principal ideals. Then $A[[X]]$ is not an SFT domain for each subring A of B with $A : B \neq 0$.*

Proof: Let $(b_nB)_{n\geq 0}$ be a strictly ascending chain of nonzero principal ideals of B. Set $a = b_0$ and $a_n = b_{n-1}/b_n$, $n \geq 1$. For $n, p \geq 1$ we have $b_{n-1} = a_n a_{n+1}\cdots a_{n+p} b_{n+p}$ for $n \geq 1$, so $b_{n-1} \in \bigcap_{p\geq 1} a_n a_{n+1}\cdots a_{n+p}B$. For $n \geq 1$, define

$$g_n = (\prod_{i=n}^{\infty}(X^i + a_i);\ b_{n-1}).$$

The coefficients of g_n belong to the ideal

$$J = \bigcup_{p\geq 1} \frac{b_{n-1}}{a_n a_{n+1}\cdots a_{n+p}} B = \bigcup_{p\geq 1} b_{n+p}B = \bigcup_{p\geq 1} b_p B.$$

For every $n \geq 1$, $g_n = (X^n + a_n)g_{n+1}$. Hence $I = \bigcup_{n\geq 1} g_n B[[X]]$ is an ideal of $B[[X]]$ contained in $J[[X]]$. Let $\pi : B[[X]] \to B$ be the canonical projection given by $\pi(f) = f(0)$. Then $\pi(I) = J$.

Let $0 \neq q \in J$. As $\pi(I) = J$, $q = h_0 + q_1 X$ for some $h_0 \in I$ and $q_1 \in B[[X]]$. Then $q_1 \in J[[X]]$, so $q_1 = h_1 + q_2 X$ for some $h_1 \in I$ and $q_2 \in B[[X]]$. Thus $q = h_0 + h_1 X + q_2 X^2$. Continuing in this way, we find a sequence of elements $h_n \in I$ such that $q = \sum_{n=0}^{\infty} h_n X^n$.

Now assume that A is a subring of B with $A : B \neq 0$ such that $A[[X]]$ is an SFT domain. Let $0 \neq y \in A : B$. We get, $qy = \sum_{n=0}^{\infty}(h_n y)X^n$ with each $h_n y$ in $A[[X]]$. As $A[[X]]$ is an SFT domain, the proof of [3, Theorem 3.7] shows that there exists some $k \geq 1$ with $(qy)^k \in \sum_{n=0}^{\infty}(h_n y)A[[X]] \subseteq I$. So there exists some $m \geq 1$ such that $(qy)^k \in g_m B[[X]] \subseteq (X^m + a_m)B[[X]]$. Hence $0 \neq (qy)^k \in \bigcap_{n\geq 1}^{\infty} a_m^n B$, which is a contradiction because B is Archimedean.
□

We retrieve [4, Theorem 4.1] as the following corollary.

Corollary 2. *Let V be a rank-one nondiscrete valuation ring containing a field K and let x a nonzero noninvertible element of V. Then $W = K + xV$ is an SFT domain whose power series extension $W[[X]]$ is not SFT.*

Proof: xV is the only prime ideal of W and $a^2 \in xW$ for each $a \in xV$. Hence W is an SFT domain. For the rest of the proof, it suffices to note that V is an Archimedean domain which does not satisfy the ascending chain condition for the principal ideals and $0 \neq x \in W : V$. The previous theorem applies. □

In [5], Kang and Park showed that for any rank-one nondiscrete valuation domain V with quotient field K, the generic fiber $V[[X]][K]$ of $V[[X]]$ over V is infinite-dimensional. We show that their proof works in a more general situation.

Theorem 3. *Let D be a characteristic zero quasi-local domain with quotient field K. Assume there exist a strictly ascending chain $(b_n D)_{n\geq 1}$ of nonzero principal ideals such that:*

(a) for every $n \geq 1$, $X - b_n$ is a prime element of $D[[X]]$ and $\bigcap_{k\geq 1} b_n^k D = 0$,

(b) for every $i \geq 1$, $\bigcap_{n\geq 1} b_1^{1^i} b_2^{2^i} \cdots b_n^{n^i} D \neq 0$.

Then the generic fiber $D[[X]][K]$ of $D[[X]]$ over D is infinite-dimensional.

Proof: Fix an $n \geq 1$. Let $D_n = D[[X]]/(X - b_n)D[[X]]$. It is well-known that D_n is the b_n-adic completion of D. Let $g(b_n)$ denote the image of $g \in D[[X]]$ in D_n. By (a), the canonical morphism $D \to D_n$ is injective and $\bigcap_{k\geq 1}(X - b_n)^k D[[X]] = 0$.

Let v_n be the $(X-b_n)$-adic discrete valuation on $D[[X]]$, that is, if $0 \neq f \in D[[X]]$, $v_n(f)$ is the greatest power of $X-b_n$ which divides f. In particular, $v_n(g)=0$ if and only if $g(b_n)\neq 0$.

Fix an $i \geq 1$ and take $0 \neq a_i \in \bigcap_{k\geq 1} b_1^{1^{i+1}} b_2^{2^{i+1}} \cdots b_k^{k^{i+1}} D$. Define

$$g_i = (\prod_{m=1}^{\infty}(X^m - b_m^m)^{m^i};\ a_i).$$

If we set $c_n = \pm a_i/(b_1^{1^{i+1}}\cdots b_n^{n^{i+1}})$, then

$$g_i = (X-b_1)^{1^i}\cdots(X^n-b_n^n)^{n^i}h \quad \text{with} \quad h = (\prod_{m=n+1}^{\infty}(X^m - b_n^m)^{m^i};\ c_n).$$

For every $d \in D\setminus\{b_n\}$ and $p \geq 1$ we have $v_n(X^p - d^p) = 0$. Since D has characteristic zero, we have $v_n(X^p - b_n^p) = 1$. So

$$v_n((X-b_1)^{1^i}\cdots(X^n-b_n^n)^{n^i}) = n^i.$$

We claim that $v_n(h) = 0$, so $v_n(g_i) = n^i$. Suppose that $v_n(h)\neq 0$, that is, $h(b_n) = 0$. For $m \geq n+1$, we have $b_n = b_m c_m$ with c_m nonunit of D. So $(b_n^m - b_m^m)^{m^i} = b_m^{m^{i+1}} u_m$ for some unit element $u_m \in D$. By definition, $h = \lim_{p\to\infty} h_p$, where

$$h_p = \frac{\pm a_i}{b_1^{1^{i+1}}\cdots b_{n+p}^{(n+p)^{i+1}}}(X^{n+1}-b_{n+1}^{n+1})^{(n+1)^i}\cdots(X^{n+p}-b_{n+p}^{n+p})^{(n+p)^i}.$$

As $h(b_n) = 0$, we get $\lim_{p\to\infty} h_p(b_n) = 0$ in the b_n-adic topology of D. But $h_p(b_n) = \pm c_n u_{n+1}\cdots u_{n+p}$ and since $u_{n+1},\ldots,u_{n+p}$ are units of D, we get that $\lim_{p\to\infty} c_n = 0$. Hence $c_n \in \bigcap_{k\geq 1} b_n^k D = 0$, which is a contradiction. So $v_n(g_i) = n^i$ for $i,n \geq 1$. Now, we apply the following proposition which seems interesting in itself. □

Proposition 4. *Let $A \subseteq B$ be an extension of domains and let K denote the quotient field of A. Let $(v_n)_{n\geq 1}$ be a sequence of discrete valuations on B which are trivial on A and $(g_i)_{i\geq 1}$ a sequence of elements of B. Assume that for every $i \geq 1$, $v_n(g_i) = n^i$ for all but finitely many $n \geq 1$. Then $B[K]$ has an infinite strictly descending chain of prime ideals.*

Proof: For each $i \geq 1$, let I_i be the set of all $f \in B$ such that $v_n(f) \geq n^i$ for all but finitely many $n \geq 1$. It is easy to see that $(I_i)_{i\geq 1}$ is a descending chain of ideals of B.

Let P be a minimal prime ideal of some I_i. Then P is lying over zero in A. Indeed, let $a \in A \bigcap P$. Since $\sqrt{I_i D_P} = PB_P$, we get $a^j s \in I_i$ for some $j \geq 1$ and

$s \in B \setminus P$. Then $v_n(a^j s) = v_n(a^j) + v_n(s) = v_n(s) \geq n^i$ for almost all $n \geq 1$, because v_n is trivial on A. So $s \in I_i \subseteq P$, which is a contradiction.

To complete the proof, it suffices to show that for each $i \geq 1$, there is no prime ideal which is simultaneously minimal over I_i and I_{i+1}. Because, once we know this, we can take a minimal prime P_1 over I_1, then take P_2 a minimal prime over I_2, $P_2 \subset P_1$, and so on, giving an infinite descending chain of prime ideals in $B[K]$.

Assume that P is a prime ideal which is simultaneously minimal over I_i and I_{i+1}. Then $\sqrt{I_i D_P} = PB_P = \sqrt{I_{i+1} D_P}$. Note that $g_i \in I_i$. So there exist $k \geq 1$ and $s \in B \setminus P$ with $g_i^k s \in I_{i+1}$. Then $n^{i+1} \leq v_n(g_i^k s) = kn^i + v_n(s)$ for almost all $n \geq 1$. So $v_n(s) \geq n^{i+1} - kn^i = n^i(n-k) \geq n^i$ for almost all $n \geq 1$. Hence $s \in I_i \subseteq P$, which is a contradiction. □

Remark 5. The well-known theorem of M. Laplaza asserting that the ring E of entire functions has an infinite strictly descending chain of prime ideals (see [6, page 146]), can be derived from Proposition 4. Indeed, for $a \in \mathbf{C}$ and $f \in E$, denote by $ord_a(f)$ the order of multiplicity of a as a zero of f. Then ord_a is a discrete valuation on E. Fix an integer $i \geq 1$. By Weierstrass' Theorem, there exists $g_i \in E$ such that $ord_n(g_i) = n^i$ for all $n \geq 1$. Proposition 4 applies.

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