Bull. Math. Soc. Sc. Math. Roumanie
Tome 49(97) No. 1, 2006, 37–46

(m, n)-purity for modules

by
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To Professor Ion D. Ion on the occasion of his 70th Birthday

Abstract

We consider (m, n)-purity for modules and show that the main properties of purity may be refined for (m, n)-purity. We give connections with (m, n)-injectivity and (n, m)-flatness of modules.

Key Words: (m, n)-purity, (m, n)-injective module, (n, m)-flat module.
2000 Mathematics Subject Classification: Primary: 16D80.

1 Introduction

Purity in module categories and its generalizations is a topic present in the literature since the 1960s, with early work by P.M. Cohn [2], B. Maddox [7], A.P. Mishina and L.A. Skornjakov [8], B. Stenström [10] and R.B. Warfield Jr. [12], to mention just few of them. Its importance became clear in the years to come, not only in module theory, but also in related fields such as model theory [9] or the theory of locally finitely presented categories [3].

In this note we consider a generalization of the purity in the sense of P.M. Cohn [2] for a short exact sequence of modules by asking for its exactness when tensored by any (n, m)-presented module. We give some basic properties and see that (m, n)-purity coincides with $\mathcal{P}$-purity in the sense of R. Wisbauer [13, p.274], where $\mathcal{P}$ is the class consisting of all (m, n)-presented modules. We characterize (m, n)-injectivity and (n, m)-flatness, introduced and studied in [1] and [14], in terms of (m, n)-purity.

Throughout m and n are non-zero natural numbers, R is an associative ring with non-zero identity and R-Mod is the category of left R-modules. By a homomorphism we mean an R-homomorphism. The injective hull of a left R-module A is denoted by $E(A)$. A right R-module M is called (n, m)-*presented* if there exists an exact sequence $R^m \to R^n \to M \to 0$ of right R-modules, or equivalently, there exists an exact sequence $0 \to K \to R^n \to M \to 0$ of right R-modules with K m-generated. Clearly, every (n,m)-presented right R-module is finitely presented.

2 (m,n)-purity

Definition 2.1. An exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{1}$$

of left R-modules is called *(m,n)-pure* if the functor $M \otimes_R -$ is left exact with respect to (1) for every (n,m)-presented right R-module M.

That is, (1) induces an exact sequence

$$0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

for every (n,m)-presented right R-module M.

In this case, f is called an *(m,n)-pure monomorphism*, g an *(m,n)-pure epimorphism* and $\mathrm{Im} f$ an *(m,n)-pure submodule* of B.

Remarks. Let $f : A \to B$ be a monomorphism.

(a) f is pure if and only if f is (m,n)-pure for every $m,n \geq 1$.

(b) f is RD-pure if and only if f is $(1,1)$-pure.

(c) A monomorphism which is $(m,1)$-pure for every $m \geq 1$ was called c-pure in [4], whereas a monomorphism which is $(1,n)$-pure for every $n \geq 1$ was called F/U-pure in [5].

(d) Let $m' \geq m$, $n' \geq n$. If f is (m',n)-pure, then it is (m,n)-pure. Also, if f is (m,n')-pure, then it is (m,n)-pure. In particular, (m,n)-purity implies RD-purity.

(e) (m,n)-purity and purity are the same over Prüfer domains. This follows by the above remarks and by the fact that RD-purity and purity are the same over Prüfer domains [12].

For a left R-module M, denote by M^* its character module, that is, $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \bar{\mathbb{Q}})$, where $\bar{\mathbb{Q}} = \mathbb{Q}/\mathbb{Z}$. Then the exact sequence (1) induces the exact sequence $0 \to C^* \to B^* \to A^* \to 0$ of right R-modules.

The main characterizations of (m,n)-purity are given in the following theorem. The proof is adapted after [11, Chapter I, Proposition 11.2] and [13, 34.5].

Theorem 2.2. *Consider the exact sequence* (1). *The following are equivalent:*

(i) The sequence (1) *is* (m,n)*-pure.*

(ii) The sequence $0 \to C^* \to B^* \to A^* \to 0$ *is* (n,m)*-pure.*

(iii) The functor $\mathrm{Hom}_R(M,-)$ *is exact with respect to* (1) *for every* (m,n)*-presented left* R*-module* M*.*

(iv) For every commutative diagram of left R-modules

$$\begin{array}{ccccc} & & R^n & \xrightarrow{q} & R^m \\ & & {\scriptstyle p}\downarrow & & \downarrow \\ 0 & \longrightarrow & A & \longrightarrow & B \end{array}$$

there exists a homomorphism $h : R^m \to A$ such that $p = hq$.

(v) Every system

$$\sum_{j=1}^{m} r_{ij} x_j = a_i$$

with $r_{ij} \in R$, $a_i \in A$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, with n equations and m unknowns which is solvable in B is already solvable in A.

Proof: $(v) \Longrightarrow (i)$ Let M be an (n, m)-presented left R-module, so that there is an exact sequence $R^m \xrightarrow{\alpha} R^n \to M \to 0$. We get the following commutative diagram with exact rows

$$\begin{array}{ccccccc} R^m \otimes_R A & \xrightarrow{\alpha \otimes 1_A} & R^n \otimes_R A & \longrightarrow & M \otimes_R A & \longrightarrow & 0 \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} & & \\ R^m \otimes_R B & \xrightarrow{\alpha \otimes 1_B} & R^n \otimes_R B & \longrightarrow & M \otimes_R B & \longrightarrow & 0 \end{array}$$

where u, v are monomorphisms.

By [11, Chapter I, Lemma 11.3], w is a monomorphism if and only if

$$\operatorname{Im}(\alpha \otimes 1_B) \cap \operatorname{Im} v = \operatorname{Im}(v(\alpha \otimes 1_A)).$$

We prove this last equality. The homomorphism α is given by an $n \times m$-matrix (r_{ij}) with entries in R. Then $z \in \operatorname{Im}(\alpha \otimes 1_B) \cap \operatorname{Im} v$ if and only if $z = \sum_{j=1}^{m} (r_{ij}) \otimes x_j \in R^n \otimes A$ is the image of an element $(x_1, \ldots, x_m) \in B^m \cong R^m \otimes_R B$ and the image of an element $(a_1, \ldots, a_n) \in A^n \cong R^n \otimes_R A$. This happens when $\sum_{j=1}^{m} r_{ij} x_j = a_i$ for $i = 1, \ldots, n$, that is, the system has a solution in B. By hypothesis, this is equivalent to the existence of a solution of the system in A, that is, z is the image of an element from $A^m \cong R^m \otimes_R A$, hence $z \in \operatorname{Im}(v(\alpha \otimes 1_A))$.

Now w is a monomorphism, showing that the sequence (1) is (m, n)-pure.

$(i) \Longrightarrow (v)$ The $n \times m$-matrix (r_{ij}) determines a homomorphism $\alpha : R^m \to R^n$. Take $M = R^n / \operatorname{Im} \alpha$ and reverse the proof for $(v) \Longrightarrow (i)$.

$(ii) \Longleftrightarrow (iii)$ Let M be an (m, n)-presented left R-module. Since M is finitely presented and $\bar{\mathbb{Q}}$ is injective, we have the isomorphism

$$\operatorname{Hom}_{\mathbb{Z}}(D, \bar{\mathbb{Q}}) \otimes_R M \cong \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_R(M, D), \bar{\mathbb{Q}})$$

for every left R-module D [13, 25.5]. Then the sequence

$$0 \to C^* \otimes_R M \to B^* \otimes_R M \to A^* \otimes_R M \to 0$$

is exact if and only if the sequence

$$0 \to (\mathrm{Hom}_R(M,C))^* \to (\mathrm{Hom}_R(M,B))^* \to (\mathrm{Hom}_R(M,A))^* \to 0$$

is exact if and only if the sequence

$$0 \to \mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C) \to 0$$

is exact, because $\bar{\mathbb{Q}}$ is an injective cogenerator. This shows the equivalence.

(iii) $\Longleftrightarrow$ *(iv)* By Homotopy Lemma [13, Lemma 7.16].

(iv) $\Longleftrightarrow$ *(v)* The homomorphisms p and q from *(iv)* are determined by the $a_i \in A$ and by an $n \times m$-matrix (r_{ij}) with entries in R respectively. Any solution of the system in B yields a homomorphism $h : R^m \to B$. Now the conclusion is immediate. □

Theorem 2.3. *Let R be commutative, Q be an R-module and consider the exact sequences*

$$0 \to A \to B \to C \to 0 \qquad (1)$$

$$0 \to \mathrm{Hom}_R(C,Q) \to \mathrm{Hom}_R(B,Q) \to \mathrm{Hom}_R(A,Q) \to 0 \qquad (2)$$

(i) If Q is injective and (1) *is (m,n)-pure, then* (2) *is (n,m)-pure.*

(ii) If Q is a cogenerator of R-Mod and (2) *is (n,m)-pure, then* (1) *is (m,n)-pure.*

Proof: Follow the proof of Theorem 2.2. □

Recall that for a non-empty class $\mathcal{P}$ of left R-modules, an exact sequence (1) is called *$\mathcal{P}$-pure* if the functor $\mathrm{Hom}_R(P,-)$ is exact with respect to (1) for every $P \in \mathcal{P}$ [13, p.274]. By Theorem 2.2, we see that (m,n)-purity means $\mathcal{P}$-purity, where the class $\mathcal{P}$ consists of all (m,n)-presented left R-modules. Hence all the general properties of $\mathcal{P}$-purity hold in our case. In what follows we discuss some specific ones.

Let us characterize (m,n)-pure sequences of modules in the case of a commutative ring R.

Theorem 2.4. *Let R be commutative. The following are equivalent:*

(i) The sequence (1) *is (m,n)-pure.*

(ii) The exact sequence

$$0 \to \mathrm{Hom}_R(C,E(S)) \to \mathrm{Hom}_R(B,E(S)) \to \mathrm{Hom}_R(A,E(S)) \to 0$$

is (n,m)-pure for every simple R-module S.

Proof: $(i) \Longrightarrow (ii)$ By Theorem 2.3.

$(ii) \Longrightarrow (i)$ Let $(S_i)_{i\in I}$ be a representative set of isomorphism classes of simple R-modules. Let M be an (m,n)-presented R-module. For an R-module X, denote $X_i = \mathrm{Hom}_R(X, E(S_i))$. Now apply the functor $M \otimes_R -$ to each exact sequence $0 \to C_i \to B_i \to A_i \to 0$, take the direct product of the obtained sequences and use the isomorphism

$$M \otimes_R (\prod_{j\in J} D_j) \cong \prod_{j\in J} (M \otimes_R D_j),$$

which holds for any finitely presented R-module M and any family $(D_j)_{j\in J}$ of R-modules [11, Chapter I, Lemma 13.2], to get the exact sequence

$$0 \to M \otimes_R (\prod_{i\in I} C_i) \to M \otimes_R (\prod_{i\in I} B_i) \to M \otimes_R (\prod_{i\in I} A_i) \to 0$$

which shows that the exact sequence

$$0 \to \prod_{i\in I} \mathrm{Hom}_R(C, E(S_i)) \to \prod_{i\in I} \mathrm{Hom}_R(B, E(S_i)) \to \prod_{i\in I} \mathrm{Hom}_R(A, E(S_i)) \to 0$$

is (n,m)-pure. Then the exact sequence

$$0 \to \mathrm{Hom}_R(C, D) \to \mathrm{Hom}_R(B, D) \to \mathrm{Hom}_R(A, D) \to 0$$

is (n,m)-pure, where $D = \prod_{i\in I} E(S_i)$ is a cogenerator of R-Mod. Now by Theorem 2.3, the exact sequence $0 \to A \to B \to C \to 0$ is (m,n)-pure. □

Recall that a monomorphism $f : A \to B$ is called *locally split* if for every $a \in A$, there exists a homomorphism $g : B \to A$ such that $g(f(a)) = a$. It is known that every locally split monomorphism is pure [6, p.163].

Theorem 2.5. *Let A be a submodule of the left R-module R^m. Then the following are equivalent:*

(i) A is (m,n)-pure in R^m.

(ii) The inclusion $i : A \to R^m$ is locally split.

(iii) A is pure in R^m.

Proof: $(i) \Longrightarrow (ii)$ Let $a_1 \in A$. Also, let $a_2, \dots, a_n \in A$ and let $\{e_1, \dots, e_n\}$ be a basis of R^n. Define the homomorphisms $p : R^n \to A$ by $p(e_k) = a_k$ for $k = 1, \dots, n$ and $q : R^n \to R^m$ by $q(e_k) = a_k$ for $k = 1, \dots, n$. Now by Theorem 2.2, there exists a homomorphism $h : R^m \to A$ such that $hq = p$. Then $h(a_1) = h(q(e_1)) = p(e_1) = a_1$. Thus i is locally split.

$(ii) \Longrightarrow (iii)$ By [6, p.163].

$(iii) \Longrightarrow (i)$ Clear. □

3 (m,n)-pure-injectivity

Definition 3.1. A left R-module M is called (m,n)*-pure-injective* (respectively (m,n)*-pure-projective*) if is injective (respectively projective) with respect to every (m,n)-pure exact sequence of left R-modules.

Clearly, every (m,n)-pure-injective (respectively (m,n)-pure-projective) left R-module is pure-injective (respectively pure-projective).

Theorem 3.2. $\mathrm{Hom}_{\mathbb{Z}}(M,\bar{\mathbb{Q}})$ *is an* (m,n)*-pure-injective left* R*-module for every* (n,m)*-presented right* R*-module* M.

Proof: Let M be an (n,m)-presented right R-module and denote

$$X = \mathrm{Hom}_{\mathbb{Z}}(M,\bar{\mathbb{Q}}).$$

Considering an (m,n)-pure exact sequence (1), we have the exact sequence

$$0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

Since D is injective, we have the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R A,\bar{\mathbb{Q}}) \to \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R B,\bar{\mathbb{Q}}) \to \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R C,\bar{\mathbb{Q}}) \to 0$$

By the adjunction we get the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(C,X) \to \mathrm{Hom}_{\mathbb{Z}}(B,X) \to \mathrm{Hom}_{\mathbb{Z}}(A,X) \to 0$$

Thus $X = \mathrm{Hom}_{\mathbb{Z}}(M,\bar{\mathbb{Q}})$ is (m,n)-pure-injective. □

Theorem 3.3. *The following are equivalent:*

(i) The sequence (1) *is* (m,n)*-pure.*

(ii) Every (m,n)*-pure-injective left* R*-module is injective with respect to the sequence* (1).

(iii) $\mathrm{Hom}_{\mathbb{Z}}(M,\bar{\mathbb{Q}})$ *is injective with respect to the sequence* (1) *for every* (n,m)*-presented right* R*-module* M.

Proof: $(i) \Longrightarrow (ii)$ Clear.

$(ii) \Longrightarrow (iii)$ By Theorem 3.2.

$(iii) \Longrightarrow (i)$ Let M be an (n,m)-presented right R-module and denote $X = \mathrm{Hom}_{\mathbb{Z}}(M,\bar{\mathbb{Q}})$. By hypothesis, we have the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(C,X) \to \mathrm{Hom}_{\mathbb{Z}}(B,X) \to \mathrm{Hom}_{\mathbb{Z}}(A,X) \to 0$$

By the adjunction we get the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R A,\bar{\mathbb{Q}}) \to \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R B,\bar{\mathbb{Q}}) \to \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R C,\bar{\mathbb{Q}}) \to 0$$

whence we get the exact sequence

$$0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

because $\bar{\mathbb{Q}}$ is a cogenerator. Thus the sequence (1) is (m,n)-pure. □

Remark. Theorem 3.3 still holds if one replaces $\mathbb{Z}$ by a commutative ring R and $\bar{\mathbb{Q}}$ by an injective cogenerator Q of R-Mod.

Theorem 3.4. *Every left R-module is an (m,n)-pure submodule of an (m,n)-pure-injective left R-module.*

Proof: By Theorem 3.3 and the dual of [10, Proposition 2.3]. □

For a left R-module N we denote by $M_{kl}(N)$ the set of formal $k \times l$-matrices with entries in N. Also, for $K \subseteq M_{hk}(N)$, denote by $\mathrm{r}_{M_{kl}(N)}(K)$ the right annihilator of K in $M_{kl}(N)$. If N is a left R-module and K is an m-generated submodule of the left R-module R^n, then there is an isomorphism

$$\varphi : \mathrm{r}_{M_{n1}(N)}(K) \to \mathrm{Hom}_R(R^n/K, N)$$

given by $\varphi(u)(r+K) = ru$ for $u \in \mathrm{r}_{M_{n1}(N)}(K)$ and $r \in R^n$ [14, p.151].

Theorem 3.5. *Let R be commutative. The following are equivalent:*

(i) The sequence (1) is (m,n)-pure.

(ii) For every m-generated submodule K of the R-module R^n and for every simple R-module S, $\mathrm{r}_{M_{n1}(E(S))}(K)$ is injective with respect to (1).

Proof: Consider the cogenerator $Q = \prod_{i\in I} E(S_i)$ of R-Mod, where $(S_i)_{i\in I}$ is a representative set of isomorphism classes of simple R-modules. By the remark following Theorem 3.3, the sequence (1) is (m,n)-pure if and only if $\prod_{i\in I} \mathrm{Hom}_R(R^n/K, E(S_i)) \cong \mathrm{Hom}_R(R^n/K, Q)$ is injective with respect to (1) for every m-generated submodule K of the R-module R^n. But this holds if and only if $\mathrm{r}_{M_{n1}(E(S))}(K) \cong \mathrm{Hom}_R(R^n/K, E(S))$ is injective with respect to (1) for every simple R-module S. □

4 On (m,n)-injective and (m,n)-flat modules

Now let us give characterizations of (m,n)-injectivity and (m,n)-flatness in terms of (m,n)-purity.

Definition 4.1. *[1]* A left R-module A is called *(m,n)-injective* if every homomorphism from an n-generated submodule I of the left R-module R^m to A extends to a homomorphism from R^m to A.

Theorem 4.2. *The following are equivalent for a left R-module A:*

(i) A is (m,n)-injective.

(ii) Every exact sequence of left R-modules $0 \to A \to B \to C \to 0$ is (m,n)-pure.

(iii) There exists an (m,n)-pure exact sequence of left R-modules $0 \to A \to B \to C \to 0$ with B (m,n)-injective (injective).

Proof: By [14, Proposition 2.3], A is (m,n)-injective if and only if every exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with C (m,n)-presented splits. But this condition is equivalent to *(ii)* and *(iii)* by [13, 35.1]. □

Corollary 4.3. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of left R-modules with B (m,n)-injective. Then the sequence is (m,n)-pure if and only if A is (m,n)-injective.*

By Theorem 4.2, we see that (m,n)-injectivity means absolute $\mathcal{P}$-purity in the sense of R. Wisbauer [13, p.297], where the class $\mathcal{P}$ consists of all (m,n)-presented left R-modules.

Definition 4.4. *[14]* A left R-module C is called *(n,m)-flat* if $i \otimes_R 1_C : I \otimes_R C \to R^n \otimes_R C$ is a monomorphism for every m-generated submodule I of the right R-module R^n.

Note that a left R-module C is (n,m)-flat if and only if the right R-module C^* is (n,m)-injective by [14, Theorem 4.3]. We use this property for giving a direct proof of the following result.

Theorem 4.5. *The following are equivalent for a left R-module C:*

(i) C is (n,m)-flat.

(ii) Every exact sequence of left R-modules $0 \to A \to B \to C \to 0$ is (m,n)-pure.

(iii) There exists an (m,n)-pure exact sequence of left R-modules $0 \to A \to B \to C \to 0$ with B (n,m)-flat (projective, free).

Proof: *(i)* $\Longrightarrow$ *(ii)* Since C is (n,m)-flat, C^* is (n,m)-injective. By Theorem 4.2, the exact sequence $0 \to C^* \to B^* \to A^* \to 0$ is (n,m)-pure. Now by Theorem 2.2 the exact sequence $0 \to A \to B \to C \to 0$ is (m,n)-pure.

(ii) $\Longrightarrow$ *(iii)* Clear.

(iii) $\Longrightarrow$ *(i)* Since the sequence $0 \to A \to B \to C \to 0$ is (m,n)-pure, the exact sequence $0 \to C^* \to B^* \to A^* \to 0$ is (n,m)-pure by Theorem 2.2. But B is (n,m)-flat, hence B^* is (n,m)-injective. Now by Corollary 4.3, it follows that C^* is (n,m)-injective, so that C is (n,m)-flat. □

Corollary 4.6. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of left R-modules with B (n,m)-flat. Then the sequence is (m,n)-pure if and only if C is (n,m)-flat.*

By Theorem 4.5, we see that (n,m)-flatness means $\mathcal{P}$-flatness in the sense of R. Wisbauer [13, p.304], where the class $\mathcal{P}$ consists of all (m,n)-presented left R-modules.

Now let us give a couple of results related to some coherence properties. Recall that a ring R is called left *(n,m)-coherent* if every m-generated submodule of the left R-module R^n is finitely presented [14, p.156].

Theorem 4.7. *(i) Suppose that every n-generated submodule of R^m is (m,n)-presented. Then for every (m,n)-pure exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with B (m,n)-injective, also C is (m,n)-injective.*

(ii) Suppose that for every (m,n)-pure exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with B (m,n)-injective, also C is (m,n)-injective. Then R is left (n,m)-coherent.

Proof: (i) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an (m,n)-pure exact sequence of left R-modules with B (m,n)-injective. Let I be an n-generated submodule of R^m and let $\alpha : I \to C$ be a homomorphism. We have the diagram

$$\begin{array}{ccccc} & 0 & \longrightarrow & I & \xrightarrow{\ i\ } & R^m \\ & & & \Big\downarrow{\scriptstyle \alpha} & & \\ & B & \xrightarrow{\ g\ } & C & \longrightarrow & 0 \end{array}$$

where i is the inclusion. Since I is (m,n)-presented by hypothesis and the epimorphism g is (m,n)-pure, by Theorem 2.2 there exists a homomorphism $\beta : I \to B$ such that $g\beta = \alpha$. Now by the (m,n)-injectivity of B, there exists a homomorphism $\gamma : R^m \to B$ such that $\gamma i = \beta$. Then $g\gamma i = \alpha$, showing that C is (m,n)-injective.

(ii) In order to prove that R is left (n,m)-coherent, by [14, Theorem 5.7] it is enough to show that every direct limit of (n,m)-injective modules is (n,m)-injective. Let $(L_i, f_{ij})_I$ be a direct system of (n,m)-injective modules and denote by L its direct limit. Then $\oplus_I L_i$ is (n,m)-injective. Since the canonical epimorphism $\oplus_I L_i \to L$ is pure, hence (n,m)-pure, we deduce that L is (n,m)-injective by hypothesis. □

Theorem 4.8. *(i) Suppose that every m-generated submodule of R^n is (m,n)-presented. Then for every (m,n)-pure exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with B (n,m)-injective, also C is (n,m)-injective.*

(ii) Suppose that for every (m,n)-pure exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with B (n,m)-injective, also C is (n,m)-injective. Then R is left (n,m)-coherent.

Proof: Similar to the proof of Theorem 4.7. □

Note added in proof. It may be possible that some of the present results have been established in: Z. Zhu, J. Chen, X. Zhang, *On (m,n)-purity of modules*, East-West J. Math. **5** (2003), No. 1, 35–44, paper unaccessible to the authors.

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Received: 9.12.2005

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