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Note on self-injective modules having finite Goldie dimension

by
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To Professor Ion D. Ion on the occasion of his 70th Birthday

Abstract

A module M_R is called em co-Hopf if every monomorphism $M \rightarrowtail M$ is an isomorphism. We prove that a self-injective module with finite Goldie dimension is co-Hopf.

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Let R be a ring with identity and denote by M_R an unital right R-module. Recall that the module M_R has *finite Goldie dimension* if M_R has no infinite independent family of non-zero submodules. M_R is said to be a *co-Hopf module* if every monomorphism $M \rightarrowtail M$ is an isomorphism. For all undefined notation and terminology on modules, the reader is referred to [2].

The aim of this note is to prove that every self-injective module with finite Goldie dimension is a co-Hopf module. In order to do this, we shall need the following well-known result, due to Jacobson (see, e.g., [1, p.9]).

Lemma 1. *Let R be a ring and let $a, b \in R$ such that $ab = 1$ and $ba \neq 1$. Consider the elements*

$$e_{ij} = b^i a^j - b^{i+1} a^{j+1}, \quad i, j \in \mathbb{N}.$$

Then, for every $i, j, k, l \in \mathbb{N}$ one has

$$e_{ij} e_{kl} = \delta_{jk} e_{il} ,$$

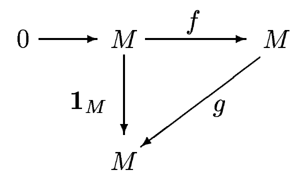
where δ_{jk} are the Kronecker deltas.

This Lemma enable us to give a simple proof to the announced statement.

Theorem 2. *Let R be a ring and M_R an unital right R-module. If M is self-injective and has finite Goldie dimension, then M is a co-Hopf module.*

Proof: Let $M \overset{f}{\rightarrowtail} M$ be a monomorphism. We have to prove that f is an isomorphism.

Since M is self-injective, there exists $g \in \mathrm{End}\,(M_R)$ such that the following diagram is commutative.



Set $S = \mathrm{End}\,(M_R)$. Thus $f, g \in S$ and $gf = 1$. Suppose that $fg \neq 1$. Then, by Lemma 1, there are some endomorphisms $e_{ij} \in S$, $i, j \in \mathbb{N}$, such that $e_{ij}e_{kl} = \delta_{jk}e_{il}$, for all $i, j, k, l \in \mathbb{N}$. Notice that $e_{ij} \neq 0$, for all i and j, by construction. Set $e_i = e_{ii}$. Then

$$e_ie_j = \delta_{ij}e_i \quad \forall\, i, j \in \mathbb{N}.$$

Consider now the submodules $M_i = e_i(M)$ of M, for $i \in \mathbb{N}$. We claim that the family $\{\, M_i \mid i \in \mathbb{N}\}$ is independent. To see this, let $k \in \mathbb{N}$ and let $x \in M_k \cap \sum_{i \neq k} M_i$. Thus $x = x_1 + \ldots + x_s$, where $x_1 \in M_{i_1}, \ldots x_s \in M_{i_s}$ for some natural numbers $i_1, \ldots i_s \neq k$. It follows that $e_ke_{i_j} = 0$ and so, $e_k(M_{i_j}) = e_ke_{i_j}(M) = 0$, for all $1 \leq j \leq s$. Consequently, $e_k(x_j) = 0$, for all $1 \leq j \leq s$. Hence $e_k(x) = e_k(x_1) + \ldots + e_k(x_s) = 0$. On the other hand, since $e_ke_k = e_k$, we see that e_k acts on $M_k = e_k(M)$ as identity, so $e_k(x) = x$. Thus $x = 0$. It follows that $M_k \cap \sum_{i \neq k} M_i = 0$ for every $k \in \mathbb{N}$, that proves our claim.

We obtain that $\{\, M_i \mid i \in \mathbb{N}\}$ is an infinite independent family of submodules of M. This is a contradiction to our hypothesis that M has finite Goldie dimension. Therefore, the supposition that $fg \neq 1$ is false. So $fg = 1$ and, consequently, f is an isomorphism. This ends the proof. □

References

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[2] R. Wisbauer, *Foundations of Module and Ring Theory.* Gordon and Breach Science Publishers, Reading, 1991.

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