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Semiprime graded rings of finite support

by
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To Professor Ion D. Ion on the occasion of his 70th Birthday

Abstract

The paper gives necessary and sufficient condition as a graded ring of finite support to be semiprime.

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1 Introduction.

Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a group graded ring. The graded ring R is said to be gr-semiprime if the intersection of all gr-prime ideals is zero, i.e. the graded prime radical,$rad_g(R)$, is zero. This is equivalent to the property that R has no nonzero nilpotent graded ideals. We are interested in the following general problem: When a gr-semiprime ring is semiprime? In [5] Fisher and Montgomery showed that if R is semiprime and it has no $|G|$-torsion then the skew group ring is semiprime. Lorenz and Passman (see [9], [10]) extended this result for crossed product $S = R * G$ and they obtained a result on the radical prime of S for the case that R has $|G|$ - torsion. For a finite group G and a ring R graded by G, it was shown in [13, Theorem 6.7], that if R has no n-torsion, where $n = |G|$, then $rad(R) = rad_g(R)$. In this paper, we will study the problem for G- graded rings of finite support, where G is an arbitrary group. The theory of graded rings of finite support has been investigated in several papers (cf.[2, 4, 12, 14]), where it has been shown that this theory does not coincide with the theory of finite group gradings. The main result (Theorem 7) extends the Lorenz and Passman's result to G-graded rings of finite support. Our proof uses different methods, in particular the graded Clifford theory is an essential ingredient, but the strategy is similar to the paper [9].

2 Preliminaries.

All rings considered in this paper will be unitary. If R is a ring, by an R module we will mean a left R - module, and we will denote the category of R - modules by $R - Mod$. Let G be a group with identity element 1, $R = \bigoplus_{\sigma \in G} R_\sigma$ a G - graded ring. The category of graded R modules will be denoted by $R - gr$. It is well known that $R - gr$ is a Grothendieck category. A graded ideal I is graded prime if whenever $JK \subseteq I$ for J, K graded ideals of R, then $J \subseteq I$ or $K \subseteq I$. The graded prime radical $rad_g(R)$ is the intersection of all graded prime ideals of R.We denote by $rad(R)$ the prime radical of R, i.e. the intersection of all prime ideals of R. A graded ring R is gr-semiprime if and only if the intersection of all gr-prime ideals is zero, i.e. $rad_g(R) = 0$. This is equivalent of the property that R has no nonzero niplotent graded ideals. If $M = \bigoplus_{\sigma \in G} M_\sigma$ is a left graded R module, we define the support of M by $supp(M) = \{\sigma \in G / M_\sigma \neq 0\}$. If a graded R module M has the property that $supp(M)$ is a finite set, then we say that M is a graded module of finite support, and we write $supp(M) < \infty$. We refer to [15] for all the definitions and basic properties of graded rings and modules.

3 Graded semiprime rings.

Proposition 1. ([3,Proposition 1.2.]) Let R be a graded ring of finite support. Suppose that $|supp(R)| = n$.

1. If A is a subring of R with $A_1 = 0$, then $A^n = 0$.
2. If A is a left (or right) ideal of R_1 with $A^d = 0$, then $(RA)^{nd} = 0$.
3. If R is gr-semiprime, then R_1 is semiprime.

We denote by $J^g(R)$ the graded Jacobson radical and by $J(R)$ the usual Jacobson radical.

Proposition 2. *Let R be a graded ring of finite support and $|supp(R)| = n$. Then:*

1. $J^g(R) \subseteq J(R)$.
2. $J(R)^n \subseteq J^g(R)$.

Proof: (1) By [12, Proposition 4.6].

(2) Follows from [12, Corollary 4.4].

□

Corollary 3. *Let R be a gr-semiprime graded ring of finite support. If R_1 is semiprimitive, then R is gr-semiprimitive.*

Proof: Indeed $J^g(R) \cap R_1 = J(R_1)$. Since $J(R_1) = 0$, then $J^g(R_l) = 0$. By Proposition 1, $J^g(R)^n = 0$. Hence $J^g(R) = 0$ since R is gr-semiprime. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a G-graded ring. If we put $R[x]_\sigma = R_\sigma[x]$, we obtain that $R[x] = \bigoplus_{\sigma \in G} R_\sigma[x]$ is a G-graded ring. □

Proposition 4. *Let R be a G - graded ring. If R is gr-semiprime (resp. gr-prime), then $R[x]$ is gr-semiprime (resp. gr-prime).*

Proof: Let $f = a_0^\sigma + a_1^\sigma x + ... + a_n^\sigma x^n \in R_\sigma[x]$ a nonzero homogeneous element. It is necessary to show that $(a_0^\sigma + a_1^\sigma x + ... + a_n^\sigma x^n)R[x](a_0^\sigma + a_1^\sigma x + ... + a_n^\sigma x^n) \neq 0$. If it is zero then $a_0^\sigma R a_0^\sigma = 0$ and $a_0^\sigma = 0$. Since x is a nonzero divizor on $R[x]$ we obtain that $(a_1^\sigma + ... + a_n^\sigma x^{n-1})R[x](a_1^\sigma + ... + a_n^\sigma x^{n-1}) \neq 0$. From this $a_1^\sigma R a_1^\sigma = 0$ and $a_1^\sigma = 0$. Continuing in this manner we obtain that the homogeneous element f is zero. Analogously we prove the case when R is gr-prime. □

Recall that a graded ideal I of a graded ring R is called gr-nil if every homogeneous element of I is nilpotent.

Lemma 5. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring of finite support. If I is nonzero graded ideal of R such that is gr-nil ideal of bounded index, then R contains a nonzero nilpotent graded ideal $J \subseteq I$.*

Proof: Let I be a gr-nil ideal of R. If $I_1 = 0$, then I is nilpotent by Proposition 1. Otherwise, I_1 is a nil ideal of R_1 of bounded index and by a theorem of Levitzki [8] there exists a nonzero nilpotent left ideal $A \subseteq I_1$. By Proposition 1, RA is a nonzero nilpotent graded left ideal. □

Let R be a graded ring of finite support. We denote $\widehat{R} = \prod_N R / \sum_N R$. Since R has finite support then $\widehat{R}$ is a G-graded ring of finite support with the grading $\widehat{R}_\sigma = \prod_N R_\sigma / \sum_N R_\sigma$, for every $\sigma \in G$.

Proposition 6. *Let R be a graded ring of finite support. The following statement are equivalent:*

i) R is gr-semiprime

ii) $\widehat{R}$ has no nonzero gr-nil graded ideals

iii) $\widehat{R}$ is gr-semiprime.

Proof: (This is a graded version of [9, Lemma 5])

i) $\Rightarrow$ *ii)* Assume H to be a nonzero gr-nil graded ideal in R. Take an homogenous element $0 \neq x \in H$. We write $x = (x_i) + \bigoplus R_i$, where the x_i's are homogeneous of the same degree as x ($deg\ x = deg\ x_i$) when $x_i \neq 0$. Since x

is nonzero, the set $A = \{i \in I/x_i \neq 0\}$ is infinite. Suppose now that for any $i \in A$, Rx_i is not a gr-nil graded ideal of bounded index. We find an homogenous element r_i such that $(r_i x_i)^i \neq 0$. Since R is finite support, there exists an infinite set of homogeneous element r_i of the same degree such that $(r_i x_i)^i \neq 0$. We take the element in $\widehat{R}$, $y = (y_i) + \bigoplus R_i$, where $y_i = r_i$ and zero elsewhere. The element $yx \in H$ is homogeneous and not nilpotent. This is a contradiction. Hence there exists an i such that Rx_i is a gr-nil graded ideal of bounded order. By Lemma 5, R is not gr-semiprime.

$ii) \Rightarrow iii)$ This is clear because if $rad_g(\widehat{R}) \neq 0$, this is a gr-nil graded ideal.

$iii) \Rightarrow i)$ If H is a nonzero nilpotent graded ideal of R, then

$$\widehat{H} = \prod_N H/\sum_N H$$

is a nilpotent graded ideal of $\widehat{R}$, contradiction. □

Theorem 7. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring of finite support with $n = |supp(R)|$. Assume that R is gr-semiprime.*

(1)If for any finite subgroup H of G with $|H| \leq n$, R is $|H|$- torsionfree group, then R is semiprime.

(2)In any case R has a unique maximal nilpotent ideal N such that $N^n = 0$.

Proof: In the hypothesis of (1) we can assume that every $|H|$ with $|H| \leq n$ is invertible in R (passing to certain ring of fractions).

We consider first the case when R_1 is gr-semiprimitive. By Proposition 2 and Corollary 3 we have $J(R)^n = 0$. In the hypothesis of (1), let $\sum$ be a gr-simple left R-module, then we have an epimorphism

$$R(\sigma) \to \sum \to 0$$

for some $\sigma \in G$. Hence $|supp(\sum)| \leq |supp(R)| = n$. If we denote by $G(\sum) = \{\sigma \in G/\sum \cong \sum(\sigma)\}$ the inertia group of $\sum$, it follows that $|G(\sum)|$ divides $|supp(\sum)|$. Therefore $|G(\sum)| \leq n$ and by the first argument $|G(\sum)|$ is invertible in R. Hence $\sum$ is semisimple by Clifford theory [6, Theorem 3.2.v.]. Every gr-semisimple graded left R module is semisimple and in this case $J(R) = 0$ since $J(R) = J^g(R)$. Hence R is semiprimitive and therefore semiprime.

The assertion 2) follows immediately. Indeed, if I is a nilpotent ideal of R, then $I \subseteq J(R)$ and thus $I^n = 0$. Therefore there exists a unique maximal ideal N such that $N^n = 0$.

Assume that R_1 has no nonzero nil ideals. Then by a result of Amitsur [7, Theorem 6.1.1] $R_1[x]$is semiprimitive. Also by Proposition 4, $R[x]$ is gr-semiprime. The preceding argument give us that in the case (1) $R[x]$ is semiprime and therefore R is semiprime. In the general case, if I is a nilpotent ideal of R, then $I[x]^n$ is a nilpotent ideal of $R[x]$ and the preceding argument tell us that $I[x]^n = 0$ and $I^n = 0$. So by Proposition 1, R_1 is semiprime.

Finally assume that R is gr-semiprime, then R_1 is semiprime by Proposition 1. By the Proposition 6 $\widehat{R}$ is gr-semiprime and $\widehat{R}_1$ has no nonzero nil ideals. The last paragraph tell us that $\widehat{R}$ is semiprime. Thus by [9, Lemma 5] R is semiprime. Finally if I is a nilpotent ideal of R, then is nilpotent and then $\widehat{I}^n = 0$. Hence $I^n = 0$. □

Corollary 8. *Let R be a G-graded ring with G finite and let $n = |G|$. Assume that R is gr-semiprime.*

i)If R has no n-torsion then R is semiprime.

ii)In any case, R has a unique maximal nilpotent ideal N and $N^n = 0$.

Corollary 9. *Let R be a G-graded ring of finite support with G torsionfree. Then R is gr-semiprime if and only if R is semiprime.*

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