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Strong Lefschetz property on algebras of embedding dimension three

by
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To Professor Ion D. Ion on the occasion of his 70th Birthday

Abstract

We show that "half" of the non-zero components of the generic ideal J of a complete intersection ideal $I = (f_1, f_2, f_3) \subset K[x_1, x_2, x_3]$, with respect to the reverse lexicographic order, are uniquely determined by the Hilbert function $H(I, -)$ of I. Moreover the whole J is uniquely given by $H(I, -)$ if and only if complete intersection standard graded K-algebras of embedding dimension 3 have strong Lefschetz property as it is conjectured in [6]. Also we give some sufficient conditions for a semi-regular sequence to remain semi-regular after a permutation.

Key Words: complete intersection, Fröberg conjecture, Lefschetz properties, semi-regular sequences.
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1 Introduction

Let K be a field of characteristic zero, $S = K[x_1, \ldots, x_n]$, $I \subset S$ a graded ideal and $A = S/I$ (such ring A is called standard graded). An important invariant of A is the Hilbert function $H(A, -) : \mathbb{N} \to \mathbb{N}$ given by $r \to \dim_K A_r$, where A_r is the r-th component of A. Let $x \in A_q$ be an element. If x is regular then the maps $A_{r-q} \to A_r$ given by multiplication with x are all injective and so $\dim_K A_{r-q} \leq \dim_K A_r$. From the exact sequence

$$0 \to A_{r-q} \xrightarrow{x} A_r \to A_r/xA_{r-q} \to 0$$

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it follows that

$$\dim(A/(x))_r = \dim A_r - \dim A_{r-q}.$$

The Hilbert series $H_A \in \mathbb{Z}[[t]]$ of A is $H_A(t) = \sum_{i\geq 0} H(A,i)t^i$. We have $H_{A/(x)}(t) = (1-t^q)H_A(t)$. If x is not regular we have only

$$\dim_K(A/(x)_r) \geq \dim_K A_r - \dim A_{r-q}$$

because $\dim_K xA_{r-q} \leq \dim_K A_r$. It follows

$$H_{A/(x)}(t) \succeq (1-t^q)H_A(t)$$

in the sense that we have $\geq$ for all corresponding coefficients of t^i. In the left side all the coefficients are non-negative, but in the right side some of them could be negative. Note also that if $H(A/(x),r) = 0$ for some r then it follows $H(A/(x),s) = 0$ for all $s \geq r$ and so the above inequality becomes

$$H_{A/(x)}(t) \succeq |(1-t^q)H_A(t)|,$$

where, given a power series $U = \sum_{i\geq 0} u_i t^i \in \mathbb{Z}[[t]]$, we set $|U| = \sum_{i\geq 0} v_i t^i$ with $v_i = u_i$ if $u_j > 0$ for all $0 \leq j \leq i$ and $v_i = 0$ otherwise. For example $|1+t-t^3| = |1+t+t^3| = 1+t$.

The element $x \in A_q$ is called *semi-regular* on A if $H_{A/(x)}(t) = |(1-x^q)H_A(t)|$.

Lemma 1.1 ([10]). *x is semi-regular if and only if the map $A_{r-q} \xrightarrow{x} A_r$ is either injective or surjective for all $r \in \mathbb{N}$.*

Certainly if x is regular then x is also semi-regular, but $A = S/(x_1,\ldots,x_n)^e$, for example, has no regular elements but all its nonzero elements are semi-regular. Indeed, if $y \in S_q$ then the multiplication by y, $A_{r-q} \xrightarrow{y} A_r$ is injective for $r < e$ and surjective for $r \geq e$ since $A_r = 0$ for $r \geq e$. If $x \in A_q$ is semi-regular then all general elements of A_q are also semi-regular, that is there exists a non-empty Zariski open set $V \subset A_q$ such that all the elements of V are semi-regular. This is because changing slightly a matrix of maximal rank it remains still of maximal rank. The following example shows that changing must be really small.

Example 1.2. Let $A = K[x_1,\ldots,x_4]/(x_1^3,\ldots,x_4^3)$, $\beta = x_1+x_2+x_3$ and $\gamma = \beta^2 + x_4\beta + x_4^2$. Then the map $A_3 \to A_5$ given by multiplication with γ is not injective and not surjective, though $\gamma + x_4\beta$ defines a bijection $A_3 \to A_5$ by Theorem 1.3. Indeed, the Hilbert power series of A is $1+4t+10t^2+16t^3+19t^4+16t^5+10t^6+4t^7+t^8$ but the Hilbert power series of $A/(\gamma)$ is $1+4t+9t^2+12t^3+9t^4+3t^5$, that is the dimension of the cokernel of $A_3 \to A_5$ defined by multiplication with γ is 3. Then the multiplication $A_3 \xrightarrow{\gamma} A_5$ is not surjective and so also not injective because $\dim_K A_3 = \dim_K A_5 = 16$.

An element $x \in A_1$ is called *weak Lefschetz* if it is semi-regular. It is called *strong Lefschetz* if all powers x^s, $s \in \mathbb{N}$ are semi-regular. A has *weak Lefschetz property* (respectively *strong Lefschetz property*) if it has a weak (respectively strong) Lefschetz element.

Theorem 1.3 (Stanley [12], Watanabe [13]). *If K has the characteristic zero then $K[x_1,\ldots,x_n]/(x_1^{a_1},\ldots,x_n^{a_n})$ has strong Lefschetz property for all positive integers $n, a_1,\ldots,a_n$.*

The proof of this result used the Hard Lefschetz Theorem (see [12]) or the representation theory of the Lie algebra $sl(2)$ (see [13]). When a very simple and clear statement in linear algebra must be proved using strong sophisticated tools as the Hard Lefschetz Theorem or the representation theory of the Lie algebra $sl(2)$ then pure algebraists feel a kind of frustration. Fortunately, finally appeared some linear algebra proofs (see [7], [8], [11]).

A very important problem in this field is the following:

Conjecture 1.4 (Fröberg [5]). *If K is an infinite field then any general sequence of polynomials $f_1,\ldots,f_r$ is a semi-regular sequence on $S = K[x_1,\ldots,x_n]$.*

The first $\min\{r,n\}$-polynomials from the above general sequence form a regular sequence and so the above conjecture is trivial for $r \leq n$. If $r = n+1$ the conjecture follows from Theorem 1.3. Thus the problem starts with $r = n+2$. When $n = 3$ the above conjecture holds by Anick [1].

Let "$\leq$" be a monomial order on the monomials of S and $\mathrm{in}_{\leq}(I)$ the initial ideal of an ideal $I \subset S$ with respect to $\leq$. By $\mathrm{Gin}_{\leq}(I)$ we denote the generic initial ideal of I. Another form of the above conjecture is the following:

Conjecture 1.5 (Pardue [10]). *Let K be an infinite field, $d_1,\ldots,d_n \in \mathbb{N}$ and $f_1,\ldots,f_n$ a general sequence in $S = K[x_1,\ldots,x_n]$. Let $\leq$" be the reverse lexicographic order on the monomials of S. Then $x_n,\ldots,x_1$ is a semi-regular sequence on $S/in_{\leq}(I)$, $I = (f_1,\ldots,f_n)$.*

Although these conjectures are still open they have already applications in Criptography (see [4]). Connected with these results are the following two theorems:

Theorem 1.6 (A. Wiebe [14]). *Let "$\leq$" be the reverse lexicographic order on the monomial of S and I a homogeneous ideal of S. Then S/I has weak (resp strong) Lefschetz property if and only if $S/Gin_{\leq}(I)$ has the weak (resp. strong) Lefschetz property too, that is if x_n is a weak (resp. strong) Lefschetz element on $S/Gin_{\leq}(I)$.*

Theorem 1.7 (Harima-Migliore-Nagel-Watanabe [6]). *A complete intersection standard graded algebra $K[x_1,x_2,x_3]/(f_1,f_2,f_3)$ has weak Lefschetz property.*

The purpose of this paper is to give a description of $J = \mathrm{Gin}(f_1,f_2,f_3)$ -the generic ideal with respect to the reverse lexicographic order. Our Corollary 2.4 says in fact that A has strong Lefschetz property if and only if J is an almost reverse lexicographic ideal in the language of [10] (an ideal T is almost reverse lexicographic if for any monomial u of the minimal system of generators of J

any monomial of the same degree which precedes u in the reverse lexicographic order is in J as well). This does not hold when A is of embedding dimension 4 as our Remark 2.5 shows. In general our Theorem 2.2 says only that "half" of the components of J are almost reverse lexicographic. The last section studies when a semi-regular sequence remains semi-regular after a permutation, this happens for sequence, which are roughly speaking almost regular.

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2 Complete intersection of embedding dimension three

We begin with a nice consequence of Theorems 1.7 and 1.6:

Corollary 2.1. *Let $A = K[x_1, x_2, x_3]/(f_1, f_2, f_3)$ be a complete intersection standard graded algebra. Then x_3, x_2, x_1 is semi-regular on $K[x_1, x_2, x_3]/Gin_{\leq}$ (f_1, f_2, f_3). In particular, the Pardue Conjecture 1.5 holds when $n = 3$.*

Proof: We show that the sequence x_3, x_2, x_1 is semi-regular on $K[x_1, x_2, x_3]/J$ for $J = \text{Gin}_{\leq}(I)$, $I = (f_1, f_2, f_3)$, $\leq$ being the reverse lexicographic order and (f_i) being a general sequence of 3 polynomials which form obviously a regular sequence. Now, by Theorem 1.7 a complete intersection K-algebra $A = K[x_1, x_2, x_3]/(f_1, f_2, f_3)$ has weak Lefschetz property and so x_3 is semi-regular on $K[x_1, x_2, x_3]/\text{Gin}_{\leq}(f_1, f_2, f_3)$ by Theorem 1.6. As the embedding dimension of $B = K[x_1, x_2, x_3]/(x_3, \text{Gin}_{\leq}(f_1, f_2, f_3))$ is two, B has even strong Lefschetz property (see e.g. [6]) and so x_2, x_1 is semi-regular on B. □

Now we study when the above generic ideal J is uniquely determined by the Hilbert function of J. The main result is given by the following:

Theorem 2.2. *Let $I, I' \subset K[x_1, x_2, x_3]$ be two graded ideals, $A = S/I$, $A' = S/I'$ and J, J' their generic initial ideals with respect to the reverse lexicographic order. Suppose that*

1. *A is Artinian and let s be a positive integer such that the Hilbert function $H(A, -)$ is maxim in s, that is $H(A, s) = max_i H(A, i)$.*
2. *A, A' have weak Lefschetz property.*
3. *A, A' have the same Hilbert function which is symmetric.*

Then $J_i = J'_i$ for any $i \leq s$. Moreover if A, A' have strong Lefschetz property then $J = J'$.

Proof: $H(A, -)$ is unimodal because A has weak Lefschetz property (see [6, Remark 3.3]). It follows $H(A, i) \leq H(A, i + 1)$ for all $i < s$. Set $B = S/J$, $B' = S/J'$. Since $H(A, -) = H(B, -)$ we get also $H(B, i) \leq H(B, i + 1)$.

By 1.6 we see that B, B' have also weak Lefschetz property and x_3 is a weak Lefschetz element on B and B'. Then the multiplication by x_3 induces an injective map $B_i \to B_{i+1}$. Thus if $x_3 u \in J_{i+1}$ for a monomial u then $u \in J_i$, that is $J_{i+1} \subset \langle \{x_1, x_2\}^{i+1}, x_3 J_i \rangle$. As $J_{i+1} \cap K[x_1, x_2]$ is a lex ideal being strongly stable, it is uniquely determined by $H(B, i+1) - \dim_K J_i$. If $J_i = J_i'$ then it follows $J_{i+1} = J_{i+1}'$. An inductive argument on $i < s$ shows the first part of the statement.

Now suppose that A, A' (and so B, B' by 1.6) have strong Lefschetz property and x_3 is a strong Lefschetz element on B, B'. Let $i > s$. Then there exists $j \le s$ such that $H(B, i) = H(B, j)$ by the symmetry of the Hilbert function. Since the multiplication by x_3^{i-j} induces an bijective map $B_j \to B_i$ we get $J_i = \langle \{x_1, x_2\}^i \cup \ldots \cup \{x_1, x_2\}^{j+1} x_3^{i-j-1} \cup J_j x_3^{i-j} \rangle$ and similarly for J_i'. Thus $J_i = J_i'$ because $J_j = J_j'$ as above. □

Remark 2.3. Let $A = K[x_1, x_2, x_3]/(f_1, f_2, f_3)$ be a complete intersection standard graded K-algebra with f_i homogeneous polynomials of degree d_i. Thus A is Artinian Gorenstein and has weak Lefschetz property by [6]. So $H(A, -)$ is unimodal and symmetric and by above theorem some of components of the generic initial ideal J of (f_1, f_2, f_3) with respect to the reverse lexicographic order depends only on $H(A, -)$ which depend only on $(d_i)_i$. Who can be J? If $d_1 = d_2 = d_3 = 3$ then J can be one of the following ideals:

$$T_1 = \langle x_1^3, x_1^2 x_2, x_1 x_2^2; x_2^4, x_2^3 x_3; x_1^2 x_3^3, x_1 x_2 x_3^3, x_2^2 x_3^3; x_1 x_3^5, x_2 x_3^5; x_3^7 \rangle,$$

$$T_2 = \langle x_1^3, x_1^2 x_2, x_1 x_2^2; x_2^4, x_1^2 x_3^2; x_2^3 x_3^2, x_1 x_2 x_3^3, x_2^2 x_3^3; x_1 x_3^5, x_2 x_3^5; x_3^7 \rangle.$$

We may see that $C_i = K[x_1, x_2, x_3]/T_i$, $i = 1, 2$ have weak Lefschetz property but only C_1 has strong Lefschetz property, because the multiplication by x_3^2 induces a map $(C_2)_2 \to (C_2)_4$ which is not injective (x_1^2 is in the kernel) and not surjective because $\dim(C_2)_2 = \dim(C_2)_4 = 6$. C_1 is in fact almost reverse lexicographic.

Corollary 2.4. *Let $A = K[x_1, x_2, x_3]/(f_1, f_2, f_3)$ be a complete intersection standard graded K-algebra with f_i homogeneous polynomials of degree d_i. The following statements are equivalent:*

1. *A has strong Lefschetz property,*
2. *The generic initial ideal of (f_1, f_2, f_3) with respect to the reverse lexicographic order depends only on $(d_i)_i$.*

Proof: (1) $\Rightarrow$ (2) follows from Theorem 2.2. For the converse note that $B = K[x_1, x_2, x_3]/(x_1^{d_1}, x_2^{d_2}, x_3^{d_3})$ has strong Lefschetz property by 1.3. Let J be the generic initial ideal of $(x_1^{d_1}, x_2^{d_2}, x_3^{d_3})$ with respect to the reverse lexicographic order. By 1.6, $C = K[x_1, x_2, x_3]/J$ has strong Lefschetz property. But (2) says that J is also the generic initial ideal of (f_1, f_2, f_3) with respect to the reverse lexicographic order. Applying again 1.6 we see that A must have strong Lefschetz property. □

Remark 2.5. If A is a complete intersection standard graded K-algebra of embedding dimension 4 then Theorem 2.2 and the above Corollary are no longer valid. For example the fourth component of

$$J = \mathrm{Gin}(x_1^3, x_2^3, x_3^3, x_4^3) \subset K[x_1, x_2, x_3, x_4]$$

is given by

$$J_4 = \langle \{x_1, x_2\}^3 \times \{x_1, x_2, x_3, x_4\}, x_1^2x_3^2, x_1x_2x_3^2, x_1x_3^3 \rangle.$$

Thus J is not almost reverse lexicographic because $x_2^2x_3^2$ precedes $x_1x_3^3 \in J_4$ but it is not in J_4. As $A = K[x_1, x_2, x_3, x_4]/(x_1^3, x_2^3, x_3^3, x_4^3)$ has strong Lefschetz property by 1.3 we see that in this case J is not uniquely determined by $H(A, -)$, because $\mathrm{Gin}(f_1, f_2, f_3, f_4)$ is almost reverse lexicographic, when f_1, f_2, f_3, f_4 are general homogeneous polynomials of degree 3 (the idea of this remark is given in [3, page 838]).

However some components of the generic initial ideal of (f_1, f_2, f_3) with respect to the reverse lexicographic order depend always only on $(d_i)_i$ as Theorem 2.2 says, but which are these? To be more explicit we need to describe precisely the Hilbert function or equivalently the Hilbert series of $K[x_1, x_2, x_3]/(f_1, f_2, f_3)$, which we will do next. First we state an elementary lemma which we add only for the sake of our completion.

Lemma 2.6. *Let $A = K[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ be a complete intersection standard graded K-algebra with f_i homogeneous polynomials of degree d_i. Then the Hilbert series of A is given by*

$$H_A(t) = \prod_{i=1}^{n} (1 + t + \ldots + t^{d_i - 1}).$$

Proof: Denote $B_i = K[x_1, \ldots, x_n]/(f_1, \ldots, f_i)$, $0 \le i \le n$. Fix $i < n$. From the exact sequence

$$0 \to B_i(-d_{i+1}) \xrightarrow{f_{i+1}} B_i \to B_{i+1} \to 0$$

we get

$$H_{B_{i+1}}(t) = (1 - t^{d_{i+1}}) H_{B_i}.$$

By recurrence it follows $H_{B_i} = (1 - t^{d_1}) \cdots (1 - t^{d_i})/(1 - t)^n = \Pi_{i=1}^{j}(1 + t + \ldots + t^{d_i - 1})/(1 - t)^{n-j}$ for all j, the case $j = 0$ being well known. □

After small computations we get the following:

Lemma 2.7. *Let $d \in \mathbf{N}$. The Hilbert series of a complete intersection $A = K[x_1, x_2, x_3]/(f_1, f_2, f_3)$ with f_i homogeneous polynomials of degree d has the form:*

$$H_A(t) = \sum_{i=0}^{d-1} \binom{i+2}{2} t^i + \sum_{j=1}^{d-2} [\binom{d+1}{2} - 2\binom{j+1}{2} + jd] t^{d+j-1} + \sum_{i=0}^{d-1} \binom{i+2}{2} t^{3d-3-i}.$$

Proposition 2.8. *Let $d \in \mathbf{N}$, $r = \lfloor d/2 \rfloor$ and $A = K[x_1, x_2, x_3]/(f_1, f_2, f_3)$ be a complete intersection with f_i homogeneous polynomials of degree d. Let J be the generic initial ideal of (f_1, f_2, f_3) with respect to the reverse lexicographic order. Then*

1. *If d is even then*

$$1 = H(A,0) = H(A,3d-3) < H(A,1) = H(A,3d-2) < \ldots$$
$$< H(A,3r-2) = H(A,3r-1).$$

2. *If d is odd then*

$$1 = H(A,0) = H(A,3d-3) < H(A,1) = H(A,3d-2) < \ldots$$
$$< H(A,3r-1) = H(A,3r+1) < H(A,3r).$$

3. *If d is even then J_i is uniquely determined by d for $i \leq 3r-1$.*

4. *If d is odd then J_i is uniquely determined by d for $i \leq 3r$.*

For the proof apply Lemma 2.7 and Theorem 2.2 (we denote by $\lfloor x \rfloor$ the biggest integer $\leq x$ and later by $\lceil x \rceil$ the smallest integer $\geq x$).

We obtain similar results to Lemma 2.7 and Proposition 2.8 in the general case as it follows.

Lemma 2.9. *Let $2 \leq d_1 \leq d_2 \leq d_3$ be positive integers. The Hilbert function of a complete intersection $A = K[x_1, x_2, x_3]/(f_1, f_2, f_3)$ with f_i homogeneous polynomials of degree d_i, for all i, with $1 \leq i \leq 3$, has the form:*

(a) If $d_1 + d_2 \leq d_3 + 1$, then

- *(1) $H(A,k) = \binom{k+2}{2}$, for $k \leq d_1 - 2$,*
- *(2) $H(A,k) = \binom{d_1+1}{2} + jd_1$, for $k = d_1 - 1 + j$, $0 \leq j \leq d_2 - d_1$,*
- *(3) $H(A,k) = \binom{d_1+1}{2} + d_1(d_2 - d_1) + \sum_{i=1}^{j}(d_1 - i)$, for $k = d_2 + j - 1$, $1 \leq j \leq d_1 - 2$,*
- *(4) $H(A,k) = d_1 d_2 = \binom{d_1+1}{2} + d_1(d_2 - d_1) + \sum_{i=1}^{d_1-1}(d_1 - i)$, for $d_1 + d_2 - 2 \leq k \leq d_3 - 1$,*
- *(5) $H(A,k) = H(A, d_1 + d_2 + d_3 - 3 - k)$, for $k \geq d_3$.*

(b) If $d_1 + d_2 > d_3 + 1$, then

- *(1) $H(A,k) = \binom{k+2}{2}$, for $k \leq d_1 - 2$,*
- *(2) $H(A,k) = \binom{d_1+1}{2} + jd_1$, for $k = d_1 - 1 + j$, $0 \leq j \leq d_2 - d_1$,*
- *(3) $H(A,k) = \binom{d_1+1}{2} + d_1(d_2 - d_1) + \sum_{i=1}^{j}(d_1 - i)$, for $k = d_2 - 1 + j$, $0 \leq j \leq d_3 - d_2$,*

(4) $H(A,k) = H(A,d_3-1) + \sum_{i=1}^{j}(d_1+d_2-d_3-2i)$, *for* $k = d_3 - 1 + j$, $0 \le j \le \lfloor \frac{d_1+d_2-d_3-1}{2} \rfloor$,

(5) $H(A,k) = H(A, d_1+d_2+d_3-3-k)$, *for* $k > d_3 - 1 + \lfloor \frac{d_1+d_2-d_3-1}{2} \rfloor$.

Proof: According to Lemma 2.6

$$H_A(t) = (1+t+\ldots+t^{d_1-1})(1+t+\ldots+t^{d_2-1})(1+t+\ldots+t^{d_3-1}),$$

i.e, $H_A(t) = (\sum_{i=1}^{d_1-2}(i+1)t^i + \sum_{i=d_1-1}^{d_2-1} d_1 t^i + \sum_{i=1}^{d_1-1}(d_1-i)t^{d_2+i-1})(1+t+\ldots+t^{d_3-1})$. Now it is obvious that we have to consider the two cases from the statement and the rest follows after some computations. □

Proposition 2.10. *Let* $2 \le d_1 \le d_2 \le d_3$ *be positive integers and* A *a complete intersection,* $A = K[x_1,x_2,x_3]/(f_1,f_2,f_3)$, *with* f_i *homogeneous polynomials of degree* d_i, *for all* i *with* $1 \le i \le 3$. *Let* J *be the generic initial ideal of* (f_1,f_2,f_3) *with respect to the reverse lexicographic order. Then:*

(a) If $d_1+d_2 \le d_3+1$, *then*

$$1 = H(A,0) = H(A,d_1+d_2+d_3-3) < H(A,1) = H(A,d_1+d_2+d_3-4) < \ldots$$
$$< d_1d_2 = H(A,d_1+d_2-2) = \ldots = H(A,d_3-1).$$

(b) If $d_1+d_2 > d_3+1$, *then*

(i) For $d_1+d_2-d_3$ *an even number*

$$1 = H(A,0) = H(A,d_1+d_2+d_3-3) < H(A,1) =$$
$$= H(A,d_1+d_2+d_3-4) < \ldots < H(A,d_3-1+\lfloor\frac{d_1+d_2-d_3-1}{2}\rfloor) =$$
$$= H(A,d_3-1+\lceil\frac{d_1+d_2-d_3-1}{2}\rceil).$$

(ii) For $d_1+d_2-d_3$ *an odd number*

$$1 = H(A,0) = H(A,d_1+d_2+d_3-3) < H(A,1) =$$
$$= H(A,d_1+d_2+d_3-4) < \ldots < H(A,d_3-1+\frac{d_1+d_2-d_3-1}{2}).$$

(c) If $d_1+d_2 \le d_3+1$, *then* J_i *is uniquely determined by* d_1,d_2,d_3 *for* $i \le d_3-1$.

(d) If $d_1+d_2 > d_3+1$, *then* J_i *is uniquely determined by* d_1,d_2,d_3, *for* $i \le \lceil\frac{d_1+d_2-d_3-1}{2}\rceil$ *if* $d_1+d_2-d_3$ *is an even number, respectively for* $i \le \frac{d_1+d_2-d_3-1}{2}$ *if* $d_1+d_2-d_3$ *is an odd number.*

Remark 2.11. Let us notice that in the case $d_1+d_2 \le d_3+1$ the maximum of the Hilbert function is d_1d_2, hence it does not depend on d_3. In the case $d_1+d_2 > d_3+1$, the maximum of the Hilbert function can be at most d_1d_2-1 and depends on d_3.

3 Semi-regular sequences

In general a permutation of a semi-regular sequence is no longer semi-regular (for example x_1^2, x_2^2, x_1x_2 is semi-regular on $K[x_1, x_2]$ but x_1^2, x_1x_2, x_2^2 is not, as it is given in [10]). Here we give some sufficient conditions for that.

The following lemma is inspired by the proof of [9, Proposition 4.3].

Lemma 3.1. *Let A be a standard graded K-algebra, f, g two homogeneous elements of A of degree k respectively q (*$\deg\ x = 1$*) which have each maxim rank on A. Then f has maxim rank on $R = A/(g)$ if and only if g has maxim rank on $A/(f)$.*

Proof: We have the following commutative diagram:

$$\begin{array}{ccccccc}
A(-q-k) & \xrightarrow{g} & A(-k) & \to & R(-k) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
A(-q) & \xrightarrow{g} & A & \to & R & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
A/(f)(-q) & \xrightarrow{g} & A/(f) & \to & R/(f) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

where the lines and columns are exact, the first above vertical maps being multiplication with f. It is enough to show one implication only. Fix a $t \in \mathbb{N}$. Suppose g has maxim rank for $A/(f)$. If the multiplication $A_{t-k} \to A_t$ by f is surjective then clearly $(A/(f))_t = 0$ and so $(A/(f,g))_t = 0$ and the multiplication $\bar{f} : (A/(g))_{t-k} \to (A/(g))_t$ by f is surjective. Thus we may suppose the multiplication $A_{t-k} \to A_t$ by f injective by our hypothesis. The same argument shows that the multiplication $(A/(g))_{t-k} \to (A/(g))_t$ by f is surjective when the multiplication $\bar{g} : (A/(f))_{t-q} \to (A/(f))_t$ by g is surjective. So we may suppose also $\bar{g}$ injective.

Now if the residue class modulo (g) of some $z \in A_{t-k}$ is in $\operatorname{Ker} \bar{f}$ then there exists $u \in A_{t-q}$ such that $fz = gu$ in A_t. So the residue class modulo (f) of u is in $\operatorname{Ker} \bar{g} = 0$, that is $u = fv$ for some $v \in A_{t-q-k}$. It follows $z = gv$ since the multiplication $A_{t-k} \to A_t$ by f was supposed injective. Thus the residue class of z modulo g is zero, which shows that $\bar{f}$ is injective. □

The following lemma is a consequence of Lemma 3.1. It follows also from [8, Lemma 1.1], or [11, Lemma 3.1].

Lemma 3.2. *Let A be a standard graded K-algebra, f a homogeneous monic polynomial of $A[x]$ of degree k and $q \in \mathbb{N}$ (*$\deg\ x = 1$*). Then f is semi-regular on $R_q = A[x]/(x^q)$ if and only if x^q is semi-regular on $A[x]/(f)$.*

Corollary 3.3. *In the notations and hypothesis of the previous lemma A has strong Lefschetz property if and only if for all positive integer q the ring R_q has weak Lefschetz property.*

For the proof apply Lemma 3.2 with $f = x + \beta$ for a generic linear form β of A.

Lemma 3.4. *Let A be a standard graded K-algebra and $f_1, \dots, f_r$ some graded forms of A with $\deg f_j = k_j$. The following statements are equivalent:*

1. *For all j, $1 \le j \le r$ and for all $t \in \mathbb{N}$ the multiplication with f_j, $(A/(f_1, \dots, f_{j-1}))_{t-k_j} \to (A/(f_1, \dots, f_{j-1}))_t$ is either surjective or injective.*
2. *For all j, $1 \le j \le r$ and for all $t \in \mathbb{N}$ it holds either $H_0(f_1, \dots, f_j; A)_t = 0$, or $H_1(f_1, \dots, f_j; A)_t = 0$.*

Proof: By [2, Corollary 1.6.13] we have the following exact sequence of Koszul homology modules in degree t:

$$H_1(f_1, \dots, f_{r-1}; A)_t \to H_1(f_1, \dots, f_r; A)_t \to$$

$$H_0(f_1, \dots, f_{r-1}; A(-k_r))_t \xrightarrow{f_r} H_0(f_1, \dots, f_{r-1}; A)_t \to H_0(f_1, \dots, f_r; A)_t \to 0.$$

Apply induction on r, the equivalence (1) $\Longleftrightarrow$ (2) being trivial for $r = 1$. Suppose $r > 1$ and that the equivalence holds for $r - 1$ by induction.

(1) $\Longrightarrow$ (2) Assume $H_0(f_1, \dots, f_r; A)_t \ne 0$, that is $((f_1, \dots, f_r)A)_t \ne A_t$. Then $((f_1, \dots, f_{r-1})A)_t \ne A_t$ and we have $H_1(f_1, \dots, f_{r-1}; A)_t = 0$ by induction hypothesis. Also the multiplication with f_r in the above sequence is not surjective and so must be injective by (1). It follows $H_1(f_1, \dots, f_r; A)_t = 0$.

(2) $\Longrightarrow$ (1) From the above exact sequence we see that if the multiplication map by f_r is not surjective it must be injective because $H_0(f_1, \dots, f_r; A)_t \ne 0$ implies $H_1(f_1, \dots, f_r; A)_t = 0$. □

Lemma 3.5. *Let A be a standard graded K-algebra and $f_1, \dots, f_r$ some graded forms of A with $\deg f_j = k_j$ such that each $r - 1$ from them form a semi-regular system on A. Let π be a permutation of $\{1, \dots, r\}$. The following statements are equivalent:*

1. *For all j, $1 \le j \le r$ and for all $t \in \mathbb{N}$ the multiplication with f_j, $(A/(f_1, \dots, f_{j-1}))_{t-k_j} \to (A/(f_1, \dots, f_{j-1}))_t$ is either surjective or injective.*
2. *For all j, $1 \le j \le r$ and for all $t \in \mathbb{N}$ the multiplication with $f_{\pi(j)}$, $(A/(f_{\pi(1)}, \dots, f_{\pi(j-1)}))_{t-k_j} \to (A/(f_{\pi(1)}, \dots, f_{\pi(j-1)}))_t$ is either surjective or injective.*

For the proof apply Lemma 3.1, which shows the case when π is a transposition.

Proposition 3.6. *Let A be a standard graded K-algebra and $f_1, \ldots, f_r$ some graded forms of A such that each $r-1$ from them form a semi-regular system on A. If $f_1, \ldots, f_r$ is semi-regular then any permutation of it is semi-regular too.*

The proof follows from the above lemma.

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