Bull. Math. Soc. Sc. Math. Roumanie
Tome 49(97) No. 1, 2006, 47–53

Cumulative sum of the Liouville function related to Gauss's problem of the tenth field

by
Alexandru Gica

To Professor Ion D. Ion on the occasion of his 70th Birthday

Abstract

We present in this paper a connection between Gauss's problem of the tenth field and some cumulative sums of the Liouville function. Namely, we compute $L(x) = \sum_{n \leq x} \lambda(n)$ for $x = \frac{p-3}{4}, x = \frac{p-1}{2}, x = p-1$, where p is a prime number, $p \equiv 19 \pmod{24}$ such that $\mathbb{Z}[\frac{1+i\sqrt{p}}{2}]$ is a principal ring. We use this result for finding another approach for the problem of the tenth field. We present also a short survey about some previous achievements related with this subject.

Key Words: Gauss's theorem of the tenth field, length, Liouville function.
2000 Mathematics Subject Classification: Primary: 11R29, Secondary 11N37.

1 Introduction

A tribute: The starting point of the present article is paper [4]. I arrived to the main idea of the paper during the "Algebraic Number Theory" seminar. The course was taught for more than 30 years by Professor Ion D. Ion. This paper is a tribute to Professor Ion. I am very indebted to him for teaching me Algebraic Number Theory.

If $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r}$ (where $a_i \in \mathbb{N}^*, (\forall) i = \overline{1,r}, p_i$ prime $(\forall) i = \overline{1,r}$), the "length" of n is the number

$$\Omega(n) = \sum_{i=1}^{r} a_i; \Omega(1) = 0.$$

The following conjecture appears in [4].

The "Strong" Conjecture: If $n \in \mathbb{N}, n > 163$, there exist $a, b \in \mathbb{N}^*$ such that $n = a + b, \Omega(a) \equiv \Omega(b) \pmod 2$ and any prime divisor q of ab satisfies the inequality $q \leq \frac{n}{4}$.

It is proved in [4] that the "Strong" conjecture implies the theorem of Gauss-Heegner-Stark-Baker: the ring of the integers for a quadratic imaginary field $\mathbb{Q}(\sqrt{d})$ is principal only for nine values of d:

$$-d = 1, 2, 3, 7, 11, 19, 43, 67, 163$$

(see [9] for a proof of this result). In [5-6] it is proved the following.

Theorem 1.1. *If $n \in \mathbb{N}, n > 3$, there exist $a, b \in \mathbb{N}^*$ such that $n = a + b$ and $\Omega(a) \equiv \Omega(b) \pmod 2$.*

The desired aim is to find another proof of the theorem of Gauss, using the "length".

We give now an euristical evidence for the truth of the "strong" conjecture.

We will use the standard notation for the Liouville function

$$\lambda(n) = (-1)^{\Omega(n)}, \forall n \in \mathbb{N}^*$$

and for the cumulative sum of the Liouville function

$$L(x) = \sum_{n \leq x} \lambda(n), \forall n \in \mathbb{N}^*.$$

The following result appears in [2], page 123.

Theorem 1.2. *We have the following asymptotical formula*

$$L(x) = O(x \cdot e^{-c \cdot (\ln x)^a})$$

for some positive constants a and c.

This result is equivalent with the celebrated Prime Number Theorem:

$$\lim_{x \to \infty} \frac{\pi(x) \cdot \ln x}{x} = 1$$

where $\pi(x)$ counts the number of primes smaller than x. The significance of the above theorem is that the number of $y \leq x$ with odd lengths is "almost the same" as the number of $y \leq x$ with even lengths. This suggest the following asymptotical formulas

$$r_o(n) = \frac{n}{8} + O(x \cdot e^{-c_1 \cdot (\ln x)^{a_1}})$$

$$r_e(n) = \frac{n}{8} + O(x \cdot e^{-c_2 \cdot (\ln x)^{a_2}})$$

$$r_b(n) = \frac{n}{4} + O(x \cdot e^{-c_3 \cdot (\ln x)^{a_3}})$$

where a_i, c_i are positive constants $\forall i = \overline{1,3}$.

In the above formulas $r_o(n)$ is the numbers of pairs (a, b) such that $a \leq b, a + b = n, a, b \in \mathbb{N}^*, \Omega(a) \equiv \Omega(b) \equiv 1 \pmod 2$, $r_e(n)$ is the numbers of pairs (a, b) such that $a \leq b, a + b = n, a, b \in \mathbb{N}^*, \Omega(a) \equiv \Omega(b) \equiv 0 \pmod 2$ and $r_b(n)$ is the numbers of pairs (a, b) such that $a \leq b, a + b = n, a, b \in \mathbb{N}^*, \Omega(a) \equiv \Omega(b) + 1 \pmod 2$. If these formulas are true then $r_o(n) > \pi(n)$ for $n \geq n_0$. This means that for n big enough then there exist $a \leq b, a+b = n, a, b \in \mathbb{N}^*, \Omega(a) \equiv \Omega(b) \equiv 1 \pmod 2$ and a, b are not primes. Therefore $\Omega(a) \geq 3, \Omega(b) \geq 3$ and if q is a prime divisor of ab then $q \leq \frac{n}{4}$. (we have $q|a$ or $q|b$; let us suppose that $q|a$. But $\Omega(a) \geq 3$. Therefore there exist q_1, q_2 prime numbers such that $q \cdot q_1 \cdot q_2 | a$. Then $4q \leq q \cdot q_1 \cdot q_2 \leq a < n, q \leq \frac{n}{4}$.) We obtained the "‘strong conjecture"’ for n big enough. The most suitable tool for trying to prove the above asymptotical formulas seems to be the circle method of Hardy and Littlewood.

2 Statements equivalent with the problem of the tenth field

Gauss-Heegner-Stark-Baker theorem is equivalent with the following three statements.

Theorem 2.1. *For every prime $p > 163$ there is a prime $q < \frac{p}{4}$ such that $\left(\frac{q}{p}\right) = 1$.*

In the above formula $\left(\frac{q}{p}\right)$ is the Legendre symbol.

Theorem 2.2. *If q is a prime number such that $4q - 1$ and $q + k \cdot (k + 1)$ are prime numbers for every $k = \overline{0, q-2}$ then $q = 2, 3, 5, 11, 17, 41$.*

Theorem 2.3. *If p is a prime number $p \equiv 3 \pmod 8, p \geq 19, A = \mathbb{Z}\left[\frac{1+i\sqrt{p}}{2}\right]$, then A is principal if and only if $\Omega(r(a^2))$ is even $\forall\, a = \overline{1, \frac{p-1}{2} - [\frac{\sqrt{3p}-1}{2}] - 1}$.*

In the last theorem we denoted by $r(n)$ the unique natural number which satisfies the conditions

$$a) 0 \leq r(n) \leq p - 1$$

$$b) r(n) \equiv n \pmod p.$$

A proof for Theorem 2.1 and 2.2 can be found in [4] and a proof for Theorem 2.3 can be found in [7].

3 The theorem

Before proving the main result of this paper, we need to prove the following

Lemma: Let p be a prime number, $p = 8k+3, k \in \mathbb{N}$. Then

$$\sum_{j=1}^{2k} \left(\frac{j}{p}\right) = 0.$$

Proof: Let us denote

$$S_1 = \sum_{j=1}^{2k} \left(\frac{j}{p}\right), S_2 = \sum_{j=2k+1}^{4k+1} \left(\frac{j}{p}\right), S_3 = \sum_{j=4k+3, j\ odd}^{8k+1} \left(\frac{j}{p}\right), S_4 = \sum_{j=4k+2, j\ even}^{8k+2} \left(\frac{j}{p}\right).$$

Then $S_1 + S_2 + S_3 + S_4 = \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) = 0$ since there are $\frac{p-1}{2}$ quadratic residues modulo p and $\frac{p-1}{2}$ quadratic nonresidues modulo p.

We have $\left(\frac{j}{p}\right) + \left(\frac{2j}{p}\right) = 0$ since $\left(\frac{2}{p}\right) = -1$ (because $p \equiv 3 \pmod 8$) We infer that $S_2 + S_4 = 0$ and (since $S_1 + S_2 + S_3 + S_4 = 0$) $S_1 + S_3 = 0$.

We have $\left(\frac{j}{p}\right) = \left(\frac{p-2j}{p}\right)$ since $\left(\frac{-2}{p}\right) = 1$ (because $p \equiv 3 \pmod 8$). From this equality we infer that $S_1 = S_3$. But $S_1 + S_3 = 0$ and the Lemma is now obvious. □

Acknowledgment: C. N. Beli gave me the hint for the proof of the Lemma.

For proving the theorem of Gauss we know (see [4]) that it is enough to show that if p is a prime number $p \equiv 19 \pmod{24}$ then $\mathbb{Z}[\frac{1+i\sqrt{p}}{2}]$ is principal only for $p = 19, 43, 67, 163$.

The aim of this paper is to show the following.

Theorem 3.1. *Let p be a prime number $p \equiv 19 \pmod{24}, p \geq 19$, such that the ring $A = \mathbb{Z}[\frac{1+i\sqrt{p}}{2}]$ is principal. Then*
i) $L(\frac{p-3}{4}) = 0$
ii) $L(\frac{p-1}{2}) = 3 - 2\left[\frac{\sqrt{p}+1}{2}\right]$
iii) $L(p-1) = 2\left(\left[\frac{\sqrt{p}+1}{2}\right] + \left[\frac{\sqrt{\frac{p}{3}}+1}{2}\right] - \left[\frac{\sqrt{3p}+1}{2}\right]\right)$

Remark: The proof of the theorem would be very easy if we apply the fact that, according to Gauss's problem of the tenth field, $p = 19, 43, 67, 163$. Some computations will supply immediately the proof. But the point is to have a proof of the theorem without using Gauss-Heegner-Stark-Baker theorem. The hope is to find another proof for this celebrated result.

Proof: We denote $p = 8k + 3, k \in \mathbb{N}$. For the proof we will use the fact that since A is principal then $\left(\frac{q}{p}\right) = -1$ for any prime $q < \frac{p}{4}$ (see [4] for the proof of this fact). From this we infer that $\left(\frac{j}{p}\right) = (-1)^{\Omega(j)} = \lambda(j), \forall j = \overline{1, 2k}$. Using the Lemma and the above remark the first statement of the theorem is now obvious.

For the second statement of the theorem we will use the fact that

$$\sum_{j=1}^{\frac{p-1}{2}} \left(\frac{j}{p}\right) = 3h,$$

where h is the class number for the field $\mathbb{Q}(i\sqrt{p})$ (see [3], page 422 for the proof of this result). But the ring A is principal and therefore $h = 1$ (see [2], page 220). We obtained the formula

$$\sum_{j=1}^{\frac{p-1}{2}} \left(\frac{j}{p}\right) = 3.$$

We noticed before that, since A is principal, $\left(\frac{j}{p}\right) = (-1)^{\Omega(j)} = \lambda(j), \forall j = \overline{1, 2k}$. We have to examine now the numbers j such that $2k + 1 \leq j \leq 4k + 1$ and $\left(\frac{j}{p}\right) \neq \lambda(j)$.

These are precisely the prime numbers $\frac{p}{4} < j < \frac{p}{2}$ which are quadratic residue modulo p. But, according to [4], this holds if and only if $j = \frac{p+x^2}{4}$, with $x \in \mathbb{N}$, x odd. We have to count now how many odd positive integers x exist such that $x^2 < p$. It is obvious that there are $\left[\frac{\sqrt{p}+1}{2}\right]$ such numbers. From all the above considerations we obtain that

$$3 = \sum_{j=1}^{4k+1} \left(\frac{j}{p}\right) = \sum_{j=1, j \neq \frac{p+x^2}{4}}^{4k+1} \left(\frac{j}{p}\right) + \left[\frac{\sqrt{p}+1}{2}\right] =$$

$$= \sum_{j=1, j \neq \frac{p+x^2}{4}}^{4k+1} \lambda(j) + \left[\frac{\sqrt{p}+1}{2}\right] = L(4k+1) + 2\left[\frac{\sqrt{p}+1}{2}\right].$$

We proved the second statement. As for the third statement of the theorem, the proof is very similar with the above. We know that $\sum_{j=1}^{p-1} \left(\frac{j}{p}\right) = 0$. The numbers j for which the equality $\left(\frac{j}{p}\right) = \lambda(j)$ does not hold are

a) $\frac{p+x^2}{4}$, where x is an odd number such that $x^2 < 3p$

b) $2\frac{p+x^2}{4}$ where x is an odd number such that $x^2 < p$

c) $3\frac{p+x^2}{4}$ where x is an odd number such that $x^2 < \frac{p}{3}$.

Now is only a matter of counting and everything follows easily. □

4 Another approach for the problem of the tenth field.

The theorem proved in the above section suggest the following path for another proof of the problem of the tenth field.

Conjecture: Let x be a positive integer such that $L(\frac{x}{4}) = 0, L(\frac{x}{2}) = 3 - 2\left[\frac{\sqrt{x}+1}{2}\right]$ and $L(x) = 2\left(\left[\frac{\sqrt{x}+1}{2}\right] + \left[\frac{\sqrt{\frac{x}{3}}+1}{2}\right] - \left[\frac{\sqrt{3x}+1}{2}\right]\right) - 1$. Then $x \leq 163$.

If such a result would be true then indeed we will have another proof for Gauss-Heegner-Stark-Baker theorem if we take into account the theorem proved in the previous section.

In connection with the L function we want to mention Pólya's conjecture:

$$L(x) \leq 0, \forall x \in \mathbb{R}, x \geq 2.$$

Pólya's conjecture implies Riemann hypothesis but Haselgrove disproved Pólya's conjecture in 1958 (see [8]). The smallest number $n \geq 2$ such that $L(n) > 0$ is $n = 906150256$ (see [10]) It is unknown if $L(x)$ changes sign infinitely often.

In the end let us mention other facts concerning the cumulative sum of the Liouville function (see [11] for them).

What is known about the L function is that

$$\liminf_{x\to\infty} \frac{L(x)}{\sqrt{x}} < \frac{1}{\zeta(\frac{1}{2})}, \limsup_{x\to\infty} \frac{L(x)}{\sqrt{x}} > \frac{1}{\zeta(\frac{1}{2})},$$

where ζ is the famous Zeta function of Riemann.

Also it is known that if we assume the Riemann hypothesis to be true then

$$L(x) = O(x^{\epsilon+\frac{1}{2}})$$

with arbitrarily small positive ϵ.

References

[1] T. Albu, I. D. Ion: Capitole de teoria algebrică a numerelor , *Editura Academiei R.S.R., Bucureşti*, 1984.

[2] R. Ayoub: An Introduction to the Analytic Theory of Numbers , *American Mathematical Society*, 1963.

[3] Z. I. Borevici, I. R. Şafarevici: Teoria Numerelor, *Editura Ştiinţifică şi Pedagogică, Bucureşti*, 1985.

[4] A. Gica, *A conjecture which implies the theorem of Gauss-Heegner-Stark-Baker*. Bull. Math. Soc. Sc. Math. Roumanie, Tome **41**(**89**) (1998), 111-118.

[5] A. Gica, *The proof of a conjecture of additive number theory.* J. Number Theory **94** (2002), 80–89.

[6] A. Gica, *Another proof of a conjecture in additive number theory.* Math. Reports **4**(**54**) (2002), 171–175.

[7] A. Gica, *A statement equivalent to a theorem of Gauss.* Rev. Roumaine Math. Pures Appl. **49** (2004), 2, 135–139.

[8] C. B. Haselgrove, *A disproof of a conjecture of Polya.* Mathematika **5** (1958), 141–145.

[9] H. M. Stark, *A complete determination of the complex quadratic fields of class number one.* Michigan Math. J. **14** (1967), 1–27.

[10] Minoru Tanaka, *A numerical investigation on cumulative sum of the Liouville function.* Tokyo J. Math. **3** (1980), 1, 187–189.

[11] Minoru Tanaka, *On the Mobius and Allied Functions.* Tokyo J. Math. **3** (1980), 2, 215–218.

Received: 5.12.2005

Faculty of Mathematics, University of Bucharest,
Str. Academiei 14, Bucharest,
RO-010014, Romania
E-mail: alex@al.math.unibuc.ro