CUBE ROOTS OF REAL 2×2 MATRICES

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Abstract. We present a simple criterion guaranteeing the existence of real cube roots for a 2×2 matrix with real entries. A complete description of such roots is then possible.

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Every real number admits an unique real cube root. In the spirit of [1] we can ask the same question for real 2×2 matrices A: When does A admit a real matrix cube root, that is a real 2×2 matrix S such that $S^3 = A$? The answer is given in the following theorem, whose proof follows naturally from two subsequent lemmas.

For any 2×2 matrix M, $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, $p, q, r, s \in \mathbb{R}$, we consider its usual invariants, trace and determinant, defined by $\operatorname{tr}(A) = p + s$ and $\det(A) = ps - qr$. It is easy to check in this case that the following Cayley-Hamilton formula in matrices holds true:

$$M^2 - \operatorname{tr}(M)M + \det(M)I = 0, \tag{1}$$

where I and 0 are the identity respectively zero 2×2 matrices. By iteration we get

$$\begin{aligned} M^2 &= \operatorname{tr}(M)M - \det(M)I, \\ M^3 &= \left(\operatorname{tr}^2(M) - \det(M)\right) M - \operatorname{tr}(M)\det(M)I. \end{aligned} \tag{2}$$

Equations (2) yield then immediately

$$\begin{aligned} \operatorname{tr}(M^2) &= \operatorname{tr}^2(M) - 2\det(M), \\ \operatorname{tr}(M^3) &= \operatorname{tr}^3(M) - 3\operatorname{tr}(M)\det(M). \end{aligned} \tag{3}$$

Theorem. a) *The 2×2 matrix O_2 admits lots of real cube roots, as shown by Lemma 2 : They are all the real 2×2 matrices S with vanishing trace and determinant.*

b) *A non-zero real 2×2 matrix A admits real cube roots if and only if $A^2 \neq O_2$.*

Lemma 1. *Assume that the real 2×2 matrix A has the property that the cubic equation $t^3 - 3\sqrt[3]{\det(A)}t - \operatorname{tr}(A) = 0$ possesses a simple real root. Then A admits real matrix cube roots.*

Proof [Proof of Lemma 1]. If A were to admit a real cube root S then $S^3 = A$, and consequently $\det(S) = \sqrt[3]{\det(A)}$. The second equation of (2)

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would give then

$$\left(\mathrm{tr}^2(S) - \sqrt[3]{\det(A)}\right) S = A + \mathrm{tr}(S)\sqrt[3]{\det(A)}I, \tag{4}$$

while the second of equations (3),

$$\mathrm{tr}^3(S) - 3\sqrt[3]{\det(A)}\mathrm{tr}(S) - \mathrm{tr}(A) = 0. \tag{5}$$

Let now t_0 be a simple real root of the cubic equation

$$t^3 - 3\sqrt[3]{\det(A)}t - \mathrm{tr}(A) = 0,$$

as stipulated in the hypothesis of the lemma. Being a simple root, we have $t_0^2 - \sqrt[3]{\det(A)} \neq 0$. As suggested by equation (4), we can now define

$$S_0 := \frac{1}{t_0^2 - \sqrt[3]{\det(A)}}\left(A + t_0\sqrt[3]{\det(A)}I\right), \tag{6}$$

and claim that this S_0 is the desired matrix cube root of A.

For starters, notice that for S_0 given by equation (6), $\mathrm{tr}(S_0)$ equals indeed t_0, via the facts

$$\begin{aligned} \mathrm{tr}(S_0) = \frac{1}{t_0^2 - \sqrt[3]{\det(A)}}&\left(\mathrm{tr}(A) + 2t_0\sqrt[3]{\det(A)}\right), \\ \mathrm{tr}(A) &= t_0^3 - 3\sqrt[3]{\det(A)}t_0. \end{aligned} \tag{7}$$

To the end of showing that $S_0^3 = A$ we have, via equations (2)

$$\begin{aligned} &(A + t_0\sqrt[3]{\det(A)}I)^3 = A^3 + 3t_0\sqrt[3]{\det(A)}A^2 + 3t_0^2\sqrt[3]{\det(A)^2}A + t_0^3\det(A)I \\ &= \left(\mathrm{tr}^2(A) - \det(A)\right)A - \mathrm{tr}(A)\det(A)I + 3t_0\sqrt[3]{\det(A)}\left(\mathrm{tr}(A)A - \det(A)I\right) \\ &\quad +3t_0^2\sqrt[3]{\det(A)^2}A + t_0^3\det(A)I = \big(\mathrm{tr}^2(A) - \det(A) + 3t_0\sqrt[3]{\det(A)}\mathrm{tr}(A) \\ &\quad +3t_0^2\sqrt[3]{\det(A)^2}\big)A + \det(A)\big(-\mathrm{tr}(A) - 3t_0\sqrt[3]{\det(A)} + t_0^3\big)I \\ &= \left(\left(t_0^3 - 3t_0\sqrt[3]{\det(A)}\right)^2 - \det(A) + 3t_0\sqrt[3]{\det(A)}\left(t_0^3 - 3t_0\sqrt[3]{\det(A)}\right)\right. \\ &\quad \left. +3t_0^2\sqrt[3]{\det(A)^2}\right)A = \left(t_0^2 - \sqrt[3]{\det(A)}\right)^3 A. \end{aligned}$$

The proof of Lemma 1 is complete. □

Lemma 2. *If A is a real 2×2 matrix, then the following statements are equivalent*:

a) $\mathrm{tr}(A) = \det(A) = 0$.

b) $A^2 = 0$.

c) $A^3 = 0$.

Proof [Proof of Lemma 2]. a) $\Longrightarrow$ b) is a consequence of the *Cayley-Hamilton* formula (1), and b) $\Longrightarrow$ c) is trivial. Assume now that $A^3 = 0$. Then $0 = \det(A^3) = \det^3(A)$ gives $\det(A) = 0$. By the second equation of (2), $A^3 = \mathrm{tr}^2(A)A = 0$, which is equivalent to $\mathrm{tr}(A) = 0$. □

Proof [Proof of the Theorem]. Part a) of the theorem is exactly the content of Lemma 2.

Assume now that a non-zero real 2×2 matrix A possesses a real cube root S. S is a non-zero matrix, since A is so. If, by contradiction, $A^2 = 0$, then $\det(S) = 0$. From equations (2) we conclude that $S^2 = \operatorname{tr}(S)S$ and $A = S^3 = \operatorname{tr}^2(S)S$, and so $0 = A^2 = \operatorname{tr}^4(S)S^2 = \operatorname{tr}^5(S)S$. Now $\operatorname{tr}^5(S)S = 0$ is equivalent to $\operatorname{tr}(S) = 0$, which by Lemma 2 forces $S^3 = A$ to be the zero matrix, contradiction.

Conversely, assume that $A^2 \neq 0$. The cubic equation

$$t^3 - 3\sqrt[3]{\det(A)}\,t - \operatorname{tr}(A) = 0$$

has always at least one real root, and we claim that it also has a simple real root. Indeed, if all its roots are real, they can all be simple, one double and one simple, or one triple. It is obvious though that the only triple real root may be 0, in which case $\operatorname{tr}(A) = \det(A) = 0$, or $A^2 = 0$, which is excluded. Now, if t_0 is such a simple root of the above cubic equation, formula (6) provides a cube root of A, cf. Lemma 1.

The proof of the theorem is complete. □

It is not hard to see that the details of the proof of Lemma 1 and a study of the nature of the possible real roots for the cubic equation

$$t^3 - 3\sqrt[3]{\det(A)}\,t - \operatorname{tr}(A) = 0$$

lead to a complete description of the possible cube roots of a given real 2×2 matrix A. We state it below as a corollary, and encourage the interested reader to supply a proof for it.

Corollary. *Let A be a real 2×2 matrix, $A \neq 0$, $A^2 \neq 0$.*

a) *If $A \neq \lambda I$, λ non-zero real number, then A admits exactly three real cube roots if $4\det(A) > \operatorname{tr}^2(A)$, and exactly one real cube root if $4\det(A) \leq \operatorname{tr}^2(A)$. The matrix cube roots of A are then given by formula (6), for the requisite single real roots of the cubic equation $t^3 - 3\sqrt[3]{\det(A)}\,t - \operatorname{tr}(A) = 0$.*

b) *If $A = \lambda I$ for some non-zero real number λ, then A possesses infinitely many matrix cube roots S, given by $S = \sqrt[3]{\lambda}\Sigma$, where Σ is an arbitrary real 2×2 matrix with $\operatorname{tr}(\Sigma) = -1$ and $\det(\Sigma) = 1$, and also $S = \sqrt[3]{\lambda} I$.*

REFERENCES

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