ARTICOLE ŞI NOTE MATEMATICE

AREA OF A RECTANGLE WITH VERTICES ON CONCENTRIC CIRCLES

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Abstract. In this paper we investigate the extreme values for the area of a rectangle with vertices on concentric circles. We present the origins of an old geometrical problem and an elementary solution for this problem.

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In [3], the following problem was proposed:

Problem 1. *Given four concentric circles $\mathcal{C}(P,x), \mathcal{C}(P,y), \mathcal{C}(P,z)$ and $\mathcal{C}(P,t)$ of center P and radii x, y, z, respectively t, where $x^2 + z^2 = y^2 + t^2$, consider four points $A \in \mathcal{C}(P,x)$, $B \in \mathcal{C}(P,y)$, $C \in \mathcal{C}(P,z)$ and $D \in \mathcal{C}(P,t)$ such that the quadrilateral $ABCD$ is a rectangle. Find the extreme values for of area of such a rectangle.*

The authors of the mentioned paper consider this problem „unusual of its surprisingly answer in spite of our rather laborious solution". The answer offered there is included in the next proposition.

Proposition 1. *The maximum area of the rectangle in Problem 1 is $xz + yt$ and the minimum area is $|xz - yt|$.*

In [1], we find a short note about the proof of the previous proposition. Since the original proof is very long (over 8 pages), another solution is proposed there. This solution is shorter and uses differential calculus. Starting from this fact, the authors of this paper investigated the origins of this problem. We searched the answers of the following two questions.

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Question 1. *What are the algebraic conditions for the existence of this rectangle?*

Question 2. *How can these values of the extremum of the area be obtained?*

Our research finds answers to both questions. In this context, the aim of this paper is to present some results collected from mathematical journals which point to the conclusion of Problem 1. Moreover, we propose an elementary solution to this problem. Supplementary, we present some concrete situations when the extremum mentioned in Problem 1 are obtained.

The starting point was represented by problem 11057 from Monthly (see [8]). We restate it:

Problem 2. *Let x, y and z be positive real numbers. What is the maximum area possible for a rectangle $ABCD$ given that it has an interior point P such that the lenghts AP, BP and CP are $x, y,$ and z, respectively?*

Ionaşcu and *Stănică* completed the conditions for the rectangle $ABCD$, but the existence of the restriction $x^2 + z^2 = y^2 + t^2$ was clarified later. In [5], *McFarland* proposed the next problem:

Problem 3. *Given four concentric circles, find a necessary and sufficient condition so that there exists a rectangle with one vertex on each circle.*

We found a solution due to *Linders* (see [4]). The results are included in the following proposition, which are completed by our proof.

Proposition 2. *Let $\mathcal{C}(P, x), \mathcal{C}(P, y), \mathcal{C}(P, z)$ and $\mathcal{C}(P, t)$ be four concentric circles, where $x > y > t > z > 0$. A necessary and sufficient condition for the existence of four points $A \in \mathcal{C}(P, x),$ $B \in \mathcal{C}(P, y),$ $C \in \mathcal{C}(P, z)$ and $D \in \mathcal{C}(P, t)$ such that the quadrilateral $ABCD$ is a rectangle is*

$$x^2 + z^2 = y^2 + t^2.$$

Proof. Let $ABCD$ be a rectangle and denote O its center. The median's theorem gives us the equality

$$PO^2 = \frac{2PA^2 + 2PC^2 - AC^2}{4} = \frac{2PB^2 + 2PD^2 - BD^2}{4}.$$

Then $PA^2 + PC^2 = PB^2 + PD^2$, hence $x^2 + z^2 = y^2 + t^2$.
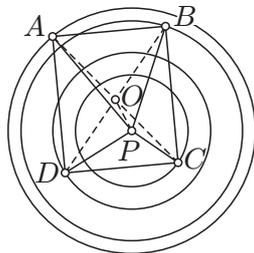
For the converse, we choose the points $B \in \mathcal{C}(P, y)$, $C \in \mathcal{C}(P, z)$ and $D \in \mathcal{C}(P, t)$ such that the triangle BCD has a right angle at C. We construct A, the fourth vertex of the rectangle $ABCD$. The median's theorem yields

$$PA^2 + PC^2 = PB^2 + PD^2.$$

We obtain

$$PA^2 = PB^2 + PD^2 - PC^2$$

and

$$PA^2 = y^2 + t^2 - z^2 = x^2.$$

We obtain $PA = x$, so $A \in \mathcal{C}(P, x)$. This concludes our proof. $\square$

Now we have an answer to Question 1. For the second question, we found an old problem in Kvant [2]).

Problem 4. *Let $ABCD$ be a rectangle and denote $A_{[ABCD]}$ its area. Let P be an interior point. Then*

$$A_{[ABCD]} \leq PA \cdot PC + PB \cdot PD. \tag{1}$$

Since, for any rectangle $ABCD$, we have $A_{[ABCD]} = AB \cdot AD$, the inequality (1) is a particular case of the inequality from the next problem due to *Mortici* (see [7]).

Problem 5. *Let $ABCD$ be a rectangle and P be an interior point. Then*

$$AB \cdot AD \leq PA \cdot PC + PB \cdot PD. \tag{2}$$

Marinescu (see [6]) removed the condition about the position of the point P and completed the inequality (2) with a bound from below. This inequality offers us the answer to Question 2.

Problem 6. *Let P be a point situated in the plane of the parallelogram $ABCD$. Then*

$$|PA \cdot PC - PB \cdot PD| \leq AB \cdot AD \leq PA \cdot PC + PB \cdot PD. \tag{3}$$

Due to the previuos results, our challenge was to connect the solution of (3) with the Proposition 1. The main tool is represented by the following lemma:

Lemma. *Let a, b, c, d be four complex numbers such that $a + c = b + d$. Then*
   a) *For any $z \in \mathbb{C}$, we have*

$$(a - b)(a - d) = (z - b)(z - d) - (z - a)(z - c).$$

   b) *For any $z \in \mathbb{C}$, we have*

$$\big||z - b| \cdot |z - d| - |z - a| \cdot |z - c|\big| \leq |a - b| \cdot |a - d| \leq |z - b| \cdot |z - d| + |z - a| \cdot |z - c|.$$

Proof. For the assertion a) we have :

$$(z - b)(z - d) - (z - a)(z - c) = z^2 - (b + d) z + bd - z^2 + (a + c) z - ac =$$

$$= bd - ac = bd - a(b + d - a) = bd - ab - ad + a^2 =$$

$$= a(a - d) - b(a - d) = (a - b)(a - d).$$

The assertion b) is a consequence of the previous equality and the inequality

$$||w| - |z|| \leq |w - z| \leq |w| + |z|,$$

which holds for any $w, z \in \mathbb{C}$. □

As a consequence of the previous Lemma, we obtain a short solution to Problem 6.

Proof. [Solution of Problem 6] We denote a, b, c, respectively d, the complex coordinates to the vertices of the parallelogram $ABCD$. Then $a + c = b + d$. Let z be the complex coordinate of the point P. Then the inequality (3) represents a geometric form of the assertion b) of the Lemma.

□

Now, we are in position to present our solution to Proposition 1.

Proof. [Proof of Proposition 1.] Consider P the origin of the complex plane. We denote a, b, c, respectively d, the complex coordinates of the points A, B, C and respectively D. We have $|a| = x$, $|b| = y$, $|c| = z$ and $|d| = t$. The quadrilateral $ABCD$ is a rectangle, so $a + c = b + d$. Problem 6 yields

$$AB \cdot AD \leq PA \cdot PC + PB \cdot PD.$$

Then

$$A_{[ABCD]} \leq xz + yt. \tag{4}$$

For the maximum value of the area, we need the equality in (4). We suppose $0 < z < t < y < x$. Let $r > 0$ be such that $r = \sqrt{x^2 + z^2} = \sqrt{y^2 + t^2}$. We choose $a = \dfrac{xt}{r} + \dfrac{yx}{r}\mathrm{i}$, $c = -\dfrac{yz}{r} - \dfrac{tz}{r}\mathrm{i}$, $b = -\dfrac{zy}{r} + \dfrac{xy}{r}\mathrm{i}$ and $d = \dfrac{xt}{r} - \dfrac{zt}{r}\mathrm{i}$. We have $|a| = x$, $|c| = z$, $|b| = y$, $|d| = t$ and $a + c = b + d$. Then $ABCD$ is parallelogram. Moreover,

$$AC = BD = \sqrt{(xt - yz)^2 + (xy + zt)^2}$$

and $ABCD$ is rectangle. Now we have

$$A_{[ABCD]} = AB \cdot AD = |b - a| \cdot |d - a| = \frac{|zy + xt|}{r} \cdot \frac{|yx + zt|}{r} =$$

$$= \frac{x^2yt + ytz^2 + xzy^2 + xzt^2}{r^2} = \frac{yt(x^2 + z^2) + xz(y^2 + t^2)}{r^2} = xz + yt.$$

On the other hand we have

$$AB \cdot AD \geq |PA \cdot PC - PB \cdot PD|,$$

so

$$A_{[ABCD]} \geq |xz - yt|. \tag{5}$$

The equality in (5) is obtained in the following case. Using the same conditions and the same notations as in previous example, we choose

$a = \dfrac{xt}{r} + \dfrac{yx}{r}\text{i},\ c = \dfrac{yz}{r} + \dfrac{tz}{r}\text{i},\ b = \dfrac{zy}{r} + \dfrac{xy}{r}\text{i}$ and $d = \dfrac{xt}{r} + \dfrac{zt}{r}\text{i}.$
We have $|a| = x,\ |c| = z,\ |b| = y,\ |d| = t$.

The conditions $a + c = b + d$ and

$$AC = BD = \sqrt{(xt - yz)^2 + (xy - zt)^2}$$

show that $ABCD$ is rectangle. Then we have

$$A_{[ABCD]} = AB \cdot AD = |b - a| \cdot |d - a| = \frac{|zy - xt|}{r} \cdot \frac{|yx - zt|}{r} =$$

$$= \frac{\left|xzy^2 + xzt^2 - x^2yt - ytz^2\right|}{r^2} = \frac{\left|xz\left(y^2 + t^2\right) - yt\left(x^2 + z^2\right)\right|}{r^2} = |xz - yt|.$$

Now, the proof is complete.

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