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THE EXTENDED BUTTERFLY THEOREM

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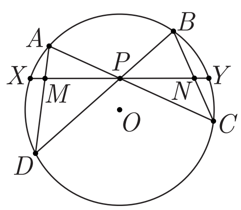
Abstract. This article presents an extension of the Butterfly Theorem and some of its applications.

Keywords: Butterfly Theorem, concyclic points.

MSC : 51M04

1. Main Results

The Butterfly Theorem. *Let P be the intersection of the diagonals of a quadrilateral $ABCD$ inscribed in circle $\mathcal{C}$ of center O. A line d through P meets cicle $\mathcal{C}$ at points X and Y. Let $d \cap AD = \{M\}$ and $d \cap BC = \{N\}$. If $PX = PY$, then $PM = PN$.*



Proof. Points A, B, C, D, X, Y are concyclic, therefore the pencils $A,(XCDY)$ and $B,(XCDY)$ have the same cross-ratio. Intersecting these two pencils with line d, we obtain that $\dfrac{MX}{MP} = \dfrac{YX}{YP}$ and $\dfrac{NP}{NY} = \dfrac{XP}{XY}$. Since $PY = PX$, this yields succesively

$$\frac{MX}{MP} = \frac{NY}{NP},\ \frac{MX}{PX} = \frac{NY}{PY},\ MX = NY,\ PM = PN. \qquad \square$$

We will extend this theorem to a more general configuration.

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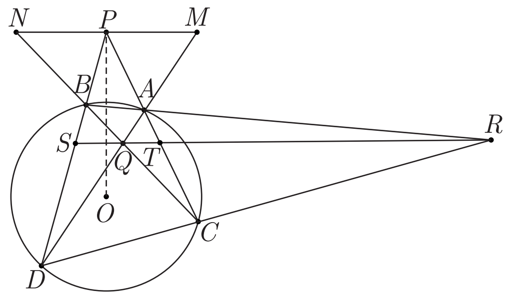
Let's notice that we can equate the condition from the problem's hypothesis to a more interesting one: „$PX = PY \Leftrightarrow OP \perp d$“. Thus, the new condition no longer requires that point P lies in the interior of circle $\mathcal{C}$. This leads us to the following result,

Extended Butterfly Theorem (EBT). *Let $\mathcal{C}$ be a circle of center O and P be a point in its plane, so that $P \notin \mathcal{C}$. Two lines through P meet the circle at points A, C and B, D, respectively. Let d be a line that contains point P and is perpendicular to PO. Let $\{M\} = d \cap AD$ and $\{N\} = d \cap BC$. Then $PM = PN$.*



Proof. Let $\{R\} = AB \cap CD$, $\{Q\} = BC \cap AD$, $\{T\} = QR \cap AC$, $\{S\} = QR \cap BD$. Since QR is the polar line of point P, QR is perpendicular to PO. Thus, $QR || MN$. We now obtain that $\triangle DQS \sim \triangle DMP$ and $\triangle CTQ \sim \sim \triangle CPN$; therefore, $\dfrac{QS}{MP} = \dfrac{DS}{DP}$ and $\dfrac{TQ}{PN} = \dfrac{CT}{CP}$. Showing that $MP = PN$ is now equivalent to proving that

$$\frac{QS \cdot DP}{DS} = \frac{TQ \cdot CP}{CT} \Leftrightarrow \frac{QS}{QT} = \frac{DS}{DP} \cdot \frac{CP}{CT} \tag{1}$$

Since (T, Q, S, R) is a harmonic division,

$$\frac{QS}{QT} = \frac{RS}{RT}. \tag{2}$$

We will now apply Menelaus' Theorem in the triangle PTS, using the transversal line $A - B - R$;

$$\frac{BP}{BS} \cdot \frac{AT}{PA} \cdot \frac{RS}{RT} = 1 \Rightarrow \frac{RS}{RT} = \frac{BS}{BP} \cdot \frac{PA}{AT}. \tag{3}$$

Given the relations (1), (2) and (3), it suffices to prove that

$$\frac{BS}{BP} \cdot \frac{PA}{AT} = \frac{DS}{DP} \cdot \frac{CP}{CT}.$$

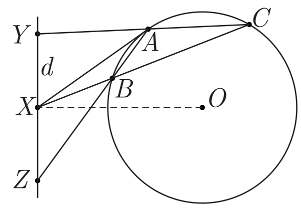
This follows from the fact that (P, A, T, C) and (P, B, S, D) are harmonic divisions, meaning that

$$\frac{AP}{AT} = \frac{CP}{CT} \text{ and } \frac{SB}{BP} = \frac{DS}{DP}. \qquad \square$$

Here is a simple, yet very beautiful application of the previous result.

Application. Consider the degenerated quadrilateral $ABAC$ inscribed in circle $\mathcal{C}$ of center O.

Let $\{X\} = AA \cap BC$. (AA represents the tangent line to $\mathcal{C}$, taken at point A.) Let d be a line that passes through X and is perpendicular to OX. Using EBT, it follows that $XY = YZ$, where $\{Y\} = d \cap AC$ and $\{Z\} = d \cap AB$.



This fact is embodied by the following problem.

Problem. *The tangent line at point A to the circumcircle of triangle ABC intersects BC at point X. d is a line that passes through X and is perpendicular XO. Prove that point X is the midpoint of line segment YZ.*

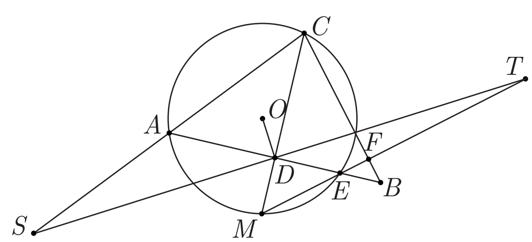
(Argentina, 2003)

We will further present some olympiad problems that can be solved using EBT.

2. Applications

1. *Let ABC be an acute-angled, isosceles triangle ($CA = CB$) and let D be the midpoint of base AB. E is an arbitrary point on line AB and O is the circumcenter of triangle ACE. Show that the parallel to AC through B, the perpendicular onto BC at E and the perpendicular onto OD at D are concurrent.*

(Bulgaria, 2000)

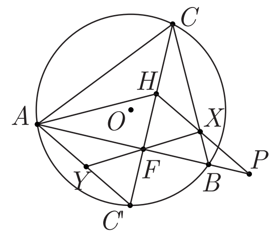


Proof. Let M be the second intersection of CD and the circumcircle of triangle ACE. Let $\{F\} = ME \cap BC$. Since $ACEM$ is a cyclic quadrilateral, $\sphericalangle CME \equiv \sphericalangle CAB$. But $\sphericalangle CAB \equiv \sphericalangle CBA$, so $\sphericalangle CME \equiv \sphericalangle CBA$, which means that points F, B, M and D are concyclic.

Consequently, $\mathrm{m}(\sphericalangle BFM) = \mathrm{m}(\sphericalangle MDB) = 90°$. We obtained that $ME \perp BC$. Let T be the intersection between ME and the perpendicular from D onto OD. Let $\{S\} = DT \cap AC$. Since $DO \perp DT$, by applying EBT in quadrilateral $ACEM$ it follows that $DS = DT$. Using this relation and the fact that $DA = DB$, we find that $ASBT$ is a parallelogram and, thus, $BT||AC$. Hence, the three lines in the problem meet at point T.

2. *Let F be the projection of point C on side AB of the triangle ABC, with $AC > BC$. Let P, O, H be the reflection of A over F, the circumcenter of triangle ABC and the orthocenter, respectively. Lines HP and BC intersect at point X. Prove that $OF \perp FX$.*

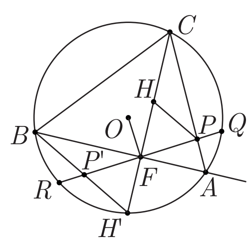
(BMO, 2008)



Proof. Let C' be the reflection of H into F. A well-known result says that C' lies on the circumcircle of $\triangle ABC$. Since $FH = FC'$ and $FA = FP$, we obtain that $HPC'A$ is a parallelogram. Let $\{Y\} = FX \cap AC'$. Then $FX = FY$ and, applying the reciprocal of EBT in quadrilateral $ACBC'$, it results that $OF \perp FX$, q.e.d.

3. *Let ABC be an acute-angled triangle with $BC > CA$. Let O, H and F be the circumcenter, orthocenter and the foot of its altitude CH, respectively. The perpendicular onto OF at F meets the side CA at point P. Prove that $\sphericalangle FHP \equiv \sphericalangle BAC$.*

(Turkey, TST)



Proof. Let the perpendicular onto OF at F meet the circle again at points Q and R, respectively. Let CH meet the circle again at H', which is the reflection of H over AB. Now, let P' be the intersection point of $H'B$ and QR. F is the midpoint of QR so, using EBT, it results that $FP = FP'$. The triangles FHP and $FH'P'$ are therefore congruent. Hence, $\sphericalangle FH'P' = \sphericalangle A = \sphericalangle FHP$.