GAZETA MATEMATICĂ

SERIA B

PUBLICAŢIE LUNARĂ PENTRU TINERET

Fondată în anul 1895

Anul CXVIII nr. 11 **noiembrie 2013**

ARTICOLE ŞI NOTE MATEMATICE

SQUARE ROOTS OF REAL 2×2 MATRICES

Nicolae Anghel[1)]

Abstract. In this note we investigate the real 2×2 matrices which admit real square roots.

Keywords: Matrix, Square Root

MSC : Primary: 15A23. Secondary: 11C20.

Only non-negative real numbers admit real square roots. Thinking of a real number as the simplest square matrix, a 1×1 matrix, an interesting question emerges. Which real $n \times n$ matrices, $n \geq 1$, admit real square roots? In other words, for which $A \in \mathcal{M}_n(\mathbb{R})$ is there an $S \in \mathcal{M}_n(\mathbb{R})$ such that $S^2 = A$?

In this short note we will give a complete answer to the above question in the case $n = 2$. As a corollary, we will also establish how many distinct real square roots a given 2×2 matrix has, and then proceed to describe them exactly.

Theorem. *For a given matrix* $A \in \mathcal{M}_2(\mathbb{R})$ *there are matrices* $S \in \in \mathcal{M}_2(\mathbb{R})$ *such that* $S^2 = A$ *if and only if* $\det A \geq 0$ *and, either* $A = -\sqrt{\det A} I$ *or* $\mathrm{tr} A + 2\sqrt{\det A} > 0$, *where* I *is the* 2×2 *identity matrix. Obviously, in the latter case,* $\mathrm{tr} A + 2\sqrt{\det A} = 0$.

Proof. Necessity. Assume that S is a real square root of A. From $S^2 = A$ we conclude that $(\det S)^2 = \det A$, so we must have $\det A \geq 0$. Any matrix $M \in \mathcal{M}_2(\mathbb{R})$ satisfies the Cayley-Hamilton equation, namely $M^2 - (\mathrm{tr} M) M + (\det M) I = 0$. In particular, $S^2 - (\mathrm{tr} S) S + (\det S) I = 0$ implies $A - (\mathrm{tr} S) S + (\det S) I = 0$. There are two cases to consider:

[1)] Professor, University of North Texas, Denton, TX

(1) $\mathrm{tr}S \neq 0$. Since for a matrix $M \in \mathcal{M}_2(\mathbb{R})$ we have $\mathrm{tr}(M^2) = (\mathrm{tr}M)^2 - 2\det M$, by taking $M = S$ we have $\mathrm{tr}A + 2\det S = (\mathrm{tr}S)^2 > 0$. However, $\mathrm{tr}A + 2|\det S| \geq \mathrm{tr}A + 2\det S$, and so we get $\mathrm{tr}A + 2|\det S| > 0$. Now $(\det S)^2 = \det A$ is equivalent to $|\det S| = \sqrt{\det A}$. In conclusion, $\mathrm{tr}A + 2\sqrt{\det A} > 0$.

(2) $\mathrm{tr}S = 0$. In this case, $A - (\mathrm{tr}S)S + (\det S)I = 0$ becomes $A = -(\det S)I$. If $\det S < 0$, $\mathrm{tr}A + 2\sqrt{\det A} = -4\det S > 0$, and there is nothing to prove. If $\det S \geq 0$, we have $\det S = \sqrt{\det A}$, and so $A = -\sqrt{\det A}I$. Clearly, then $\mathrm{tr}A + 2\sqrt{\det A} = 0$. Therefore, there are no matrices A with $\mathrm{tr}A + 2\sqrt{\det A} < 0$, which admit real square roots.

Sufficiency. If $\det A \geq 0$ and $\mathrm{tr}A + 2\sqrt{\det A} > 0$, a direct calculation taking into account that $A^2 = (\mathrm{tr}A)A - (\det A)I$ shows that

$$S := \frac{1}{\sqrt{\mathrm{tr}A + 2\sqrt{\det A}}}(A + \sqrt{\det A}I)$$

is a real square root of A. Suitable equations, found in the necessity part of the proof, show that this is the only possible real square root of A with positive trace and non-negative determinant.

If $\det A \geq 0$ and $A = -\sqrt{\det A}I$, then it is easily seen that

$$S := \begin{pmatrix} 0 & 1 \\ -\sqrt{\det A} & 0 \end{pmatrix}$$

is a square root of A. □

Corollary. *As the existence of real square roots goes, the following is true about any matrix* $A \in \mathcal{M}_2(\mathbb{R})$:

(1) *If* $A \neq aI$, $a \in \mathbb{R}$, *then* A *admits only finitely many real square roots, as follows:*

(a) *If* $\det A > 0$ *and* $\mathrm{tr}A - 2\sqrt{\det A} > 0$, *there are exactly four distinct square roots, given by*

$$S = \pm\frac{1}{\sqrt{\mathrm{tr}A + 2\sqrt{\det A}}}(A + \sqrt{\det A}I)$$

or

$$S = \pm\frac{1}{\sqrt{\mathrm{tr}A - 2\sqrt{\det A}}}(A - \sqrt{\det A}I).$$

(b) *If* $\det A < 0$, *or* $\det A \geq 0$ *and* $\mathrm{tr}A + 2\sqrt{\det A} \leq 0$, *there are no real square roots.*

(c) *Otherwise, there are exactly two distinct real square roots, given by* $S = \pm\dfrac{1}{\sqrt{\mathrm{tr}A + 2\sqrt{\det A}}}(A + \sqrt{\det A}I)$.

(2) *If* $A = aI$, $a \in \mathbb{R}$, *then* A *admits infinitely many real square roots. Regardless of* a, *the doubly infinite family* $S = \begin{pmatrix} s & t \\ \dfrac{a - s^2}{t} & -s \end{pmatrix}$, $s, t \in \mathbb{R}$, $t \neq 0$, *is in. If* $a = 0$ *we add to the above family the matrices* $S = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix}$, $s \in \mathbb{R}$, *while if* $a > 0$ *we add the family* $S = \begin{pmatrix} \pm\sqrt{a} & 0 \\ s & \mp\sqrt{a} \end{pmatrix}$, $s \in \mathbb{R}$, *plus the two more matrices given by* $S = \begin{pmatrix} \pm\sqrt{a} & 0 \\ 0 & \pm\sqrt{a} \end{pmatrix}$ *(the signs correspond).*

Proof. If we are in case (1)(a), a direct calculation, similar to that in the sufficiency part of the Theorem, shows that indeed the four proposed matrices are real square roots of A. Conversely, let S be a real square root of A. As in the proof of the Theorem, we then have $(\det S)^2 = \det A$, $(\mathrm{tr} S)^2 = \mathrm{tr} A + 2 \det S$, and $A - (\mathrm{tr} S)S + (\det S)I = 0$.

If $\det S = \sqrt{\det A}$, then $\mathrm{tr} S = \pm\sqrt{\mathrm{tr} A + 2\sqrt{\det A}}$, and so

$$S = \pm \frac{1}{\sqrt{\mathrm{tr} A + 2\sqrt{\det A}}} (A + \sqrt{\det A} I).$$

Similarly, if $\det S = -\sqrt{\det A}$ we get the other two matrices in the family of square roots.

The four matrices are distinct because, for instance, their traces are distinct:

$$\sqrt{\mathrm{tr} A + 2\sqrt{\det A}} > \sqrt{\mathrm{tr} A - 2\sqrt{\det A}} > -\sqrt{\mathrm{tr} A - 2\sqrt{\det A}} > -\sqrt{\mathrm{tr} A + 2\sqrt{\det A}}.$$

The case (1)(b) follows immediately from the Theorem, by negation, since $A \neq aI$, $a \in \mathbb{R}$.

In case (1)(c) „otherwise“ means after some „detective work“, $A \neq aI$, $a \in \mathbb{R}$ and in addition, $\det A > 0$ and $\mathrm{tr} A + 2\sqrt{\det A} > 0$ and $\mathrm{tr} A - 2\sqrt{\det A} \leq$ ≤ 0, or $\det A = 0$ and $\mathrm{tr} A > 0$. The claimed conclusion can then be reached as in (1)(b).

Finally, the case (2) is an easy ‘by hand’ calculation, given the simple structure of A.