C.O:5114. Să se determine funcţiile $f:[0,1]\to\mathbb{R}$ derivabile cu derivata continuă, astfel încât $f(0)=0$, $f(1)=1$ şi:

$$\int_0^1\left(x^2\left(f'(x)\right)^2+\left(f(x)\right)^2\right)\mathrm{d}x=\frac{\sqrt{5}-1}{2}.$$

Cantemir Iliescu, Piteşti

PROBLEMS FOR COMPETITIONS AND OLYMPIADS

Junior Level

C.O:5107. Find all integer values of n such that 13 divides $\underbrace{55\ldots5}_{n\text{ times}}1$.

Ovidiu Ţâţan, Rm. Sărat

C.O:5108. Show that $\dfrac{a^4}{b}+\dfrac{b^4}{c}+\dfrac{c^4}{a}\geq ab^2+bc^2+ca^2$, for all $a,b,c>0$.

Gabriela Boeriu, Făgăraş

C.O:5109. Let $ABCD$ be a rhombus of side lenght 1, with $\mathrm{m}(\sphericalangle ABC)=60^\circ$. Consider point E, on the other side of line AB as C, such that $CE\perp AB$ and $CE=AB$. Let F be a point on line AB such that $DF=DE$. Find the lenght of the segment AF.

Neculai Stanciu, Buzău

C.O:5110. Six prime numbers are in arithmetical progression. Find the minimum value of the largest number.

Dan Nedeianu, Drobeta Tr. Severin

Senior Level

C.O:5111. Find all positive integers p for which there exist integers x and y such that $x^3+y^3=(p+3)!+4$.

Marian Cucoaneş, Mărăşeţi

C.O:5112. Let a, b, c be real numbers such that $c\geq a\geq b$ and $a^2\geq bc$. Prove that: $\dfrac{a}{\sqrt{a+b}}+\dfrac{b}{\sqrt{b+c}}+\dfrac{c}{\sqrt{c+a}}\geq\dfrac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{2}}$.

Tuan Le, Fairmont H. S. Anaheim, U. S. A.

C.O:5113. The real sequence $\{x_n\}_{n\geq0}$ is defined by $x_0\in(0,\pi)$ and $x_{n+1}=$ $=\sin x$, $n\geq 0$. Evaluate $\lim\limits_{n\to\infty}nx_n^2$.

Neculai Stanciu, Buzău

C.O:5114. Find all differentiable functions $f:[0,1]\to\mathbb{R}$, with continuous derivative and with the properties: $f(0)=0$, $f(1)=1$ and:

$$\int_0^1\left(x^2\left(f'(x)\right)^2+\left(f(x)\right)^2\right)\mathrm{d}x=\frac{\sqrt{5}-1}{2}.$$

Cantemir Iliescu, Piteşti

THE DISTRICT ROUND OF THE NATIONAL OLYMPIAD 2010

7^{th} Grade

1. **(i)** Factorize $xy - x - y + 1$.

(ii) Prove that if integers a and b satisfy $|a+b| > |1+ab|$, then $ab = 0$.

2. Let n be an integer, $n \geq 2$. Find the remainder of the division of the number $n(n+1)(n+2)$ by $n-1$.

3. Let ABC be a triangle with $AB = AC$ and $\sphericalangle BAC = 40°$. Points S and T lie on the sides AB and BC, respectively, such that $\sphericalangle BAT = \sphericalangle BCS = 10°$. Lines AT and CS intersect at point P. Show that $BT = 2PT$.

4. Consider a quadrilateral $ABCD$ with $AD = DC = CB$ and $AB \parallel CD$. Points E and F lie on the sides CD and BC such that $\sphericalangle ADE = \sphericalangle AEF$. Prove that:

(i) $4CF \leq CB$. **(ii)** If $4CF = CB$, then AE is the bisector line of $\sphericalangle DAF$.

8^{th} Grade

1. **(i)** Prove that one cannot assign to each vertex of a cube 8 distinct numbers from the set $\{0, 1, 2, 3, \ldots, 11, 12\}$ such that, for every edge, the sum of the two numbers assigned to its vertices is even.

(ii) Prove that one can assign to each vertex of a cube 8 distinct numbers from the set $\{0, 1, 2, 3, \ldots, 11, 12\}$ such that, for every edge, the sum of the two numbers assigned to its vertices is divisible by 3.

2. Let x, y be distinct positive integers. Show that the number $\dfrac{(x+y)^2}{x^3 + xy^2 - x^2y - y^3}$ is not an integer.

3. Consider the cube $ABCDA'B'C'D'$. The bisector lines of the angles $\sphericalangle A'C'A$ and $\sphericalangle A'AC'$ intersect AA' and $A'C'$ in points P and Q, respectively. Point M is the foot of the perpendicular from A' to $C'P$ while N is the foot of the perpendicular from A' to AS. Point O is the center of the face $ABB'A'$.

(i) Prove that the planes (MNO) and $(AC'B)$ are parallel.

(ii) Given that $AB = 1$, find the distance between the planes (MNO) and $(AC'B)$.

4. Find all non negative integers (a, b) such that:

$$a + 2b - b^2 = \sqrt{2a + a^2 + |2a + 1 - 2b|}.$$

9^{th} Grade

1. A line passing through the incenter I of a triangle ABC intersects the sides AB and AC at points P and Q respectively. Let $BC = a$, $AC = b$, $AB = c$ and $\dfrac{PB}{PA} = p$, $\dfrac{QC}{QA} = q$.

(i) Prove that $a(1+p)\overrightarrow{IP} = (a - pb)\overrightarrow{IB} - cp\overrightarrow{IC}$. **(ii)** Prove that $a = bp + cq$.

(iii) Prove that if $a^2 = 4bcpq$, then lines AI, BQ and CP are concurrent.

2. Consider the sequence $(x_n)_{n\geq 0}$ given by $x_n = 2^n - n$, $n \geq 0$. Find all integers $p \geq 0$ such that the number $s_p = x_0 + x_1 + x_2 + \cdots + x_p$ is a power of 2.

3. Let x be a real number. Prove that x is an integer if and only if

$$[x] + [2x] + [3x] + \cdots + [nx] = \frac{n\left([x] + [nx]\right)}{2}$$

holds for all positive integers n.

Here, $[a]$ denotes the integer part of the real number a.

4. Find all functions $f : \mathbb{N}^* \to \mathbb{N}^*$ such that $f(n)+f(n+1)+f(f(n)) = 3n+1$, for all $n \in \mathbb{N}^*$.

10^{th} Grade

1. Prove that:

(i) $\{x \in \mathbb{R} \mid \log_2 [x] = [\log_2 x]\} = \bigcup_{m\in\mathbb{N}} [2^m,\ 2^m + 1)$.

(ii) $\left\{x \in \mathbb{R} \mid 2^{[x]} = [2^x]\right\} = \bigcup_{m\in\mathbb{N}} [m,\ \log_2 (2^m + 1))$.

Here, $[a]$ denotes the integer part of the real number a.

2. Suppose $a \in [-2, \infty)$, $r \in [0, \infty)$ and let n be a positive integer. Show that:

$$r^{2n} + ar^n + 1 \geq (1 - r)^{2n}.$$

3. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that:

$$3f (f (f (n))) + 2f (f (n)) + f (n) = 6n, \text{ for all } n \in \mathbb{N}.$$

4. Consider a complex number z, $z \neq 0$ and the real sequence $a_n = \left|z^n + \dfrac{1}{z^n}\right|$, $n \geq 1$,

(i) Show that if $a_1 > 2$, then $a_{n+1} < \dfrac{a_n + a_{n+2}}{2}$, for all $n \in \mathbb{N}^*$.

(ii) Prove that if there exists $k \in \mathbb{N}^*$ such that $a_k \leq 2$, then $a_1 \leq 2$.

11^{th} Grade

1. Prove that any continuous function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} a_1x + b_1, & \text{for } x \leq 1 \\ a_2x + b_2, & \text{for } x > 1 \end{cases}$, with $a_1, a_2, b_1, b_2 \in \mathbb{R}$, can be represented as $f(x) = m_1x + n_1 + \varepsilon|m_2x + n_2|$, for $x \in \mathbb{R}$, where $m_1, m_2, n_1, n_2 \in \mathbb{R}$ and $\varepsilon \in \{-1, +1\}$.

2. Consider the matrices $A, B \in \mathcal{M}_3(\mathbb{C})$ with $A = -{}^tA$, $B = {}^tB$. Prove that if the polynomial $f(x) = \det(A + xB)$ has a double root, then $\det(A + B) = \det B$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function such that $f \circ f$ is continuous. Prove that f is continuous.

4. Show that there exist sequences $(a_n)_{n\geq 0}$ with $a_n \in \{-1, +1\}$ for all $n \geq 0$ such that $\lim\limits_{n\to\infty} (\sqrt{n + a_1} + \sqrt{n + a_2} + \cdots + \sqrt{n + a_n} - n\sqrt{n + a_0}) = \dfrac{1}{2}$.

12^{th} Grade

1. Let S be the sum of all invertible elements of a finite ring. Prove that $S^2 = S$ or $S^2 = 0$.

2. Let G be a group in which $a^2b = ba^2$ implies $ab = ba$.

(i) Show that if G has 2^n elements, then G is an abelian group.

(ii) Exhibit an example of a nonabelian group with the given propriety.

3. Let $a < c < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be a function continuous at c. Show that if f is the derivative of function on $[a, c)$ and on $(c, b]$, then f is a derivative of a function on $[a, b]$.

4. Let $f:[0,1]\to\mathbb{R}$ be a differentiable function with $f(0)=f(1)$, $\int\limits_0^1 f(x)\mathrm{d}x=0$ and $f'(x)\neq 1$, for every $x\in[0,1]$.

(i) Prove that the function $g:[0,1]\to\mathbb{R}$ given by $g(x)=f(x)-x$ is decreasing.

(ii) Prove that for any integer $n\geq 1$ the following inequality holds:

$$\left|\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\right| < \frac{1}{2}.$$

RUBRICA REZOLVITORILOR DE PROBLEME

În perioada 1 martie-31 martie 2010, au trimis soluţii la problemele propuse următorii elevi:

ALBA IULIA (ALBA) *C. Tehnic „Apulum“* cl.VI Constantinescu Ioana (80); *C. N. „Horea, Cloşca şi Crişan“* cl.IX Macarie Manase (90), Pârvu Ioana Alexandra (80), Trifon Alexandra (100).
BACĂU (BACĂU) *Ş.g. „A.I. Cuza“* cl.V Albu Crina Luciana (300), Boronea Ştefan (210), Coandei Georgiana (300+180), Gavrilă Şerban (320), Guţu Vlad (330), Huiban Raluca (340), Ladoriu Mihai (280), Mârza Alexandra (290), Nicolae Gabriel (320), Tanasov Andrei (330), cl.VI Boronea Anna (90), Chiticaru Oana (110), Chebac Ioana (110), Dărăuţă Goagă Alexandra (110), Iosif Rareş (70), Lăptoiu Ştefan (110), Luca Veronica (100), Nădejde Xenia (110), Paveluc Sabin (100), Păduraru Ambra (120), Popa Vlad Octavian (90), Roman Ana Maria (100), Şorea Ilinca (100); *Ş. g. „C-tin Platon“* cl.VII Botezatu Paul (50); *Ş.g. „I. Creangă“* cl. VI Cojoc Raluca Elena (60), Bianca Mihaela (60), Obreja Diana (60), cl.VII Breahnă Cristina (40), Hozu Vlad Crina Ionela (40), Lazăr Mariana (40), Miron Roxana (40), Năstac Bianca (40), Ostaci Bogdan (40); *Ş. g. „G. Enescu“* cl.VI Horasemiuc Raluca Elena (110); *C. N. „V. Alecsandri“* cl.V Cojan Bogdan Alexandru (180).
BAIA MARE (MARAMUREŞ) *Ş. g. „Lucian Blaga“* cl.VI Ţânţar Oana Alexandra (50); *Ş. g.* 11 *„Nicolae Iorga“* cl.VIII Giurgiu Roxana (60); *Ş. g. „Dr. Victor Babeş“* cl.VI Sabadăr Oana (80); *Lic. de Artă* cl.V Păcurar Radu (50); *C. N. „Vasile Lucaciu“* cl.VIII Ţârlea Olivia (80).
BEIUŞ (BIHOR) *Colegiul „Ioan Ciordaş“* cl.V Matei Răzvan (90), Tirla Darius (110), Tirla Denis (110), Ungur Tabita Ioana (130).
BISTRIŢA (BISTRIŢA NĂSĂUD) *Ş. g.* 1 cl.VIII Câmpan Andra (70), Marţian Cristina (50); *Ş. g. „Lucian Blaga“* cl.VIII David Ana Adriana (60), Isaiu Ioana (80); *Ş.g. „Ştefan cel Mare“* cl.VI Ignătuş Raul(70), Moisil Adriana(70), Şandor Lari Paul (80).
BOTOŞANI (BOTOŞANI) *C. N. „Mihai Eminescu“* cl.V Anichitoaie Beatrice Roxana (100), Horodică Paula Antonia (100), Pricope Tudor (120), Sandu Dan (100), Stâncescu Sebastian (100).
BOZOVICI (CARAŞ-SEVERIN) *Ş. g.* cl.V Băin Oriana (100), Hotac Roberto (90), Melcescu Florina (70+100), cl.VII Mitocaru Patricia (60), Ruva Mihaela (60).
BRĂILA (BRĂILA) *Ş. g. „Mihu Dragomir“* cl.VI Chelbosu Ştefan Claudiu (40); *C. N. „Gheorghe M. Murgoci“* cl.V Staicu Cristina (80), cl.VII Şelaru Andreea Laura (70).
BUCUREŞTI *Ş. g.* 56 *„José Marti“* cl.V Guşter Andreea (40); *Ş. g.* 79 cl.V Ichim Adrian Cosmin (70), Tufan Andrei (80); *Ş. g.* 98 cl.IV Orban Irina Mariana (90); *Ş. g.* 111 *„ G. Bacovia“* cl.IV Ghinescu Alexandru Rareş (100); *Ş. g.* 190 *„Marcela Peneş“* cl.VI Ivan Vlad (80), Manea Alexandru (80), Nedelcu Nicolae (80), Scafaru Andrei (80), Ţopană