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## ARTICOLE ŞI NOTE MATEMATICE

### A RESPONSE TO A MATHEMATICAL NOTE

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**Abstract**. The article proves that every so-called pseudo-decreasing sequence has a limit

**Keywords:** monotonic sequence, pseudo-decreasing sequence, limit

**MSC :** 40A99

In [1], conf. univ. dr. *I. C. Drăghicescu* writes a mathematical note concerning pseudo-decreasing $PD(m)$ sequences of order $m$, generalizing the notion introduced (for $m = 2$) by *Laurenţiu Panaitopol*. The author's open question, at the end of his note, proves to have been readily answered in the affirmative, by the following:

**Theorem.** *Given a sequence $(a_n)_{n\geq 1}$ made of real numbers, and some natural number $m \geq 2$, such that*

$$a_{\sum\limits_{i=1}^{m} n_i} \leq \frac{\sum\limits_{i=1}^{m} a_{n_i}}{m} \quad \text{for all} \;\; n_i \geq 1,\; 1 \leq i \leq m$$

(*call this a „pseudo-decreasing“ $PD(m)$ sequence of order $m$*), $\lim\limits_{n\to\infty} a_n$ *exists.*

*Proof.* We will start with the following:

**Lemma.** *Given $k, r \geq 1$, then $a_{(m-1)kn+r} \leq a_k + \dfrac{a_r - a_k}{m^n}$, for any $n \geq 1$.*

*Proof of Lemma* goes by simple induction. For $n = 1$ we have $a_{(m-1)k+r} \leq \leq \dfrac{(m-1)a_k + a_r}{m} = a_k + \dfrac{a_r - a_k}{m}$. For the induction step, one will compute:

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$$a_{(m-1)k(n+1)+r} \leq \frac{(m-1)a_k + a_{(m-1)kn+r}}{m} \leq a_k + \frac{1}{m}\left(a_k + \frac{a_r - a_k}{m^n} - a_k\right) =$$

$$= a_k + \frac{a_r - a_k}{m^{n+1}}. \qquad \square$$

But for any $k \geq 1$ and $\varepsilon > 0$ there exists $n_{k,\varepsilon} > (m-1)k$, such that for any $n \geq n_{k,\varepsilon}$ to have $\frac{a_r - a_k}{m^t} \leq \varepsilon$, where $n - 1 = (m-1)kt + (r-1)$, with $1 \leq r \leq (m-1)k$ (from *Euclid*'s algorithm), therefore $a_n \leq a_k + \frac{a_r - a_k}{m^t} \leq$ $\leq a_k + \varepsilon$ (according with the Lemma).

Therefore $\limsup\limits_{n\to\infty} a_n \leq a_k + \varepsilon$, hence $\limsup\limits_{n\to\infty} a_n \leq \liminf\limits_{k\to\infty}(a_k + \varepsilon) =$ $= \liminf\limits_{n\to\infty} a_n + \varepsilon$ for all $\varepsilon > 0$, therefore $\limsup\limits_{n\to\infty} a_n \leq \liminf\limits_{n\to\infty} a_n$. It follows:

$$\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

(all this having been done in $\overline{\mathbb{R}}$). $\blacksquare$

**Remarks. 1.** The value of $\lim\limits_{n\to\infty} a_n$ can be either $-\infty$ or any real number $\ell$, since a trivial example for a $PD(m)$ sequence is any decreasing sequence.

By the same token, a pseudo-increasing sequence also has a limit (just multiply the terms of the sequence by $-1$, in order to turn it into a pseudo-decreasing sequence).

**2.** Unfortunately, the claim given in the before mentioned note, that for $a \in \left[1 + \frac{2}{m}, 1 + \frac{2}{m-1}\right)$ the sequence $x_n = \frac{a + (-1)^n}{n}$ for $n \geq 1$ is $PD(m+1)$ but not $PD(m)$, is, as a whole, wrong. It is true for $m$ even, but we leave it to you (as a useful exercise) to prove that, for $m$ odd, whenever it is $PD(m+1)$ it also is $PD(m)$ (we will therefore only use it as the original example, for $a \geq 3$, of a $PD(2)$ sequence which is not a decreasing sequence).

A simpler example is the sequence $y_1 = a - 1$, $y_n = \frac{a+1}{n}$ for $n \geq 2$. Now the claim that for $m \geq 2$ and $a \in \left[1 + \frac{2}{m}, 1 + \frac{2}{m-1}\right)$ the sequence is $PD(m+1)$, but not $PD(m)$, is easily proved by what follows.

On one hand, for $a \geq 1 + \frac{2}{k-1}$, we have:

$$\frac{\sum_{i=1}^{k} y_{n_i}}{k} \geq \frac{\sum_{i=1}^{k} \frac{a-1}{n_i}}{k} = \frac{a-1}{k} \sum_{i=1}^{k} \frac{1}{n_i} \geq \frac{a-1}{k} \cdot \frac{k^2}{\sum_{i=1}^{k} n_i} \geq \frac{a+1}{\sum_{i=1}^{k} n_i} \geq y_{\sum_{i=1}^{k} n_i},$$

hence the sequence is $PD(k)$. On the other hand, if the sequence is $PD(k)$, then $\frac{a+1}{k} = y_k \leq \frac{\sum_{i=1}^{k} y_1}{k} = y_1 = a-1$, whence $a \geq 1 + \frac{2}{k-1}$. All holds for any $k \geq 2$. ■

More generally, study the sequences $z_n = \frac{a+\varepsilon_n}{n}$, with $\varepsilon_n \in \{-1, 1\}$, for $n \geq 1$. As above, if $a \geq 1 + \frac{2}{k-1}$, then the sequence is $PD(k)$. By playing with the sequence of values for $\varepsilon_n$, one can obtain other interesting examples, of $PD(m+1)$ sequences which are not $PD(m)$ even ultimately.

## REFERENCES

[1] I. C. Drăghicescu, *În legătură cu şirurile pseudomonotone*, G. M. - B nr. 6/2010.

# DERIVABILITATEA FUNCŢIEI-PUNCT INTERMEDIAR DIN TEOREMA LUI LAGRANGE

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**Abstract**. If the function $f : I \to \mathbb{R}$ is differentiable on the interval $I \subseteq \mathbb{R}$, and $a \in I$, then for each $x \in I \backslash \{a\}$, according to the mean value theorem, there exists a point $c(x)$ belonging to the open interval determined by $x$ and $a$, and there exists a real number $\theta(x) \in ]0, 1[$ such that

$$f(x) - f(a) = (x-a) f^{(1)}(c(x))$$

and

$$f(x) - f(a) = (x-a) f^{(1)}(a + (x-a)\theta(x)).$$

In this paper we shall study the differentiability of the functions $c$ and $\theta$ in a neighbourhood of $a$.

**Keywords:** intermediate point, mean-value theorem.
**MSC :** 26A24

Fie $I \subseteq \mathbb{R}$ un interval, $f : I \to \mathbb{R}$ o funcţie derivabilă pe $I$ şi $a \in I$. Atunci, în baza teoremei de medie a lui *Lagrange*, pentru fiecare $x \in I \setminus \{a\}$

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