PROBLEMS FOR COMPETITIONS AND OLYMPIADS

Junior Level

C.O:5083. Find all functions $f : \mathbb{N}^* \to \mathbb{N}^*$ such that $x \cdot 3^{f(y)}$ divides $f(x) \cdot 3^y$, for all $x, y \in \mathbb{N}^*$.

Florin Rotaru, Focşani

C.O:5084. Prove that $1 < xy + (1-y)(1-z) + (1-x)(1-t) + zt < 2$, for all $x, y, z, t \in \left(0, \dfrac{1}{2}\right)$.

Manuela Prajea, Drobeta Turnu-Severin

C.O:5085. Let $x, y, z > 0$, with $x + y + z = 1$. Prove that:

$$\frac{1+xy}{x+y} + \frac{1+yz}{y+z} + \frac{1+zx}{z+x} \geq 5.$$

Ion Nedelcu, Ploieşti

C.O:5086. Show that the equation $x^2 + y^2 = 2z^3 + 8$ has infinitely many integer solutions.

Ion Nedelcu, Ploieşti

Senior Level

C.O:5087. The medians from A, B, C meet again the circumcircle of the triangle ABC at points A', B', C', respectively. Show that:

$$3\left(AA'^2 + BB'^2 + CC'^2\right) \geq 4\left(AB^2 + BC^2 + CA^2\right).$$

Cătălin Cristea, Craiova

C.O:5088. Prove that:

$$\frac{4}{3}\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) + \sqrt[3]{\frac{abc}{(a+b)(b+c)(c+a)}} \geq \frac{5}{2},$$

for all $a, b, c \geq 0$.

Tuan Le, Anaheim, California, U.S.A.

C.O:5089. Let AD, BE, CF be the angle bisectors of the triangle ABC. The perpendicular bisectors of the segments AD, BE, CF intersect the lines BC, CA, AB at points A', B', C' respectively. Prove that points A', B', C' are collinear.

Dan Nedeianu, Drobeta-Turnu Severin

C.O:5090. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous bounded function. Define $g(x) = \inf\{f(t) \mid t \leq x\}$, $x \in \mathbb{R}$ and $h(x) = \sup\{g(t) \mid t \geq x\}$, $x \in \mathbb{R}$. Show that if h is strictly increasing, then $f = g = h$.

Petru Todor, Sebeş, Alba