THE 26TH BALKAN MATHEMATICAL OLYMPIAD, 2009

Kragujevac, Serbia, April 28 - May 4, 2009

presented by Marian Andronache[1)], Bogdan Enescu[2)] and Dan Schwarz[3)]

The 26th Balkan Mathematical Olympiad was held in Kragujevac, Serbia, between April 28th and May 4th. A number of 20 teams attended the competition: 11 from the member countries (Albania, Bosnia and Herzegovina, Bulgaria, Cyprus, Greece, FYR Macedonia, Moldova, Montenegro, Romania, Serbia and Turkey) and 9 from the invited countries (Azerbaijan, City of Brno, France, Italy, Kazakhstan, Serbia 2, Tajikistan, Turkmenistan and the United Kingdom).

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The members of the Romanian team were: *Mădălina Persu* (I.C.H.B.), *Andrei Deneanu* (I.C.H.B.), *Tudor Pădurariu* (C. N. „Grigore Moisil“, Oneşti), *Radu Bumbăcea* (C.N. „Tudor Vianu“, Bucureşti), *Omer Cerrahoglu* (C.N. „Vasile Lucaciu“, Baia Mare) and *Horia Mania* (C.N. „Tudor Vianu”, Bucureşti).

All six Romanian students were awarded medals: silver for Mădălina, Tudor, Radu and Omer, bronze for Andrei and Horia. In the unofficial country ranking, Romania came on fourth place, after Serbia, Turkey and Bulgaria.

The competitors had four and a half hours to solve four problems. Here are the problems, with solutions.

Problem 1. Solve in positive integers the equation

$$3^x - 5^y = z^2.$$

Greece

Solution. Modulo 3, $-(-1)^y$ is a quadratic residue, hence y must be odd: $y = 2b + 1$.

Modulo 4, $(-1)^x - 1$ is a quadratic residue, hence x must be even: $x = 2a$. Then $5^{2b+1} = (3^a - z)(3^a + z)$. But $\gcd(3, z) = 1$, and $\gcd(3^a - z, 3^a + z) | \gcd(2 \cdot 3^a, 2z) = 2\gcd(3^a, z) = 2$, and since z is even, $\gcd(3^a - z, 3^a + z) = 1$. Since $3^a + z > 3^a - z$, this implies $3^a - z = 1$ and $3^a + z = 5^{2b+1}$, hence $2 \cdot 3^a - 1 = 5^{2b+1} = 5 \cdot 25^b$.

Modulo 24 this yields $2 \cdot 3^a - 1 \equiv 5$, or $6(3^{a-1} - 1) \equiv 0$, hence a must be odd, since for a even we always have $3^{a-1} \equiv 3$. Thus a must be odd: $a = 2c + 1$, and the equation writes $6 \cdot 9^c - 1 = 5 \cdot 25^b$, with obvious solution $c = b = 0$, whence $(x, y, z) = (2, 1, 2)$. Take then $c \geq 1$.

Modulo 9 we have $5 \cdot 7^b \equiv -1$, only valid for $b = 3d + 1$, since $7^3 \equiv 1$. Then we have $6 \cdot 9^c - 1 = 125 \cdot 25^{3d}$.

Modulo 7 this yields $2^c + 1 \equiv 4^{3d} \equiv 1$, absurd.

Therefore the only solution is $(x, y, z) = (2, 1, 2)$.

An alternative way to deal with the second part of the solution was found by Tudor Pădurariu: after obtaining the equality $2 \cdot 3^a - 1 = 5^{2b+1}$, he observed that it can be written as

$$2 \cdot 3^a = 1 + 5^{2b+1} = (1 + 5)\left(5^{2b} - 5^{2b-1} + \ldots - 5 + 1\right).$$

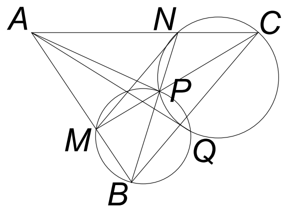
But, modulo 3 we have

$$5^{2b} - 5^{2b-1} + \ldots - 5 + 1 \equiv 1 - (-1) + 1 - \ldots - (-1) + 1 \equiv 2b + 1.$$

Now, if $a \geq 2$, we obtain $3 | 2b + 1$, hence $y = 6n + 3$. But then $2 \cdot 3^a = 1 + 5^{6n+3} = 1 + \left(5^3\right)^{2n+1}$, therefore $126 = 1 + 5^3$ divides $2 \cdot 3^a$, hence 7 divides $2 \cdot 3^a$, a contradiction.

Problem 2. Let MN be a line parallel with the side BC of the triangle ABC, meeting the sides AB and AC at the points M and N respectively. The lines BN and CM meet at the point P. The circumcircles of the triangles BMP and CNP have a common point Q (other than P). Prove that $\angle BAQ = \angle CAP$.

Moldova



Solution. The quadrilaterals $BMPQ$ and $CNPQ$ are cyclic. We therefore obtain the relations $\sphericalangle BQN = \sphericalangle BQP + \sphericalangle NQP =$ $= \sphericalangle PMA + \sphericalangle NCP = \pi - \sphericalangle MAC =$ $= \pi - \sphericalangle BAN$. It follows that the quadrilateral $ABQN$ is cyclic.

We denote by r_1 and r_2 the radii of the circumcircles of the triangles BMP and CNP, respectively. The quadrilateral $ABQN$ being cyclic

$$\frac{\sin \angle BAQ}{\sin \sphericalangle CAQ} = \frac{BQ}{NQ} = \frac{2r_1 \sin \sphericalangle BPQ}{2r_2 \sin \sphericalangle NPQ} = \frac{r_1}{r_2}.$$

From $MN \parallel BC$ follow the relations

$$\frac{\sin \sphericalangle CAP}{\sin \sphericalangle MAP} = \frac{PC}{MP} \cdot \frac{AM}{AC} = \frac{BC}{MN} \cdot \frac{AM}{AC} = \frac{AM}{AN} = \frac{BM}{CN} = \frac{2r_1 \sin \sphericalangle BPM}{2r_2 \sin \sphericalangle CPN} = \frac{r_1}{r_2}.$$

From the relations above follows the equality

$$\frac{\sin \sphericalangle BAQ}{\sin \sphericalangle CAQ} = \frac{\sin \sphericalangle CAP}{\sin \sphericalangle MAP}.$$

Let $\sphericalangle BAQ = \alpha$, $\sphericalangle CAP = \beta$ and $\sphericalangle PAQ = x$. From the last relation we conclude that

$$\frac{\sin \alpha}{\sin(\beta + x)} = \frac{\sin \beta}{\sin(\alpha + x)} \;\Leftrightarrow\; \sin(\alpha + x) \cdot \sin \alpha = \sin(\beta + x) \cdot \sin \beta \Leftrightarrow$$

$$\cos x - \cos(2\alpha + x) = \cos x - \cos(2\beta + x) \;\Leftrightarrow\; \cos(2\alpha + x) = \cos(2\beta + x).$$

Since $\alpha + x + \beta = \sphericalangle BAC < \pi$, we obtain the equality $\alpha = \beta$.

Alternative solution. (*T. Pădurariu*) The quadrilaterals $MBQP$ and $PQCN$ being cyclic, it follows $\sphericalangle MQB = \sphericalangle MPB = \sphericalangle NPC = \sphericalangle NQC$ and $\sphericalangle BMQ = \sphericalangle BPQ = \sphericalangle NCQ$, hence triangles BMQ and NCQ are similar. For T being the meeting point of lines AP and BC, Ceva's theorem yields $\dfrac{AM}{MB} \cdot \dfrac{BT}{TC} \cdot \dfrac{CN}{NA} = 1$, hence $BT = TC$, therefore T is the midpoint of BC.

Consider complex coordinates with origin at A, and denote by x the affix of a point X. From the givens of the problem, there exists $\lambda \in \mathbb{R}_+^*$ such that $m = \lambda b$ and $n = \lambda c$. Triangles BMQ and NCQ being similar, it follows $\dfrac{m - b}{q - b} = \dfrac{c - n}{q - n}$, whence $q = \dfrac{mn - bc}{m + n - b - c} = \dfrac{bc(1 + \lambda)}{b + c}$. Since $t = \dfrac{b + c}{2}$, it follows $\dfrac{q}{b} = \dfrac{c(1+\lambda)}{b+c} = \dfrac{1+\lambda}{2} \cdot \dfrac{c}{t}$, therefore, passing to arguments of the affixes, $\arg\left(\dfrac{q}{b}\right) = \arg\left(\dfrac{c}{t}\right)$, i.e. $\sphericalangle BAQ = \sphericalangle CAP$.

(It is seen then that points M and N could be anywhere on lines AB and AC respectively, with $MN \parallel BC$).

Another solution. (T. von Burg) From $MN \parallel BC$ follows the similitude of triangles ABC and AMN, therefore $\dfrac{AM}{AB} = \dfrac{AN}{AC}$, or $AM \cdot AC = AN \cdot AB = \rho^2$. Consider the inversion of center A and radius ρ, and denote by accented letters the images of points through it.

Then $AM' = AC$, $AN' = AB$, $AB' = AN$, $AC' = AM$. On the other hand, point Q is the *Miquel point* of the complete quadrilateral $AMBPNC$. Therefore Q' is the meeting point of lines $B'N'$ and $C'M'$. Lastly, P' is the second meeting point of circles $(AB'N')$ and $(AC'M')$ (or, equivalently, of circles $(B'M'Q')$ and $(C'N'Q')$, since P' is the Miquel point of the transformed figure). It follows that the transformed figure is symmetrical to the initial one, with respect to the angle bisector of $\angle BAC$, therefore $\sphericalangle BAQ = \sphericalangle B'AQ' = \sphericalangle CAP$.

Problem 3. A rectangle 9×12 is partitioned into unit squares, and the centers of all the squares, except for the four corner squares and the eight squares orthogonally adjacent to them, are colored in red. Is it possible to draw a closed broken line that has the following properties:

1) has all the 96 red points as its vertices, and them only;
2) has all its edges of length $\sqrt{13}$;
3) has central symmetry?

Bulgaria

Solution. Such a broken line does not exist. To show this, color the red point squares in a check pattern (black and white, so that every two red points at distance 1 lie in squares of different color). It is easy to see then that any two red points at distance $\sqrt{13}$ lie on squares of different color, so black and white alternate along the broken line. Also, the center of symmetry of the line must coincide with that of the set of points, and thus with that of the rectangle.

Consider now the points $A(2;2)$ and $B(8;11)$ (as usual, the point $(i;j)$ is the center of the unit square in the i-th row and the j-th column). The line can be divided in two parts – one leading from A to B, and the other from B to A. If they are symmetric to each other, each of them must consist of $96/2 = 48$ edges. So an even number of edges connects A to B, hence A and B must lie in squares of same color, untrue.

So, each part is symmetric to itself (since the symmetrical of the part leading from A to B can only be the other part, case dismissed in the above, or itself; and same for the part leading from B to A), and each part contains an odd number of edges. Since the edges can be divided in symmetric pairs, each part must contain some edge symmetric to itself. Only two such edges are possible: one joining $(4;5)$ and $(6;8)$; the other joining $(6;5)$ and $(4;8)$.

Consider now the point $(2;2)$. It can only be joined to $(5;4)$ and $(4;5)$, so the line must include this two edges. A similar consideration for the points $(8;2)$, $(8;11)$ and $(2;11)$ shows that the line must include the edges $(4;5)-(2;2)-(5;4)-(8;2)-(6;5)-(4;8)-(2;11)-(5;9)-(8;11)-(6;8)-(4;5)$. But this is a closed broken line that does not contain all the points, a contradiction.

Remark. (*D. Schwarz*) The issue of the existence of a (non-central symmetric) Hamiltonian circuit has been settled by the computer-found result (due to *C. Grosu*) presented in the Table 1 below.

		37	74	31	8	81	68	39	48		
	1	24	57	76	63	18	51	46	55	90	
73	32	9	80	67	38	49	44	7	82	69	40
36	75	30	19	96	23	56	89	62	17	52	47
25	58	77	2	43	10	85	64	91	50	45	54
12	79	72	33	4	27	66	93	70	41	6	83
29	20	35	86	59	14	95	22	53	88	61	16
	3	26	11	78	71	42	5	84	65	92	
		13	28	21	34	87	60	15	94		

TABLE 1. A Hamiltonian circuit on the 9×12 reduced array.

The site `http://www.ktn.freeuk.com/`, compiled by *George Jellis* is a comprehensive monography on such topics.

Problem 4. Find all functions $f : \mathbb{Z}_+^* \to \mathbb{Z}_+^*$ such that

$$f(f(m)^2 + 2f(n)^2) = m^2 + 2n^2 \quad \text{for any } m, n \in \mathbb{Z}_+^*.$$

Bulgaria

Solution. Notice that f is injective (for any fixed n, if $f(m_1) = f(m_2)$ then $m_1^2 + 2n^2 = f(f(m_1)^2 + 2f(n)^2) = f(f(m_2)^2 + 2f(n)^2) = m_2^2 + 2n^2$, whence $m_1^2 = m_2^2$ and so $m_1 = m_2$ for positive integers), hence

$$f(m)^2 + 2f(n)^2 = f(p)^2 + 2f(q)^2 \Leftrightarrow m^2 + 2n^2 = p^2 + 2q^2. \qquad (1)$$

Setting $f(1) = a$ one has $f(3a^2) = 3$. By (1)

$$f(5a^2)^2 + 2f(a^2)^2 = f(3a^2)^2 + 2f(3a^2)^2 = 3f(3a^2)^2 = 27.$$

Since the solutions of the equation $x^2 + 2y^2 = 27$ in positive integers are $(x, y) = (3, 3)$ and $(x, y) = (5, 1)$, it follows that $f(a^2) = 1$ and $f(5a^2) = 5$. By (1)

$$2f(4a^2)^2 - 2f(2a^2)^2 = f(5a^2)^2 - f(a^2)^2 = 24.$$

Since the only solution of the equation $x^2 - y^2 = 12$ in positive integers is $(x, y) = (4, 2)$, it follows that $f(2a^2) = 2$ and $f(4a^2) = 4$. By (1) again

$$f((k+4)a^2)^2 = 2f((k+3)a^2)^2 - 2f((k+1)a^2)^2 + f(ka^2)^2$$

(this is based on the identity $(k+4)^2 + 2(k+1)^2 = k^2 + 2(k+3)^2$) and therefore $f(ka^2) = k$ by induction on k. Then $f(a^3) = a = f(1)$ and thus $a = 1$.

It is clear that the function $f(k) = k$ satisfies the given condition.