PROBLEMS FOR COMPETITIONS AND OLYMPIADS

Junior Level

C.O:5019. Prove that 2009 divides $2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2008 - 1 \cdot 3 \cdot 5 \cdot \ldots \cdot 2007$.

Neculai Stanciu, Berca, Buzău

C.O:5020. A point P is given inside an equilateral triangle ABC with $AB = 1$. Show that $PA + PB + PC < 3$.

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C.O:5021. Let p be a prime number and let a, b, c be integers, $a < b < c < p$, such that a^3, b^3, c^3 have the same remainder at the division by p. Show that p divides $a + b + c$.

Costin Raicu, student, Blaj

C.O:5022. The integers a, b satisfy $\sqrt{a} + \dfrac{10}{\sqrt{b}} \in \mathbb{Z}$. Find the number b.

Costin Raicu, student, Blaj

Senior Level

C.O:5023. Let ABC be a triangle, r the inradius and I_a, I_b, I_c centers of the excircles of the triangle. Prove that $I_aB \cdot I_aC + I_bA \cdot I_bC + I_cA \cdot I_cB \geq 36r^2$.

Neculai Roman, Mirceşti

C.O:5024. Let $n \geq 2$ and $a_1, a_2, \ldots, a_n \in (1, \infty)$. Prove that:

$$a_1^{\log_{a_2} a_1} \cdot a_2^{\log_{a_3} a_2} \cdot \ldots \cdot a_n^{\log_{a_1} a_n} \geq a_1^{\sqrt{\log_{a_2} a_1}} a_2^{\sqrt{\log_{a_3} a_2}} \cdot \ldots \cdot a_n^{\sqrt{\log_{a_1} a_n}} \geq a_1 a_2 \cdot \ldots \cdot a_n.$$

Mihai Opincariu, Brad, Hunedoara

C.O:5025. Consider f a monic polynomial with integer coefficients and $\deg f \geq 2$, so that its roots are real not integer numbers. Prove that f has two roots x_1, x_2 with $x_2 - x_1 > 1$.

Marius Cavachi, Constanţa

C.O:5026. Consider $A_1A_2A_3A_4$ a parallelogram and P_0 a point. Construct inductively the sequence of points $P_1, P_2, \ldots$ as follows: P_{k+1} is the image of P_k under the clockwise rotation around A_{k+1} with $90°$ ($A_k = A_{k-4}$, $\forall\, k \geq 5$).

Prove that if $P_{2008} = P_0$, then $A_1A_2A_3A_4$ is a square.

Diana Lascu, student, Timişoara