PROBLEMS FOR COMPETITIONS AND OLYMPIADS

Junior Level

C.O:5011. Consider the numbers $a = 1 + 2 + 2^2 + \ldots + 2^{2007}$ and $b = 1+3+3^2+\ldots+3^{2007}$. Find the greatest number from $(a+1)^3$ and $(2b+1)^2$.

Constantin Rusu, Râmnicu Sărat

C.O:5012. Let $ABCD$ be a tetrahedron inscribed in a sphere of radius R, such that $MA^2 + MB^2 + MC^2 + MD^2 = 8$, for any point M on the sphere.

a) Find the radius R.

b) Show that the line segments AB, AC, AD can form a triangle.

Marcel Chiriţă, Bucharest

C.O:5013. Let $a \in \mathbb{R}$. Show that the area of the triangle with side lengths $\sqrt{a^2-a+1}$, $\sqrt{a^2+a+1}$, $\sqrt{4a^2+3}$ does not depend on a.

Marcel Chiriţă, Bucharest

C.O:5014. 12 points are given on the sides of a square, so that none is a vertex. Find the maximum number of triangles with vertices in these points.

Gabriel Săndoiu, Rm. Vâlcea

Senior Level

C.O:5015. Prove that:

$$((x+y+z)(xy+yz+zx))^2 \leq 3\left(x^2+xy+y^2\right)\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right),$$

for all $x, y, z > 0$.

Lucian Petrescu, Tulcea

C.O:5016. Consider the real numbers $a_i, b_i \geq 0$, $i = 1, 2, \ldots, n$. The following statements are equivalent:

1) $a_i < b_i$, for all $i = 1, 2, \ldots, n$;

2) there are real numbers $x_i > 0$, $i = 1, 2, \ldots, n$, so that:

$$\max_{1\leq i\leq n} a_i x_i < \min_{1\leq i\leq n} b_i x_i.$$

George Stoica, Saint John, Canada

C.O:5017. Let $\alpha, \beta, \alpha_1, \beta_1 \in \mathbb{R}$, not all equal to 0. Consider A the set of all functions $f : [a,b] \to \mathbb{R}$, three times differentiable, with the third derivative f''' continuous, so that $f(a) = \alpha$, $f(b) = \beta$, $f'(a) = \alpha_1$, $f'(b) = \beta_1$.

Determine $\min\limits_{f\in A} \int\limits_a^b \left(f''(x)\right)^2 \mathrm{d}x$.

A. Corduneanu, Iaşi

C.O:5018. Let ABC be a triangle. A circle externally tangent to the circumcircle of the triangle is also tangent to the rays $(AB$ and $(AC$ at M and N respectively. Points $P \in (BC$, $Q \in (BA$ and $L \in (CA$, $K \in (CB$ are defined similarly. Prove that:

$$\frac{AB}{MN} + \frac{BC}{PQ} + \frac{AC}{LK} \geq \frac{3}{2}.$$

Neculai Roman, Mirceşti, Iaşi