THE 2^{ND} ROMANIAN MASTER OF MATHEMATICS COMPETITION

Bucharest 2009

presented by DAN SCHWARZ [1)]

The second Romanian Mathematical Master in Mathematics took place in the period February 26th - March 1st 2009. The organizer and the host of the competition was „T. Vianu" High School, from Bucharest. This was an international competition in mathematics, where teams with high scores in the IMO's were invited.

This year the following teams accepted the invitation and participated officially (each team consisted of up to 6 students, one leader and one deputy leader): Bulgaria, China, Italy, Russia, Serbia, The United Kingdom, The United States of America and three Romanian teams: Romania A, Romania B, and Vianu, the last one representing the host school.

The problems were selected by a Romanian committee and were discussed in the Jury formed by all leaders, where each problem was voted as suitable.

In the sequel we shall present the given problems and their solutions. Also, some comments will be made.

Problem 1. For any positive integers $a_1, \ldots, a_k$, let $n = a_1 + \ldots + a_k$, and consider the multinomial coefficient

$$\binom{n}{a_1, \ldots, a_k} = \frac{n!}{\prod_{i=1}^{k}(a_i!)}.$$

Let $d = \gcd(a_1, \ldots, a_k)$ denote the greatest common divisor of $a_1, \ldots, a_k$.

Prove that $\dfrac{d}{n}\dbinom{n}{a_1, \ldots, a_k}$ is an integer.

Romania, *Dan Schwarz*[2)]

Solution. The key idea is the fact that the greatest common divisor is a linear combination with integer coefficients of the numbers involved, i.e. there exist $u_i \in \mathbb{Z}$ such that $d = \sum_{i=1}^{k} u_i a_i$. But

$$\binom{n}{a_1, \ldots, a_k} = \frac{n}{a_i}\binom{n-1}{a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_k},$$

[1)] Profesor, Liceul Internaţional de Informatică, Bucureşti.

[2)] Based on a property of quasi-Catalan numbers of J. Conway, see [GUY, R.K., *Unsolved Problems in Number Theory*].

so

$$\frac{d}{n}\binom{n}{a_1, \ldots, a_k} = \sum_{i=1}^{k} u_i \binom{n-1}{a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_k},$$

which clearly is an integer, since multinomial coefficients are known to be integer.

Problem 2. A set S of points in space satisfies the property that all pairwise distances between points in S are distinct. Given that all points in S have integer coordinates (x, y, z), where $1 \le x, y, z \le n$, show that the number of points in S is less than $\min\left((n+2)\sqrt{\frac{n}{3}}, n\sqrt{6}\right)$.

Romania, *Dan Schwarz*

Solution. The critical idea is to estimate the total number possible T of distinct distances realized by pairs of points (x, y, z), of integer coordinates $1 \le x, y, z \le n$. However, any such distance is also realized by a pair anchored at $(1, 1, 1)$, from symmetry considerations.

But the number of distinct distances to points with no coordinates x, y, z equal is at most $\binom{n}{3} = \frac{1}{6}\, n(n-1)(n-2)$, the number of distinct distances to points with two of the three coordinates x, y, z equal is at most $2\binom{n}{2} = n(n-1)$, while the number of distinct distances to points with all three coordinates x, y, z equal is $n-1$, hence

$$T \le \frac{1}{6}n(n-1)(n-2) + n(n-1) + (n-1) < \frac{1}{6}(n^3 + 3n^2 + 2n).$$

On the other hand, the total number of distinct distances between the N points in S is $\binom{N}{2} = \frac{1}{2}N(N-1) \le T$, yielding

$$(2N-1)^2 < \frac{1}{3}(4n^3 + 12n^2 + 8n) + 1 \le \frac{1}{3}(2n\sqrt{n} + 3\sqrt{n})^2,$$

hence $N < \frac{1}{2}\left((2n+3)\sqrt{\frac{n}{3}} + 1\right) \le (n+2)\sqrt{\frac{n}{3}}$ for $n \ge 3$. One can easily check that the inequality is true for $n = 2$ also, since then $T = 3$.

On the other hand, since the squares of the distances can only take the integer values between 1 and the trivial upper bound $3(n-1)^2$ (for the diagonal of the cube), it follows that $T \le 3(n-1)^2$, yielding $N < n\sqrt{6}$.

Remark. As a matter of fact, the trivial upper bound $3n^2$ for T is better from $n \ge 15$, offering an asymptotic of $n\sqrt{6}$. The true asymptotic is probably unknown, since it is not known even for the 2-dimensional case (reported in [GUY, R.K. – *Unsolved Problems in Number Theory*]).

Problem 3. Given four points A_1, A_2, A_3, A_4 in the plane, no three collinear, such that $A_1A_2 \cdot A_3A_4 = A_1A_3 \cdot A_2A_4 = A_1A_4 \cdot A_2A_3$, denote by O_i the circumcenter of $\Delta A_jA_kA_\ell$, with $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$.

Assuming $A_i \neq O_i$ for all indices i, prove that the four lines A_iO_i are concurrent or parallel.

Bulgaria, *Nikolai Ivanov Beluhov*

Solution. The given triple equality being invariated by any permutation in $\mathcal{S}_4$, it is enough to prove that the lines A_iO_i for $2 \leq i \leq 4$ are concurrent or parallel. The relations can then be written

$$\frac{A_1A_2}{A_1A_3} = \frac{A_4A_2}{A_4A_3}, \quad \frac{A_1A_3}{A_1A_4} = \frac{A_2A_3}{A_2A_4}, \quad \frac{A_1A_4}{A_1A_2} = \frac{A_3A_4}{A_3A_2}.$$

Consider the Apollonius circles Γ_k of centers $\omega_k \in A_iA_j$, for $\{i, j, k\} = \{2, 3, 4\}$, determined by the point A_1, which therefore lies on all three, while the points A_k lie on Γ_k. Moreover, the points ω_k are collinear, since the point A'_k which is the other meeting point (than A_1, if any) of Γ_i and Γ_j fulfills

$$\frac{A'_kA_j}{A'_kA_k} = \frac{A_iA_j}{A_iA_k} \text{ and } \frac{A'_kA_i}{A'_kA_k} = \frac{A_jA_i}{A_jA_k}, \text{ thus } \frac{A'_kA_i}{A'_kA_j} = \frac{A_kA_i}{A_kA_j},$$

therefore A'_k also lies on Γ_k, hence all three circles Γ_k share the same meeting point(s), thus their centers are collinear.

Now, the circumcenters O_i and O_j, as well as the point ω_k, lie on the perpendicular bisector of the segment A_1A_k, for $\{i, j, k\} = \{2, 3, 4\}$. It follows that the pairs of lines A_iA_j, O_iO_j meet at the collinear points ω_k. Finally, Desargues' theorem for the perspective triangles $\Delta A_iA_jA_k$ and $\Delta O_iO_jO_k$ yields the claim.

Remark. There exists a particular (degenerate) case, when the points are the vertices of a kite of $\dfrac{\pi}{6}$ equal angles, hence one of the associated ratios is 1, so a corresponding Apollonius circle degenerates to the perpendicular bisector.

Problem 4. For a finite set X of positive integers, let

$$\Sigma(X) = \sum_{x \in X} \arctan \frac{1}{x}.$$

Given a finite set S of positive integers for which $\Sigma(S) < \dfrac{\pi}{2}$, show that there exists a finite set T of positive integers for which $S \subset T$ and $\Sigma(T) = \dfrac{\pi}{2}$.

United Kingdom, *Kevin Buzzard*

Solution. (*D. Schwarz*) We will step-by-step augment the set S with positive integers t_n, by taking each time t_n as the least positive integer larger than $\max(S)$, and not already used, such that $\Sigma(S \cup \{t_1, t_2, \ldots, t_n\})$ remains at most $\dfrac{\pi}{2}$ (this is possible since $\arctan \dfrac{1}{t} \to 0$ when $t \to \infty$).

If, at some point, we get exactly $\dfrac{\pi}{2}$ we are through, since we have augmented S to a set T as required, so assume the process continues indefinitely.

Clearly the sequence $(t_n)_{n\geq 1}$ is built strictly increasing, so for all $n \geq 1$ we have $t_{n+1} > t_n > \max(S)$.

We will make some useful notations. Take $S_0 = S$, $S_{n+1} = S_n \cup \{t_{n+1}\}$, for $n \in \mathbb{N}$. Also take $x_n = \tan\left(\dfrac{\pi}{2} - \Sigma(S_n)\right)$. One can easily prove by simple induction that a lesser than $\dfrac{\pi}{2}$ sum of arcs of rational tangents is as well an arc of rational tangent, therefore $x_n = \dfrac{p_n}{q_n}$, with $p_n, q_n \in \mathbb{N}^*$, $(p_n, q_n) = 1$. Since arctan is increasing, we need take $t_{n+1} \geq \left\lceil \dfrac{1}{x_n} \right\rceil$, in order that $\Sigma(S_{n+1}) \leq \dfrac{\pi}{2}$.

Assume that for all $n \geq 1$ we have $\dfrac{1}{x_n} \leq t_n$. Since we need both $t_{n+1} \geq \left\lceil \dfrac{1}{x_n} \right\rceil$ and $t_{n+1} > t_n \geq \dfrac{1}{x_n}$, it follows that $t_{n+1} = t_n + 1$ (the least available value), so $t_{k+1} = t_1 + k$ for all $k \geq 0$. But then

$$\frac{\pi}{2} > \Sigma(\{t_1, t_2, \ldots, t_n\}) = \sum_{k=0}^{n-1} \arctan \frac{1}{t_1 + k} > \frac{1}{2} \sum_{k=0}^{n-1} \frac{1}{t_1 + k} \to \infty$$

when $n \to \infty$, absurd (see *Lemma*).

Therefore there exists some $N \geq 1$ for which $\dfrac{1}{x_N} > t_N$, so $\left\lceil \dfrac{1}{x_N} \right\rceil$ is available for t_{N+1}.

Moreover, for any $n \geq N$ with $t_{n+1} = \left\lceil \dfrac{1}{x_n} \right\rceil$, we have

$$x_{n+1} = \frac{x_n - \dfrac{1}{t_{n+1}}}{1 + x_n \dfrac{1}{t_{n+1}}} = \frac{x_n t_{n+1} - 1}{t_{n+1} + x_n} < \frac{x_n}{t_{n+1} + x_n} < \frac{1}{t_{n+1}},$$

since $t_{n+1} = \left\lceil \dfrac{1}{x_n} \right\rceil$ implies $x_n t_{n+1} - 1 < x_n$; and so we can take $t_{n+1} = \left\lceil \dfrac{1}{x_n} \right\rceil$ indefinitely for $n \geq N$. Now we use the fact that $x_n = \dfrac{p_n}{q_n}$. Then

$$\frac{p_{n+1}}{q_{n+1}} = \frac{\dfrac{p_n}{q_n} - \dfrac{1}{t_{n+1}}}{1 + \dfrac{p_n}{q_n} \dfrac{1}{t_{n+1}}} = \frac{p_n t_{n+1} - q_n}{q_n t_{n+1} + p_n},$$

hence $p_{n+1} \leq p_n t_{n+1} - q_n < p_n$, since $t_{n+1} = \left\lceil \dfrac{q_n}{p_n} \right\rceil$, and so $t_{n+1} < \dfrac{q_n}{p_n} + 1$.

Therefore the sequence $(p_n)_{n \geq 1}$ of the numerators of x_n eventually becomes strictly decreasing, absurd for any sequence of positive integers.

Lemma. For $x \in \left(0, \dfrac{\pi}{2}\right)$ one has $\arctan x > \dfrac{x}{2}$.

Proof. We start by proving that under given condition one has $\sin x > \tan \dfrac{x}{2}$, in turn equivalent to $2 \sin \dfrac{x}{2} \cos \dfrac{x}{2} > \dfrac{\sin \dfrac{x}{2}}{\cos \dfrac{x}{2}}$, $2 \cos^2 \dfrac{x}{2} - 1 > 0$, and finally $\cos x > 0$, patently true.

Now, arctan is increasing, hence applied to the above, together with the well-known inequality $x > \sin x$, true for all $x > 0$, yields $\arctan x > \arctan \sin x > \arctan \tan \dfrac{x}{2} = \dfrac{x}{2}$.

PROBLEME

REZOLVAREA PROBLEMELOR DIN GAZETA MATEMATICĂ Nr.9/2008

PROBLEME PENTRU GIMNAZIU

Clasa a V-a

E:13695. *Aflaţi restul împărţirii numărului* $A = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 2008 - 3$ *la* 8.

Alfred Eckstein şi *Viorel Tudoran*, Arad

Soluţie. Avem $A = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots \cdot 2008 - 8 + 5 = 8 \cdot (1 \cdot 3 \cdot 5 \cdot \ldots \cdot 2008 - 1) + 5$. În concluzie, restul împărţirii lui A la 8 este 5.

E:13696. *Determinaţi numerele naturale* a, b, c *ştiind că* $a \cdot b = 18$*,* $a \cdot c = 30$*, iar* $5b + 2c = 75$.

E. Blăjuţ, Bacău

Soluţie. Din $5b + 2c = 75$, deoarece $5b$ şi 75 sunt multipli de 5 deducem că $2c$ este multiplu de 5, deci c este multiplu de 5. Din $a \cdot c = 30$ rezultă $c \leq 30$. Pentru $c = 5$, din $a \cdot c = 30$ găsim $a = 6$, iar din $a \cdot b = 18$ găsim $b = 3$. Soluţia găsită nu convine pentru că nu verifică relaţia $5b + 2c = 75$. Procedând analog pentru $c = 10$, $c = 15$, $c = 20$, $c = 25$ şi $c = 30$ constatăm că singura soluţie convenabilă este $c = 15, a = 2$ şi $b = 9$.

E:13697. *Aflaţi numărul maxim de pagini ale unei cărţi, ştiind că cifra* 3 *s-a folosit la numerotarea paginilor sale de* 71 *de ori.*

* * *

Soluţie. De la 1 la 100, cifra 3 s-a folosit de 20 de ori: câte o dată la fiecare din cele 10 zeci pe locul unităţilor şi de 10 ori pe locul zecilor de la 30 până la 39. De încă 20 de ori apare 3 de la 101 la 200 şi de 20 de ori de la 201 la 299. Până