THE „MATH STARS" CONTEST
6-7 December 2008

presented by DAN SCHWARZ [1)]

The contest was organized by the International Computer High School from Bucharest, on the 6^{th} and 7^{th} of December 2008 and consisted of 6 IMO-style questions.

1. Prove that, for every positive integer m, the equation

$$\frac{n}{m} = \left\lfloor \sqrt[3]{n^2} \right\rfloor + \left\lfloor \sqrt{n} \right\rfloor + 1$$

has at least a positive integer solution n_m.

C. Lupu & D. Schwarz

2. The 2^N vertices of the N-dimensional hypercube $\{0,1\}^N$ are labeled with numbers from 0 to $2^N - 1$ such that $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ gets the label

$$v(\mathbf{x}) = \sum_{k=1}^{N} x_k 2^{k-1}.$$

Find all n, $2 \le n \le 2^N$, so that the vertices from $V = \{v \mid 0 \le v \le n-1\}$ can be visited in a circuit using only edges of the hypercube which connect vertices of V (each vertex is visited exactly once).

E. Băzăvan & C. Tălău

3. Consider a convex quadrilateral and the incircles of the two triangles obtained drawing one of its diagonals. Prove that the tangency points of these two circles to the diagonal are symmetrical with respect to the midpoint of the diagonal if and only if the centers of the circles and the meeting point of the diagonals are collinear.

D. Schwarz[2)]

4. Find the maximum number of consecutive integer values which can be assumed by a polynomial $P \in \mathbb{Z}[X]$ of degree $n > 1$, for integer values of the variable.

B. Berceanu

5. Suppose that $\sqrt{23} > \dfrac{m}{n}$, where m, n are positive integers.

i) Show that $\sqrt{23} > \dfrac{m}{n} + \dfrac{3}{mn}$.

ii) Show that $\sqrt{23} < \dfrac{m}{n} + \dfrac{4}{mn}$ infinitely many times and find at least three such cases.

D. Schwarz

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1) Teacher, International Computer High School of Bucharest.

2) See IMO 2008 Problem 6 in GM-B nr. 7-8 2008.

6. Consider the infinite array consisting of the first quadrant of the plane divided into unit squares, each square containing a real number. The array will be called $k-$*balanced* (where $k > 1$ is an integer) if not all its numbers are equal and the sum of the elements of each of its $k \times k$ subsquares is the same v_k. An array which is both $p-$balanced and $q-$balanced will be called $(p, q)-$*balanced*. If p, q are relatively prime, the array will be called *coprime*.

We will call $(M \times N)-$*seed* of a $(p, q)-$balanced array a $M \times N$ rectangle, having its left corner in the origin and which, extended periodically in both dimensions of the plane, generates the array.

(i) Prove that, in a $(p, q)-$balanced array, $q^2 v_p = p^2 v_q$.

(ii) Prove that a coprime $(p, q)-$balanced array contains at least three different numbers. Show that this is not necessarily true if the array is not coprime $((p, q) > 1)$.

(iii) Prove that every coprime $(p, q)-$balanced array has a seed.

(iv) Show that for every p, q there exists a $(p, q)-$balanced array containing only three different numbers.

(v) Prove that a $k-$balanced array and a $(p, q)-$balanced array which is not coprime $(p, q > 1)$ do not necessarily have a seed.

(vi) Find the minimum T for which there exists a $(T \times T)-$seed of a given coprime $(p, q)-$balanced array, when p, q are given primes.

(vii) Show that, for every relatively prime p, q, there exists a coprime $(p, q)-$balanced array which has a square $(T \times T)-$seed and does not have "smaller" $(M \times N)-$seeds ($M \leq T$, $N \leq T$ and $MN < T^2$).

D. Schwarz

Solutions

1. The sequence $a_n = \lfloor \sqrt[3]{n^2} \rfloor + \lfloor \sqrt{n} \rfloor + 1$ is made of positive integers and non-decreasing. For $n \geq (2m)^6$ we clearly have $a_n < \dfrac{n}{m}$, since

$$\frac{a_n}{n} \leq \frac{\sqrt[3]{n^2} + \sqrt{n} + 1}{n} \leq \frac{1}{4m^2} + \frac{1}{8m^3} + \frac{1}{64m^6} \leq \frac{1}{m}\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{64}\right) < \frac{1}{m},$$

so take k the largest n for which $a_n \geq \dfrac{n}{m}$ (clearly this occurs for $n = 1$, so the set is not empty). Assume $a_k > \dfrac{k}{m}$; then $ma_{k+1} \geq ma_k > k$, so $ma_{k+1} \geq k + 1$, hence $a_{k+1} \geq \dfrac{k+1}{m}$, in contradiction with the maximality of k. Therefore we must have $a_k = \dfrac{k}{m}$ and we may take

$$n_m = \max\left\{n \in \mathbb{N}^* \mid a_n \geq \frac{n}{m}\right\}.$$

2. Consider the *Hamming distance* between N-vectors, given by the number of differing coordinates between the two vectors. In our case this is given by

$$\mathrm{d}(\mathbf{x},\mathbf{y}) = \sum\sum_{k=1}^{N} |x_k - y_k|,$$

and since all edges are hypercube edges, distances between adjacent vertices are all 1. Assume a *Hamiltonian* circuit $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}, \mathbf{x}_n = \mathbf{x}_0)$ exists. Notice that modulo 2 we have

$$\sum_{v=0}^{n-1} \mathrm{d}(\mathbf{x}_v, \mathbf{x}_{v+1}) = \sum_{v=0}^{n-1}\sum_{k=1}^{N} |x_{v,k} - x_{v+1,k}| \equiv \sum_{k=1}^{N}\sum_{v=0}^{n-1} (x_{v,k} - x_{v+1,k}) = 0,$$

and since

$$\sum_{v=0}^{n-1} \mathrm{d}(\mathbf{x}_v, \mathbf{x}_{v+1}) = \sum_{v=0}^{n-1} 1 = n,$$

in order for a Hamiltonian circuit to exist, n must be even.

To show that for any even n there exists a Hamiltonian circuit, we start by building one for $n = 2^M$, $1 \le M \le N$. Consider the mapping $\rho(a_1, a_2, \ldots, a_m) = a_m, \ldots, a_2, a_1$ which reverses the order of symbols in a sequence, and the translation τ_t which adds t to each term of a numeric sequence. Build now the iterative sequences $W_0 = 0$, $W_{k+1} = W_k, \rho(\tau_{2^k}(W_k)$ for $k \ge 0$. For instance, $W_4 = 0, 1, 3, 2, 6, 7, 5, 4, 12, 13, 15, 14, 10, 11, 9, 8$.

It is obvious that $W_M, 0$ is a Hamiltonian circuit. Moreover, notice that any pair $\{2k, 2k+1\}$, with $0 \le k \le 2^{M-1} - 1$, appears as adjacent terms in the above circuit. Take $M = \lfloor \log_2 n \rfloor$. Since $n - 2^M$ is even, it adds pairs of vertices to be considered in the circuit. Now, any pair $\{2^M + 2k, 2^M + 2k + 1\}$, with $0 \le k \le 2^{M-1} - 1$, may be injected within the above sequence $W_M, 0$, inside the $\{2k, 2k+1\}$ pair, such that the even, respectively odd, terms are adjacent, and so that the new sequence still observes the requirement that adjacent terms are connected by edges in the hypercube.

3. Denote by $ABCD$ the quadrilateral, with $[AC]$ being the diagonal in question, and by P, respectively Q, the tangency points with AC of the incircle γ_B of the triangle ABC, respectively of the incircle γ_D of the triangle ADC.

If BC and AD are parallel, and BA and CD are also parallel, then $ABCD$ is a parallelogram, with central symmetry, and all is clear. We thus may assume $(AD$ and $(BC$ are crossing, beyond D, respectively C (otherwise we either swap the pair of labels B and D, or the pair A and C, or both pairs – while these moves invariate both stated conditions). Consider the circle Γ tangent to $(AD$ and $(CD$ beyond D, and $(BC$ beyond C. Let us also denote by $H^{\pm}_{\gamma_2\gamma_1}(X)$ the homothety of center X which transforms a circle γ_1 into another circle γ_2, bearing the sign of the homothety ratio at superscript.

For the direct, we are told that $|AP| = |CQ|$. We have

$$|AP| = \frac{1}{2}(|AC| + |AB| - |BC|) \quad \text{and} \quad |CQ| = \frac{1}{2}(|AC| + |CD| - |AD|),$$

so $|AP| = |CQ|$ is equivalent to $|AB| + |AD| = |CB| + |CD|$. The other tangent from B to Γ than BC meets AD at A'. Then a variant to Pithot's Theorem yields $|A'B|+|A'D| = |CB|+|CD|$, so $|AB| = |A'B| \pm |AA'|$, hence the triangle ABA' is degenerated, and so A' must coincide with A (in fact a proof of the converse to Pithot). This means that Γ is also tangent to $(BA$ beyond A.

Then, with R the center of the negative ratio homothety between γ_B and γ_D (i.e. the crossing point of their internal common tangents), we have

$$H^{+}_{\gamma_B \Gamma}(B) = H^{-}_{\gamma_B \gamma_D}(R) \circ H^{-}_{\gamma_D \Gamma}(D),$$

so D, R and B are collinear, hence $R \in BD$. But AC is one of the internal common tangents of γ_B and γ_D, so $R \in AC$. Finally, $R = AC \cap BD$ is on the line of the incenters.

For the reciprocal, we are told that $R = AC \cap BD$. Then, with B' the center of the positive ratio homothety between γ_B and Γ (i.e. the crossing point of their external common tangents), we have

$$H^{+}_{\gamma_B \Gamma}(B') = H^{-}_{\gamma_B \gamma_D}(R) \circ H^{-}_{\gamma_D \Gamma}(D),$$

so D, R and B' are collinear, hence $B' \in RD = BD$. But BC is one of the external common tangents of γ_B and Γ, so $B' \in BC$, hence B' must coincide with B. This means that Γ is also tangent to $(BA$ beyond A.

Finally, applying the variant to Pithot's Theorem, we get $|AB|+|AD| = |CB| + |CD|$, equivalent to $|AP| = |CQ|$.

Remark. The same result is valid for the excircles (with respect to AC) of the triangles ABC and ADC, with analogous proof, since a known fact yields that their tangency points with AC are symmetrical with respect to the midpoint of $[AC]$ with those of the incircles, which are given so.

4. Let a_k, $1 \le k \le m$, $a_{k+1} = a_k + 1$ for $1 \le k \le m-1$, be m consecutive integer values taken for $P(x_k) = a_k$, with $x_k \in \mathbb{Z}$. It is easy to prove that $b - a \mid P(b) - P(a)$ for $a, b \in \mathbb{Z}$.

Then $x_{k+1} - x_k \mid P(x_{k+1}) - P(x_k) = a_{k+1} - a_k = 1$, so $|x_{k+1} - x_k| = 1$ for $1 \le k \le m-1$, therefore $(x_k)_{1 \le k \le m}$ is an arithmetic progression of ratio $r = \pm 1$ (since the values of x_k need be distinct).

But the polynomial $Q(x) = r(x - x_1) + a_1$ also satisfies $Q(x_k) = a_k$ for $1 \le k \le m$, so $m \le n$, otherwise $P - Q$ vanishes in more than $\deg P$ arguments, therefore $P \equiv Q$, absurd, since $n > 1$.

On the other hand, the polynomials

$$P(x) = \lambda \prod_{k=0}^{n-1} \big(x - (a+k)\big) \pm x + b$$

with $\lambda, a, b \in \mathbb{Z}$, take values $P(a+k) = (b \pm a) \pm k$ for $0 \le k \le n-1$, which are consecutive, therefore the maximal value n is reached (and in fact this is the general form of polynomials reaching the maximal possible value).

5. From $\sqrt{23} > \dfrac{m}{n}$ follows $23n^2 \ge m^2 + 1$. But the quadratic residues modulo 23 are $0, 1, 25 \equiv 2, 49 \equiv 3, 4, 121 \equiv 6, 100 \equiv 8, 9, 81 \equiv -11, 36 \equiv -10, 16 \equiv -7$ and $64 \equiv -5$, so the tightest inequality available would seem to be $23n^2 \ge m^2 + 5$, only that modulo 5 we have $23 \equiv 3$, and 3 is not a quadratic residue modulo 5. Therefore the tightest inequality is $23n^2 \ge m^2 + 7$, for which there exists the principal solution $23 \cdot 1^2 = 4^2 + 7$.

i) Hence $23n^2 \ge m^2 + 7$, or

$$23 \ge \left(\frac{m}{n}\right)^2 + \frac{7}{n^2} = \left(\frac{m}{n} + \frac{3}{mn}\right)^2 + \frac{1}{n^2} - \frac{9}{m^2n^2},$$

while $\dfrac{1}{n^2} - \dfrac{9}{m^2n^2} = \dfrac{m^2-9}{m^2n^2} > 0$, as long as $m > 3$.

Now, for $m = 1$ or 3, $\left(\dfrac{m}{n} + \dfrac{3}{mn}\right)^2 = \dfrac{16}{n^2} \le 16 < 23$; while for $m = 2$, $\left(\dfrac{m}{n} + \dfrac{3}{mn}\right)^2 = \dfrac{49}{4n^2} \le \dfrac{49}{4} < 23$.

Therefore in all cases $\sqrt{23} > \dfrac{m}{n} + \dfrac{3}{mn}$.

ii) For $23n^2 = m^2 + 7$ we have

$$23 = \left(\frac{m}{n}\right)^2 + \frac{7}{n^2} = \left(\frac{m}{n} + \frac{4}{mn}\right)^2 - \frac{1}{n^2} - \frac{16}{m^2n^2} < \left(\frac{m}{n} + \frac{4}{mn}\right)^2,$$

hence $\sqrt{23} < \dfrac{m}{n} + \dfrac{4}{mn}$, with the example $(m, n) = (4, 1)$ provided by the principal solution above.

It remains to show that we infinitely often may have $23n^2 = m^2 + 7$ (1), a Pell equation. Since the classical Pell equation $23y^2 = x^2 - 1$ has the principal solution $(24, 5)$, the solutions for (1) are given by

$$m + n\sqrt{23} = (\pm 4 \pm \sqrt{23})(24 + 5\sqrt{23})^s, \ \ s \in \mathbb{N},$$

enough for exhibiting the required examples.

6. *Lemma.* If a sequence $(x_n)_{n\ge 0}$ is both p-periodic and q-periodic, with p, q relatively prime, then it must be constant.

Proof. There exist positive integers u, v such that $1 = up - vq$. Then, for all $n \ge 0$, $x_n = x_{n+up} = x_{(n+up)-vq} = x_{n+(up-vq)} = x_{n+1}$, hence $(x_n)_{n\ge 0}$ is of period 1, therefore the sequence is constant.

As an extension, if $d = (p, q) > 1$, the result is that the sequence must be d-periodic. This follows from the proof above, by grouping together „molecules“ of d consecutive ”atom” terms. Now for $p = dp'$ and $q = dq'$ we have $(p', q') = 1$ and we apply the Lemma. This means the ”molecules” are equal, i.e. the sequence is d-periodic.

We will now present some mostly trivial consequences of the definitions. If an array is k-balanced, then it is mk-balanced for any positive integer m. Hence, if it is (p,q)-balanced, then it is (mp,nq)-balanced for any positive integers m, n.

The set $\mathcal{L}$ of all $(M\times N)$-seeds for a (p,q)-balanced array is a lattice. If there exists a $(M\times N)$-seed, then it canonically yields a $(mM\times nN)$-seed, by extension through periodicity, therefore a square $([M,N]\times[M,N])$-seed does exist. If there exists both a $(M\times N)$-seed and a $(M'\times N')$-seed, then there exist a $([M,M']\times[N,N'])$-seed, and a $((M,M')\times(N,N'))$-seed (by the extension to the Lemma). Finally, if $\mathcal{L}$ is not empty, there exists a minimal $(M_0\times N_0)$-seed, in the sense that for any other $(M\times N)$-seed we have $M_0|M$ and $N_0|N$, and a minimal square $([M_0,N_0]\times[M_0,N_0])$-seed.

(i) Clearly the k-balanced property is hereditary to any rectangular $m\times n$ sub-array. Consider any $pq\times pq$ square sub-array. A little bit of double counting shows that the sum of its elements is both q^2v_p (partition it in $p\times p$ sub-squares), and p^2v_q (partition it in $q\times q$ sub-squares), hence $q^2v_p=p^2v_q$.

(ii) Assume only two real values $x\neq y$ are used. Let us transform a value v into $\dfrac{v-x}{y-x}$, which turns x into 0 and y into 1. The sums of elements of $k\times k$ sub-squares (for k equal to p or q) remain constant, in fact v_k turns into $\dfrac{v_k-k^2x}{y-x}$, the number of y's in the $k\times k$ sub-square. Now, the formula from point (a) implies that $p^2|v_p$ and $q^2|v_q$, but the array being non-constant we must have $0<v_p<p^2$ and $0<v_q<q^2$, contradiction.

If $(p,q)>1$, build the array by using as seed any $(p,q)\times(p,q)$ square block made of two values only (of course the array will be "constant" at the coarser level of granularity of these blocks, but non-constant at the level of granularity of unit squares).

(iii) We will show that a square $(pq\times pq)$-seed will do, therefore $\mathcal{L}\neq\emptyset$.

Denote ${}_kr_{i,j}:=\displaystyle\sum_{\ell=0}^{k-1}a_{i,j+\ell}$, the sum of k consecutive elements on row i, starting from column j. Similarly, denote ${}_kc_{i,j}:=\displaystyle\sum_{\ell=0}^{k-1}a_{i+\ell,j}$, the sum of k consecutive elements on column j, starting from row i.

We have ${}_pr_{i,k}={}_pr_{i+p,k}$ for any non-negative integers i, j, and $k\geq j$, because of the p-balanced property. By iterating, this yields

$$ {}_pr_{i,k}={}_pr_{i+qp,k}. $$

Denote $x_n:=a_{i,n}-a_{i+qp,n}$, hence $\displaystyle\sum_{\ell=0}^{p-1}x_{k+\ell}=0$. Similarly $\displaystyle\sum_{\ell=0}^{q-1}x_{k+\ell}=0$.

But this means the sequence $(x_k)_{k\geq j}$ is both p-periodic and q-periodic, so according to the Lemma it must be constant. Now, that constant must be 0, since $\sum_{\ell=0}^{p-1} x_{j+\ell} = 0$. So $x_j = 0$, hence $a_{i,j} = a_{i+qp,j}$ for all non-negative integers i, j.

In a totally similar manner (working along the other dimension) we get $a_{i,j} = a_{i,j+pq}$ for all non-negative integers i, j, and therefore we have proven the array originates from a square $(pq \times pq)$-seed.

(iv) Let us present a model of a $(p \times q)$-seed using only three different values, which results into a (p,q)-balanced array with $v_p = v_q = 0$.

0	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
0	0	0	$\cdots$	0
-1	1	0	$\cdots$	0
1	-1	0	$\cdots$	0

Table 1. A general $(p \times q)$-seed.

(v) We can build a k-balanced array starting with two orthogonal strips of width $k-1$ made of arbitrary values, having the origin as lower left crossing point, choosing an arbitrary value for v_k, and filling up the rest of the array, step by step, in the uniquely possible manner. For $(p,q) > 1$ just take $k = (p,q)$; the $p \times p$ and $q \times q$ sub-squares are made of $k \times k$ sub-squares, so the array is doubly-balanced, with $v_p = \frac{p^2}{k^2} v_k$ and $v_q = \frac{q^2}{k^2} v_k$. Then $\mathcal{L} = \emptyset$.

$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$
$a_{k-1,0}$	$\cdots$	$a_{k-1,k-2}$	up to v_k	$\cdots$
$a_{k-2,0}$	$\cdots$	$a_{k-2,k-2}$	$a_{k-2,k-1}$	$\cdots$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$
$a_{0,0}$	$\cdots$	$a_{0,k-2}$	$a_{0,k-1}$	$\cdots$

Table 2. A general k-balanced array starting from $(k-1)$-wide strips.

(vi) We have seen at point (iii) that $T = pq$ yields a square seed. Any least value will have to be a divisor of pq.

Let us prove that for $(p,q) = 1$, $p > q \geq 2$, there can never exist a $(q \times q)$-seed. Assume the converse; then ${}_pr_{i,j} = {}_pr_{i+mq,j}$ and ${}_pc_{i,j} = {}_pc_{i,j+p}$, for any positive integer m, from periodicity. But in a (p,q)-balanced array we need have ${}_pr_{i,j} = {}_pr_{i+mp,j}$ and ${}_pc_{i,j} = {}_pc_{i,j+mp}$. Since $(p,q) = 1$, there exist positive integers u, v such that $1 = up - vq$. Then ${}_pr_{i,j} = {}_pr_{i+up,j} = {}_pr_{(i+up)-vq,j} = {}_pr_{i+1,j}$, hence, by iteration, all ${}_pr_{i+m,j}$ are equal, for $m \geq 0$.

But $\sum_{k=0}^{p-1}({}_pr_{i+k,j+1} - {}_pr_{i+k,j}) = \sum_{k=0}^{p-1}({}_pr_{i,j+1} - {}_pr_{i,j}) = p({}_pr_{i,j+1} - {}_pr_{i,j})$, and this is ${}_pc_{i,j+p} - {}_pc_{i,j} = 0$, hence we will have all ${}_pr_{i,j+m}$ equal, for $m \geq 0$, by iteration. This means that any row i is both q-periodic and p-periodic, hence, according to the Lemma, constant, so there exists a $(q \times 1)$-seed. But then any column j will be both q-periodic and p-periodic, hence, according again to the Lemma, constant, so there exists a (1×1)-seed, i.e. the array would be constant, absurd.

Similarly, there can never exist a $(p \times p)$-seed. Take $p' = (p-1)q > p = q'$, so $(q', p') = 1$, and clearly a q-balanced array is also $(p-1)q$-balanced, hence the array would be (p', q')-balanced, so we can apply the above result.

Therefore $T = pq$ is minimal for p, q both primes.

Otherwise take $Q|q$. If a $(Q \times Q)$-seed would exist, then a $(q \times q)$-seed would exist, absurd. Similarly for $P|p$. Therefore a least square seed need contain factors P, Q from both p, respectively q, and we have seen at point (iv) how to build a $(P \times Q)$-seed resulting in a (P, Q)-balanced array, which will also be (p, q)-balanced, so this shows there exist co-prime (p, q)-balanced arrays originating from $(P \times Q)$-seeds, hence from a $(PQ \times PQ)$-seed. The minimal value is achieved when P, Q are the least prime factors of p, respectively q.

(vii) Let us exhibit the general structure of a minimal square (6×6)-seed for a $(2, 3)$-balanced array with $v_2 = v_3 = 0$. We use arbitrary real a, b, c and x, with $\mu = \frac{1}{3}(a+b+c)$ (of course, we can add any value v to all cells). A general minimal (2×3)-seed is obtained for $a = b = c = \mu$. A general minimal (3×2)-seed is obtained for $\mu = x = 0$.

$b-2\mu$	$-b-x+2\mu$	$b+x-\mu$	$-b$	$b-x$	$-b+x+\mu$
c	$-c+x$	$c-x-\mu$	$-c+2\mu$	$c+x-2\mu$	$-c-x+\mu$
$a-2\mu$	$-a-x+2\mu$	$a+x-\mu$	$-a$	$a-x$	$-a+x+\mu$
b	$-b+x$	$b-x-\mu$	$-b+2\mu$	$b+x-2\mu$	$-b-x+\mu$
$c-2\mu$	$-c-x+2\mu$	$c+x-\mu$	$-c$	$c-x$	$-c+x+\mu$
a	$-a+x$	$a-x-\mu$	$-a+2\mu$	$a+x-2\mu$	$-a-x+\mu$

TABLE 3. The minimal square (6×6)-seed for a $(2, 3)$-balanced array.

This suggests there exist co-prime (p, q)-balanced arrays for which the minimal seed is the square $(pq \times pq)$-seed prescribed at point (iii). Moreover, it was obvious already from the result at point (iii) that the number of different values such an array uses is at most p^2q^2, and from the result at point (iv) that it is at least 3. (For the particular case, one can see that the number of different values can indeed be $2^2 3^2 = 36$.)

Indeed, the conjecture formulated in the above turns to be true! The linear combination, with $\alpha\beta \neq 0$, of the array A generated by the $(p \times q)$-seed at point (d), and the array ${}^tA$ generated by the transposed $(q \times p)$-seed, is an array $B = \alpha\, A + \beta\, {}^t\!A$, clearly (p, q)-balanced, and with minimal seed the square $(pq \times pq)$-seed thus obtained.