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ARTICLES

Partition regularity properties of equilateral, rectangular, and complete polygonal geometric configurations

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Abstract. The theme of the article is related to van der Waerden and Ramsey theorems, the paradigms of Ramsey theory. Under specific assumptions, the main results identify partition regularity properties of some geometric configurations in the Euclidean plane, and point out upper bounds of the associated partition regularity numbers.

Keywords: van der Waerden theorem, Ramsey theorem, partition regularity properties, partition regularity numbers.

MSC: 05C15, 05D10, 05E45.

1. Introduction

The concepts of partition regularity and partition regularity numbers originated from the theorems on arithmetic progressions and complete graphs proved almost a hundred years ago by B. L. van der Waerden [7], 1927, and F. P. Ramsey [6], 1930. The two discoveries prompted over time the development of Ramsey theory, a far–reaching, intricate, and ongoing research area, part of the broad field of combinatorics, with significant applications in number theory, graph theory, algebra, geometry, and analysis. As excellent sources of information in this regard we refer to Graham, Rothschild, Spencer [2], Katz, Reimann [3], and Landman, Robertson [4].

The article is addressing a theme related to uses of van der Waerden and Ramsey numbers in geometry. Subject to appropriate definitions and assumptions, the main results point out partition regularity properties of some geometric configurations in the Euclidean plane, providing upper bounds of the associated partition regularity numbers. Section 2 records a few requisites. The three other sections describe and investigate equilateral, rectangular, and complete polygonal configurations. As a highlight of our approach

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that deserves mentioning, the proofs are elementary. Several particular issues have been successfully analyzed by students with major in mathematics or computer science enrolled in the Problem Solving Seminar at Baker University. Perhaps other students and their professors would be interested in reading the present article.

2. Partition regularity

Throughout the article we adopt the following definition of partition regularity properties and partition regularity numbers, introduced in a general framework that involves finite sets.

Definition 1. *Suppose that $\mathfrak{C}$ is a collection of finite sets characterized by a defining property* P*, and $\kappa \geq 1$ is a given integer. A set $\Sigma \in \mathfrak{C}$ with cardinal number $|\Sigma|$ has the (κ, P)–partition regularity property provided there exists a positive integer $\mathfrak{s}(\kappa)$ such that, if $|\Sigma| \geq \mathfrak{s}(\kappa)$ and $[\,\Sigma_1, \Sigma_2, \ldots, \Sigma_\kappa\,]$ is an arbitrary κ–partition of Σ, then at least one class Σ_ι, $1 \leq \iota \leq \kappa$, is in $\mathfrak{C}$. The lowest integer $\mathfrak{s}(\kappa)$ satisfying this requirement, denoted by $\mathfrak{s}(\kappa, \mathrm{P})$, is called the (κ, P)–partition regularity number of collection $\mathfrak{C}$.*

In applications, κ–partitions of sets Σ are usually defined by assigning κ distinct colors, $c_1, c_2, \ldots, c_\kappa$, to the elements of Σ and assuming that class Σ_ι, $1 \leq \iota \leq \kappa$, consists of elements of color c_ι.

The case $\kappa = 1$, which is not ruled out, reduces to finding $s(1, \mathrm{P})$, the smallest size of sets from $\mathfrak{C}$. However, we should note that when $\kappa \geq 2$, in most situations the proofs of the existence theorems concerned with partition regularity properties only provide upper bounds $\mathfrak{s}(\kappa)$ of the partition regularity numbers $\mathfrak{s}(\kappa, \mathrm{P})$, without indicating their actual values.

We next state the specific definitions of van der Waerden and Ramsey numbers, which quantify the van der Waerden and Ramsey theorems.

Definition 2. *Let $\kappa \geq 1$ and $\nu \geq 2$ be two integers. The van der Waerden number $\mathfrak{w}(\kappa, \nu)$ is the least positive integer uniquely determined by the requirement that, if $n \geq \mathfrak{w}(\kappa, \nu)$, then for any κ–partition $[\,\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\kappa\,]$ of the set $\mathcal{S}(n) = \{1, 2, \ldots, n\}$ there exists a class $\mathcal{S}_\iota$, $1 \leq \iota \leq \kappa$, that includes ν numbers, terms of an arithmetic progression.*

Definition 3. *Let $\nu_1, \nu_2, \ldots, \nu_\kappa \geq 2$ be $\kappa \geq 1$ integers. The Ramsey nymber $\mathfrak{r}(\nu_1, \nu_2, \ldots, \nu_\kappa)$ is the least positive integer determined by the requirement that, if $n \geq \mathfrak{r}(\nu_1, \nu_2, \ldots, \nu_\kappa)$ and Γ_n is a complete graph of order $\mathrm{ord}(\Gamma_n) = n$, then for any κ–partition $[\,\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_\kappa\,]$ of the set $\mathcal{E}(\Gamma_n)$ of edges of graph Γ_n there exists a class $\mathcal{E}_\iota, 1 \leq \iota \leq \kappa$, that includes the edges of a complete subgraph $\Gamma_\nu \subseteq \Gamma_n$ of order $\mathrm{ord}(\Gamma_\nu) = \nu \in \{\nu_1, \nu_2, \ldots, \nu_\kappa\}$.*

We recall that a graph Γ with sets of vertices and edges denoted by $\mathcal{V}(\Gamma)$ and $\mathcal{E}(\Gamma)$ is complete if any two vertices are connected by a unique edge. The order $\mathrm{ord}(\Gamma)$ of a complete graph Γ equals $|\,\mathcal{V}(\Gamma)\,|$, and $|\,\mathcal{E}(\Gamma)\,| = \mathrm{ord}(\Gamma)\,[\,\mathrm{ord}(\Gamma) - 1\,]/2$.

As a matter of fact, not too many exact values of van der Waerden and Ramsey numbers are known. For instance, in addition to $\mathfrak{w}(1,\nu) = \nu$ and $\mathfrak{w}(\kappa,2) = \kappa+1$, if $\kappa \geq 2$ and $\nu \geq 3$ only seven van der Waerden numbers $\mathfrak{w}(\kappa,\nu)$ have been found. The first four correspond to $\kappa = 2$, $3 \leq \nu \leq 6$, and the last three to $\kappa = 3$, $\nu = 3, 4$, and $\kappa = 4$, $\nu = 3$. With regard to Ramsey numbers, the obvious cases are $\mathfrak{r}(\nu) = \mathfrak{r}(2,\nu) = \nu$, $\nu \geq 2$. If $\kappa = 2$ and $\nu_1, \nu_2 \geq 3$, the known values of $\mathfrak{r}(\nu_1,\nu_2)$ correspond to $\nu_1 = 3$, $3 \leq \nu_2 \leq 9$, and $\nu_1 = 4$, $\nu_2 = 4, 5$. The list ends with two special cases, when $\kappa = 3$ and $\nu_1 = \nu_2 = 3$, $\nu_3 = 3, 4$.

3. Equilateral configurations

A set $\Delta_n = \{\, X_{i,j} : 1 \leq i \leq j,\, 1 \leq j \leq n \,\}$, $n \geq 2$, of points in the Euclidean plane $\mathbb{E}$ is called an equilateral configuration provided the triangles $\Delta\,[X_{i,j}, X_{i,j+1}, X_{i+1,j+1}]$, $1 \leq i \leq j$, $1 \leq j \leq n-1$, and $\Delta\,[X_{i,j}, X_{i-1,j-1}, X_{i-1,j}]$, $2 \leq i \leq j$, $2 \leq j \leq n$, are equilateral and congruent to each other. For each $1 \leq j \leq n$, the subset $\mathcal{R}_j = \{\, X_{i,j} : 1 \leq i \leq j \,\}$ of an equilateral configuration Δ_n is referred to as row j. We note that Δ_n is uniquely determined by n and the points $X_{1,1}, X_{1,n}, X_{n,n}$, which are vertices of an equilateral triangle. We will assume that the line passing through $X_{1,n}$ and $X_{n,n}$ is horizontal and vertex $X_{1,1}$ is in the upper half–plane bounded by that line. For instance, an equilateral configuration Δ_4 has the representation

$$\begin{array}{ccccccc} & & & * & & & \\ & & * & & * & & \\ & * & & * & & * & \\ * & & * & & * & & *\,. \end{array}$$

The following result is a geometric consequence of van der Waerden theorem.

Theorem A – Partition regularity property of configurations Δ_n
Let $\mathfrak{n}(\kappa)$, $\kappa \geq 1$, be the sequence of integers defined by

$$\mathfrak{n}(1) = 2, \quad \mathfrak{n}(\kappa+1) = \mathfrak{w}(\kappa+1, \mathfrak{n}(\kappa)+1), \quad \kappa \geq 1, \tag{3.1}$$

where $\mathfrak{w}(\cdot\,,\cdot)$ are van der Waerden numbers. Assume that the points of an equilateral configuration Δ_n with $n \geq \mathfrak{n}(\kappa)$ are colored using $\kappa \geq 1$ colors. Then Δ_n includes three points of the same color, vertices of an equilateral triangle.

Before proceeding with a proof, we note that consistent with Definition 2.1 Theorem A is about the collection $\mathfrak{C}$ of finite sets $\Sigma \subseteq \mathbb{E}$ with the defining property $\mathrm{P}(\Delta)$ which singles out as relevant the sets Σ that include vertices of an equilateral triangle. Accordingly, we intoduce the $(\kappa, \mathrm{P}(\Delta))$–partition regularity numbers $\mathfrak{n}(\kappa, \mathrm{P}(\Delta))$ defined, for each $\kappa \geq 1$, as the least integer n such that any equilateral configuration Δ_n has the $(\kappa, \mathrm{P}(\Delta))$–partition regularity property. The conclusion of Theorem A amounts to the statement

$$\mathfrak{n}(\kappa, \mathrm{P}(\Delta)) \leq \mathfrak{n}(\kappa), \quad \kappa \geq 1. \tag{3.2}$$

Proof of Theorem A. We have to show that sequence (3.1) provides upper bounds $\mathfrak{n}(\kappa)$ for the partition regularity numbers $\mathfrak{n}(\kappa, \mathrm{P}(\Delta))$. As expected, the proof is by induction on κ. If $\kappa = 1$, (3.2) is true because $\mathfrak{n}(1, \mathrm{P}(\Delta)) = 2 = \mathfrak{n}(1)$.

Suppose next that $\mathfrak{n}(\kappa)$ satisfies (3.2) for a certain $\kappa \geq 1$, and consider a coloring of an equilateral configuration Δ_n, $n \geq \mathfrak{n}(\kappa+1)$, with $\kappa+1$ colors denoted by $c_1, c_2, \dots, c_\kappa, c_{\kappa+1}$. Since row $\mathcal{R}_n$ of Δ_n consists of n equidistant points on a horizontal line, and the recurrence relation in (3.1) implies $n \geq \mathfrak{w}(\kappa+1, \mathfrak{n}(\kappa)+1)$, by van der Waerden theorem we get that $\mathcal{R}_n$ includes $\mathfrak{n}(\kappa)+1$ equidistant points of the same color, which for convenience is assumed to be color $c_{\kappa+1}$. We list those points from left to right using the labels $Y_{i,\mathfrak{n}(\kappa)+1}$, $1 \leq i \leq \mathfrak{n}(\kappa)+1$, and then select from Δ_n the additional points $Y_{i,j}$, $1 \leq i \leq j$, $1 \leq j \leq \mathfrak{n}(\kappa)$, such that the set $\{\, Y_{i,j},\ 1 \leq i \leq j,\ 1 \leq j \leq \mathfrak{n}(\kappa)+1 \}$ is an equilateral configuration with $\mathfrak{n}(\kappa)+1$ rows, denoted by $\Delta_{\mathfrak{n}(\kappa)+1}$. Let $\Delta_{\mathfrak{n}(\kappa)}$ be the equilateral configuration with $\mathfrak{n}(\kappa)$ rows obtained by removing the last row from $\Delta_{\mathfrak{n}(\kappa)+1}$. Obviously, $\Delta_{\mathfrak{n}(\kappa)} \subseteq \Delta_{\mathfrak{n}(\kappa)+1} \subseteq \Delta_n$. We conclude the proof of the induction step based on two remarks. If at least one point $Y_{i,j}$ in $\Delta_{\mathfrak{n}(\kappa)}$ is of color $c_{\kappa+1}$, then $\Delta[Y_{i,j}, Y_{i,\mathfrak{n}(\kappa)+1}, Y_{\mathfrak{n}(\kappa)+i-j+1,\mathfrak{n}(\kappa)+1}]$ is an equilateral triangle with vertices of color $c_{\kappa+1}$. In the opposite case, when all points of $\Delta_{\mathfrak{n}(\kappa)}$ are colored by using colors $c_1, c_2, \dots, c_\kappa$, then the existence of three points in $\Delta_{\mathfrak{n}(\kappa)}$ vertices with the same color of an equilateral triangle is a consequece of the induction hypothesis. Therefore, configuration Δ_n has the $(\kappa+1, \mathrm{P}(\Delta))$–partition regularity property. The proof of Theorem A is complete. $\square$

From (3.1) we get that $\mathfrak{n}(2) = \mathfrak{w}(2,3)$. Since $\mathfrak{w}(2,3) = 9$, Theorem A implies that $\mathfrak{n}(2, \mathrm{P}(\Delta)) \leq 9$. Actually, a direct approach will refine the estimate.

Proposition 4. *If* $\kappa = 2$, *then* $\mathfrak{n}(2, \mathrm{P}(\Delta)) = 5$.

Proof. There are two tasks we need to complete, first

(i) to determine all essentially different colorings of equilateral configurations Δ_4 that yield counterexamples when $\kappa = 2$, and then

(ii) to prove that any equilateral configuration Δ_5 has the $(2, \mathrm{P}(\Delta))$–partition regularity property.

We rely on the following concept. Assuming that the colored points are represented as $\bullet$ or $\circ$, two distinct colorings of a configuration Δ_4 or Δ_5 are called correlated provided we get one from the other by either changing $\bullet$ to

$\circ$ and $\circ$ to $\bullet$ everywhere, or by reversing the order of colored points in every row. It is quite obvious that a coloring of Δ_4 yields a counterexample, only if each correlated coloring does the same. Therefore, related to task (i) it would be enough to find a complete list of representative uncorrelated colorings, all with $\bullet$ as the single point in row 1. Moreover, we may use $\circ\,\circ$ or $\bullet\,\circ$ in row 2, since $\bullet\,\bullet$ is inappropriate and $\circ\,\bullet$ is redundant. Accordingly, the number of cases which we should analyze turns out to be rather small.

When the colors in row 2 are $\circ\,\circ$, the second point in row 3 must be $\bullet$. This remark leads to two essentially different situations.

Case 1.1 – The points in row 3 are $\circ\ \bullet\ \circ$. A trial and error investigation shows that there are no colorings of row 4 which result in counterexamples.

Case 1.2 – The points in row 3 are $\bullet\ \bullet\ \circ$. In this case there is just one coloring of row 4 that provides a counterexample, namely $\circ\ \circ\ \bullet\ \bullet$.

We continue the search for counterexamples by assuming that the colors in row 2 are $\bullet\,\circ$ and by observing, based on a quick inspection, that the first point in row 3 must be $\circ$. Since $\circ\ \circ\ \circ$ is inappropriate, the possible colorings of row 3 might be $\circ\ \bullet\ \circ$, $\circ\ \circ\ \bullet$, or $\circ\ \bullet\ \bullet$. Thus we end up with three distinct situations.

Case 2.1 – The points in row 3 are $\circ\ \bullet\ \circ$. We rely on brute force and discover that the only counterexample occurs when row 4 is $\circ\ \bullet\ \circ\ \bullet$.

Case 2.2 – The points in row 3 are $\circ\ \circ\ \bullet$. A careful examination shows that in this case we get four new counterexamples for which row 4 is either $\circ\ \bullet\ \circ\ \circ$, or $\circ\ \bullet\ \circ\ \bullet$, or $\bullet\ \bullet\ \circ\ \circ$, or $\circ\ \bullet\ \bullet\ \circ$.

Case 2.3 – The points in row 3 are $\circ\ \bullet\ \bullet$. Under this assumption we get the last two counterexamples when the colorings of row 4 are $\circ\ \bullet\ \circ\ \circ$, or $\bullet\ \bullet\ \circ\ \circ$.

We just completed task (i). Task (ii) is equivalent to showing that there are no colored configurations Δ_5 which disprove the claim $\mathfrak{n}(2, \mathrm{P}(\Delta)) = 5$. We rely once more on the concept of uncorrelated colorings trying to avoid occurences in Δ_5 of vertices of an equilateral triangle with the same color. Consistent with such attempts, the configuration Δ_4 obtained by removing row 5 from Δ_5 must be among the eight counterexample identified earlier. Accordingly, the completion of tak (ii) reduces to eight case by case investigations.

Case 1.2, for instance, yields the subsequent partial coloring of Δ_5,

$$\begin{array}{ccccccccc}
 & & & & \bullet & & & & \\
 & & & \circ & & \circ & & & \\
 & & \bullet & & \bullet & & \circ & & \\
 & \circ & & \circ & & \bullet & & \bullet & \\
* & & * & & * & & * & & *.
\end{array}$$

The attempt of disproving Proposition 3.1 forces the second and fourth points in row 5 to be • and ∘, and then the first and fifth points to be • and ∘. The attempt fails, no matter what color we use for the third point. In Case 2.1, the corresponding partial coloring of Δ_5 is

$$
\begin{array}{ccccccccc}
 & & & & \bullet & & & & \\
 & & & \bullet & & \circ & & & \\
 & & \circ & & \bullet & & \circ & & \\
 & \circ & & \bullet & & \circ & & \bullet & \\
* & & * & & * & & * & & *.
\end{array}
$$

This time the only colors assigned successively to the third, second, and first point in row 5, which would prevent for a while the occurence in Δ_5 of vertices with the same color of an equilateral triangle, are •, ∘, and •. However, the choices of either • or ∘ for the fifth point both result in colorings of Δ_5 that confirm the $(2, \mathrm{P}(\Delta))$–partition regularity property. Cases 2.2 and 2.3 are dealt with by using similar reasonings. The proof of Proposition 3.1 is complete. □

4. Rectangular configurations

A set $\Box_{m,n} = \{\, X_{i,j} : 1 \leq i \leq m,\ 1 \leq j \leq n \,\}$, $m,\ n \geq 2$, of points in the Euclidean plane $\mathbb{E}$ is a rectangular configuration provided the quadrilaterals $\Box\,[X_{i,j}, X_{i,j+1}, X_{i+1,j+1}, X_{i+1,j}]$, $1 \leq i \leq m$, $1 \leq j \leq n-1$, are rectangles, congruent to each other. The subsets $\mathcal{R}_i = \{\, X_{i,j} : 1 \leq j \leq n \,\}$, $1 \leq i \leq m$, and $\mathcal{C}_j = \{\, X_{i,j} : 1 \leq i \leq m \,\}$, $1 \leq j \leq n$, each consisting of colinear, equidistant points, are called the rows and columns of $\Box_{m,n}$. For convenience, we assume that the rows are horizontal and, consequently, the columns are vertical. The subsequent diagram depicts $\Box_{3,6}$:

$$
\begin{array}{cccccc}
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & *.
\end{array}
$$

The counterpart of Theorem A for rectangular configurations is about the collection $\mathfrak{C}$ of finite sets $\Sigma \subseteq \mathbb{E}$ defined by the property $\mathrm{P}(\Box)$ that characterizes the sets Σ which include vertices of a rectangle.

Theorem B – Partition regularity property of configurations $\Box_{m,n}$

Let $\mathfrak{w}(\,\cdot\,,\,\cdot\,)$ denote the function that assigns to each $\kappa \geq 1$ and $\nu \geq 2$ the van der Waerden number $\mathfrak{w}(\kappa, \nu)$. For any given κ, define the functions $\mathfrak{w}_{i\,|\,\kappa}(\,\cdot\,)$ by

$$\mathfrak{w}_{1\,|\,\kappa}(\,\cdot\,) = \mathfrak{w}(\kappa,\,\cdot\,), \quad \mathfrak{w}_{i+1\,|\,\kappa}(\,\cdot\,) = \mathfrak{w}(\kappa, \mathfrak{w}_{i\,|\,\kappa}(\,\cdot\,)\,), \quad i \geq 1. \tag{4.1}$$

Let $\mathfrak{m}(\kappa)$ and $\mathfrak{n}(\kappa)$, $\kappa \geq 1$, be the sequences of integers given by

$$\mathfrak{m}(\kappa) = \kappa + 1, \quad \mathfrak{n}(\kappa) = \mathfrak{w}_{\kappa+1\,|\,\kappa}(2). \tag{4.2}$$

Assume that $\square_{m,n}$ is a rectangular configuration with $m \geq \mathfrak{m}(\kappa)$ and $n \geq \mathfrak{n}(\kappa)$. Then $\square_{m,n}$ has the $(\kappa, \mathrm{P}(\square))$–partition regularity property for each $\kappa \geq 1$.

Proof of Theorem B. Suppose that the κ–partitions of $\square_{m,n}$ come from colorings with colors denoted by $c_1, c_2, \ldots, c_\kappa$, and observe that it is enough to prove the $(\kappa, \mathrm{P}(\square))$–partition regularity property for $\square_{\mathfrak{m}(\kappa),\mathfrak{n}(\kappa)}$, $\kappa \geq 1$.

Since by (4.2) and (4.1), $\mathfrak{m}(1) = 1 + 1 = 2$ and $\mathfrak{n}(1) = \mathfrak{w}_{2\,|\,1}(2) = \mathfrak{w}(1, \mathfrak{w}(1, 2)) = \mathfrak{w}(1, 2) = 2$, the case $\kappa = 1$ is obvious. All four points in $\square_{\mathfrak{m}(1),\mathfrak{n}(1)}$, vertices of a rectangle, have color c_1. The proof when $\kappa = 2$ and the colors are c_1, c_2 requires a brief reasoning. Note that $\square_{\mathfrak{m}(2),\mathfrak{n}(2)}$ has $\mathfrak{m}(2) = 3$ rows and $\mathfrak{n}(2) = \mathfrak{w}_{3\,|\,2}(2) = \mathfrak{w}(2, \mathfrak{w}_{2\,|\,2}(2))$ columns. Actually, there is no need to compute $\mathfrak{n}(2)$. Van der Waerden theorem and the definition of van der Waerden numbers imply the existence in the first row of $\square_{3,\mathfrak{n}(2)}$ of $\mathfrak{w}_{2\,|\,2}(2)$ equidistant points, all of the same color $c(1) \in \{\, c_1, c_2 \,\}$. Select from $\square_{3,\mathfrak{n}(2)}$ the columns that include those points, remove the other columns, and get a subconfiguration $\square_{3,\mathfrak{w}_{2\,|\,2}(2)}$ of $\square_{3,\mathfrak{n}(2)}$, with $\mathfrak{w}_{2\,|\,2}(2) = \mathfrak{w}(2, \mathfrak{w}_{1\,|\,2}(2))$ columns and with points in the first row of color $c(1)$. Since the second row of $\square_{3,\mathfrak{w}_{2\,|\,2}(2)}$ has $\mathfrak{w}(2, \mathfrak{w}_{1\,|\,2}(2))$ colored points, $\mathfrak{w}_{1\,|\,2}(2)$ among them must be of a certain color $c(2) \in \{\, c_1, c_2 \,\}$. As in the previous step, select the columns that include those points, remove the others, and get a subconfiguration $\square_{3,\mathfrak{w}_{1\,|\,2}(2)}$ of $\square_{3,\mathfrak{n}(2)}$ with $\mathfrak{w}_{1\,|\,2}(2) = \mathfrak{w}(2, 2)$ columns, in which the points in the first row have color $c(1)$ and the points in the second row have color $c(2)$. The last step is similar. Specifically, there are two points in the third row of $\square_{3,\mathfrak{w}_{1\,|\,2}(2)}$, both of a color denoted by $c(3)$. By selecting the two columns that include these points, we end up with a configuration $\square_{3,2} \subseteq \square_{3,\mathfrak{n}(2)}$, which has in each row two points of the same color, $c(1)$ for row 1, $c(2)$ for row 2, and $c(3)$ for row 3. Since $c(1), c(2), c(3) \in \{\, c_1, c_2 \,\}$, two rows in $\square_{3,2}$ have points of the same color, either c_1 or c_2. Those rows form a configuration $\square_{2,2} \subseteq \square_{3,\mathfrak{n}(2)}$, that proves the $(2, \mathrm{P}(\square))$–partition regularity property of $\square_{\mathfrak{m}(2),\mathfrak{n}(2)}$.

The just completed select–remove 3–step process for $\kappa = 2$ could be extended to a $\kappa + 1$–step process for arbitrary values of κ. Starting with a configuration $\square_{\mathfrak{m}(\kappa),\mathfrak{n}(\kappa)}$, where $\mathfrak{m}(\kappa) = \kappa + 1$ and $\mathfrak{n}(\kappa) = \mathfrak{w}_{\kappa+1\,|\,\kappa}(2)$, the select–remove steps yied successively subconfigurations of $\square_{\mathfrak{m}(\kappa),\mathfrak{n}(\kappa)}$, each with less columns and with more parts of the original rows consisting of points that have the same color. The first step, for instance, yields a subconfiguration $\square_{\mathfrak{m}(\kappa),\mathfrak{w}_{\kappa\,|\,\kappa}(2)}$, with all points in row 1 of a color denoted by $c(1)$. A repeated use of (4.2) implies that the completion of step i, $2 \leq i \leq \kappa$, yieds a configuration $\square_{\mathfrak{m}(\kappa),\mathfrak{w}_{\kappa-i+1\,|\,\kappa}(2)} \subseteq \square_{\mathfrak{m}(\kappa),\mathfrak{n}(\kappa)}$ in which the rows $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_i$ have points of colors $c(1), c(2), \ldots, c(i)$, respectively. In particular, step κ yields a subconfiguration $\square_{\mathfrak{m}(\kappa),\mathfrak{w}_{1\,|\,\kappa}(2)}$, with each row $\mathcal{R}_i$, $1 \leq i \leq \kappa$, of a specific color $c(i)$. Since $\mathfrak{m}(\kappa) = \kappa + 1$ and $\mathfrak{w}_{1\,|\,\kappa}(2) = \mathfrak{w}(\kappa, 2)$, the last row, $\mathcal{R}_{\kappa+1}$, in $\square_{\mathfrak{m}(\kappa),\mathfrak{w}_{1\,|\,\kappa}(2)}$ has lenght $\mathfrak{w}(\kappa, 2)$, hence two of its points are of a certain color $c(\kappa+1)$. It remains to select the colomns that include these points, and get a configuration $\square_{\kappa+1,2} \subseteq \square_{\mathfrak{m}(\kappa),\mathfrak{n}(\kappa)}$ that has $\kappa+1$ rows, with two points in each row of the same color. Since $c(1), c(2), \ldots, c(\kappa+1) \in \{\, c_1, c_2, \ldots, c_\kappa \,\}$, the colors of two rows in $\square_{\kappa+1,2}$ must coincide. Those rows make up a configuration $\square_{2,2} \subseteq \square_{\mathfrak{m}(\kappa),\mathfrak{n}(\kappa)}$ which consists of four points of the same color, vertices of a rectangle. Therefore, $\square_{\mathfrak{m}(\kappa),\mathfrak{n}(\kappa)}$ has the $(\kappa, \mathrm{P}(\square))$–partition regularity property. The proof of Theorem B is complete. $\square$

Theorem B and the coloring with κ distinct colors $c_1, c_2, \ldots, c_\kappa$ of configurations $\square_{\kappa,n}$ in which row $\mathcal{R}_i$, $1 \le i \le \kappa$, has all its points of color c_i, clearly imply that in order to get the $(\kappa, \mathrm{P}(\square))$–partition regularity property for configurations $\square_{m,n}$ we need to assume, as a requirement, that $m \ge \kappa+1$. Of course, for each such m we still need to determine the values of n. We should note that anything proved for $\square_{m,n}$ would be true for $\square^{\dagger}_{m,n} = \square_{n,m}$. In particular, $n \ge \kappa + 1$, too.

These comments indicate the right ways of defining pairs of related row and column $(\kappa, \mathrm{P}(\square))$–partition regularity numbers. Specifically, the row partition regularity numbers form a parametrized family of integers $\mathfrak{m}_\alpha(\kappa, \mathrm{P}(\square)) = \kappa+\alpha$, where the values of α are from a uniquely determined set $A(\kappa)$, which depends on κ. Each $\alpha \in A(\kappa)$ has an associated column partition regularity number $\mathfrak{n}_\alpha(\kappa, \mathrm{P}(\square))$, defined as the least integer n for which configurations $\square_{\kappa+\alpha,n}$ have the $(\kappa, \mathrm{P}(\square))$–partition regularity property. The process start with $\alpha_{\min}(\kappa)$, the smallest integer in $A(\kappa)$, and ends with $\alpha_{\max}(\kappa)$, the greatest integer in $A(\kappa)$. Set $A(\kappa)$ makes it possible to determine all minimal configurations with the $(\kappa, \mathrm{P}(\square))$–partition regularity property. According to a previous remark, we know that $\square_{m,n}$ is minimal only if $\square^{\dagger}_{m,n} = \square_{n,m}$ is minimal. As a consequence, $\mathfrak{m}_{\alpha_{\max}(\kappa)}(\kappa, \mathrm{P}(\square)) = \mathfrak{n}_{\alpha_{\min}(\kappa)}(\kappa, \mathrm{P}(\square))$, hence $\kappa + \alpha_{\max}(\kappa)) = \mathfrak{n}_{\alpha_{\min}(\kappa)}(\kappa, \mathrm{P}(\square))$. Since $\alpha_{\min}(\kappa) = 1$, we end up with the following general result.

Proposition 5. *The row and column $(\kappa, \mathrm{P}(\square))$–partition regularity numbers are given by $\mathfrak{m}_\alpha(\kappa, \mathrm{P}(\square)) = \kappa + \alpha$ and $\mathfrak{n}_\alpha(\kappa, \mathrm{P}(\square)) = \kappa + (\alpha_{\max}(\kappa) - \alpha + 1)$, where $1 \le \alpha \le \alpha_{\max}(\kappa)) = \mathfrak{n}_1(\kappa, \mathrm{P}(\square)) - \kappa$.* $\square$

In the case $\kappa = 2$, Theorem B shows that $\mathfrak{n}_1(2, \mathrm{P}(\square)) \le \mathfrak{w}_{3\,|\,2}(2)$. Using (4.1), (4.2) and the known van der Waerden numbers $\mathfrak{w}(2, 2) = 3$, $\mathfrak{w}(2, 3) = 9$, we get

$$\mathfrak{w}_{3\,|\,2}(2) = \mathfrak{w}(2, \mathfrak{w}(2, \mathfrak{w}(2, 2))) = \mathfrak{w}(2, \mathfrak{w}(2, 3)) = \mathfrak{w}(2, 9).$$

The only information about $\mathfrak{w}(2, 9)$ is the lower bound $\mathfrak{w}(2, 9) \ge 41,266$ found by Rabung, Lotts [5]. However, a direct approach makes the point.

Proposition 6. *If $\kappa = 2$, then $\mathfrak{n}_1(2, \mathrm{P}(\square)) = 7$ and $A(2) = \{1, 2, 3, 4, 5\}$.*

Proof. Suppose that the colored points are $\bullet$ or $\circ$. Refer to two distinct colorings of a configuration $\square_{m,n}$ as correlated if one is derived from the other by either changing $\bullet$ to $\circ$ and $\circ$ to $\bullet$, or by permuting columns. Our task, determining $\mathfrak{n}_1(2, \mathrm{P}(\square))$, reduces to proving that configuration $\square_{3,7}$ is minimal. We first observe that the colored configuration

$$\begin{matrix} \bullet & \bullet & \bullet & \circ & \circ & \circ \\ \bullet & \circ & \circ & \bullet & \bullet & \circ \\ \circ & \bullet & \circ & \bullet & \circ & \bullet \end{matrix}$$

excludes all configuration $\square_{3,n}$, $3 \le n \le 6$. Next, we note that when $n = 7$, at least four points in the first row of $\square_{3,7}$ have the same color. Trying to avoid the occurence of a configuration $\square_{2,2} \subseteq \square_{3,7}$ with points of the same color, we may assume that, up to a correlation transformation, the points in rows 1 and 2 are $\bullet\ \bullet\ \bullet\ \bullet\ *\ *\ *$ and $\circ\ \circ\ \circ\ *\ *\ *\ *$. The attempt fails, because among the first three points in row 3 there are two, either both $\bullet$, or both $\circ$. Therefore, $\mathfrak{n}_1(\kappa, \mathrm{P}(\square)) = 7$, $\alpha_{\max}(2) = 7 - 2 = 5$, and $A(2) = \{1, 2, 3, 4, 5\}$. The proof of Proposition 4.2 is complete. $\square$

5. Complete polygonal configurations

The result proved in this section is concerned with finding upper bounds of the Ramsey numbers $\mathfrak{r}(\nu_1, \nu_2, \dots, \nu_\kappa)$ when $\nu_1 = \nu_2 = \cdots = \nu_\kappa = 3$ and $\kappa \ge 1$, denoted by $\mathfrak{r}(\kappa \,|\, 3)$. Though the result might be well known, maybe some readers would be seeing it for the first time.

Let Π_n be a convex polygon with $n \ge 3$ vertices. The complete polygonal configuration $\Gamma(\Pi_n)$ is the complete graph of order n defined by using the vertices of Π_n as the set $\mathcal{V}(\Gamma(\Pi_n))$ of vertices of $\Gamma(\Pi_n)$. The set $\mathcal{E}(\Gamma(\Pi_n))$ of edges of $\Gamma(\Pi_n)$ consists of the sides and diagonals of Π_n.

Theorem C – Partition regularity property of configurations $\Gamma(\Pi_n)$

Let $\mathfrak{n}(\kappa)$, $\kappa \ge 1$, be the sequnce of integers defined by

$$\mathfrak{n}(1) = 3, \quad \mathfrak{n}(\kappa+1) = (\kappa+1)\,(\mathfrak{n}(\kappa) - 1) + 2, \ \ \kappa \ge 1. \tag{5.1}$$

Assume that the edges of a complete polygonal graph $\Gamma(\Pi_n)$ with $n \ge \mathfrak{n}(\kappa)$ are colored using $\kappa \ge 1$ colors. Then $\Gamma(\Pi_n)$ includes three edges of the same color, sides of a triangle. Consequently, $\mathfrak{r}(\kappa \,|\, 3) \le \mathfrak{n}(\kappa)$ for any $\kappa \ge 1$.

Proof of Theorem C. The proof is by induction on κ. The base case for $\kappa = 1$ is obvious. Assuming that the theorem is true for $\kappa \ge 1$, suppose that the edges of a polygonal graph $\Gamma(\Pi_n)$ with $n \ge \mathfrak{n}(\kappa+1)$ are colored using $\kappa+1$ colors, $c_1, c_2, \dots, c_{\kappa+1}$. Let $\mathcal{E}(\Gamma(\Pi_n), X_0) \subseteq \mathcal{E}(\Gamma(\Pi_n))$ be the set of edges incident to a vertex X_0 of Π_n, and note that $|\,\mathcal{E}(\Gamma(\Pi_n), X_0)\,| = n - 1 \ge \mathfrak{n}(\kappa+1) - 1$. By (5.1), $|\,\mathcal{E}(\Gamma(\Pi_n), X_0)\,| \ge (\kappa+1)\,(\mathfrak{n}(\kappa) - 1) + 1$. Using the pigeonhole principle we conclude that $\mathfrak{n}(\kappa)$ edges in $\Gamma(\Pi_n), X_0)$ have the same color, which we assume to be color $c_{\kappa+1}$. These edges connect X_0 with

$\mathfrak{n}(\kappa)$ vertices of Π_n denoted by $X_1, X_2, \ldots, X_{\mathfrak{n}(\kappa)}$. Those vertices determine a polygon $\Pi_{\mathfrak{n}(\kappa)}$, and $\Gamma(\Pi_{\mathfrak{n}(\kappa)})$ is a complete subgraph of $\Gamma(\Pi_n)$. If one edge of $\Gamma(\Pi_{\mathfrak{n}(\kappa)})$ has color $c_{\kappa+1}$, then $\Gamma(\Pi_n)$ includes three edges of color $c_{\kappa+1}$, sides of a triangle. If no edges of $\Gamma(\Pi_{\mathfrak{n}(\kappa)})$ have color $c_{\kappa+1}$, then by the induction hypothesis three edges from $\mathcal{E}(\Gamma(\Pi_{\mathfrak{n}(\kappa)})$ are sides with the same color of a triangle. Thus, $\Gamma(\Pi_n)$ has the required partition regularity property. The proof of Theorem C is complete. $\square$

The only known Ramsey numbers $\mathfrak{r}(\kappa \,|\, 3)$ are $\mathfrak{r}(1 \,|\, 3) = 3$, $\mathfrak{r}(2 \,|\, 3) = 6$, and $\mathfrak{r}(3 \,|\, 3) = 17$, which turn out to be the first three terms of the sequence defined by (5.1), $\mathfrak{n}(1) = 3$, $\mathfrak{n}(2) = 6$, $\mathfrak{n}(3) = 17$. Theorem C implies the estimate $\mathfrak{r}(4 \,|\, 3) \leq \mathfrak{n}(4) = 66$. Actually, more elaborate techniques developed by Fettes, Kramer, Radziszowski [1] provide the upper bound $\mathfrak{r}(4 \,|\, 3) \leq 62$. It is expected that computer assisted research in progress will lead to the exact value of $\mathfrak{r}(4 \,|\, 3)$ in a not too distant future.

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Yet another take on the Basel problem

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Abstract. This article revisits Euler's 1741 solution of the Basel problem and extends it to obtain a new evaluation of a series involving central binomial coefficients, expressed in terms of Catalan's and Apéry's constants. We also highlight a framework for analyzing other series of this type and emphasise some connections to log-sine integrals.

Keywords: Basel problem, Aperry's constant, Catalan's constant, log-sine integral.

MSC: Primary 40A10; Secondary 40A05.

1. The Basel problem

The Basel problem asks for the numerical evaluation of the infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots . \tag{1}$$

The problem was addressed by Jakob Bernoulli (1654–1705) in Basel in 1689 and has since become known as the *Basel problem.* Leonhard Euler (1707–1783) was the first to solve it, in 1734. For a leisurely presentation of Euler's original work, see Dunham [8, Ch. 3]. For a more serious take, see the chapter "Euler's Solution of the Basel Problem–The Longer Story" in Bradley et al. [5]. Many other proofs by various methods have been found since then; see e.g. Campbell and Levrie [6].

In this article, we revisit one of Euler's classical approaches to the Basel problem and provide a related evaluation of (1) that may be familiar to some, but does not appear to have been previously recorded, at least not in this exact form. Using the same mathematical framework, we also provide a new evaluation of a certain series in terms of Catalan's and Apéry's constants, and offer a brief incursion into the realm of log-sine integrals.

2. Solving the Basel problem yet again

Our proof relies on the following identity, valid for $-1 < p < 1$,

$$\int_0^{\pi/2} \frac{\mathrm{d}\theta}{1 + p\cos\theta} = \frac{\arccos p}{\sqrt{1-p^2}}. \tag{2}$$

To establish (2), we use the following *rational parametrization* of the unit circle: if $t = \tan(\theta/2)$ then

$$\cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2}, \quad \text{and} \quad \tan\theta = \frac{2t}{1-t^2}. \tag{3}$$

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Thus, $\theta = 2\arctan t$, so that $\mathrm{d}\theta = 2\mathrm{d}t/(1+t^2)$ and then, the integral in (2) equals

$$\frac{2}{1-p}\int_0^1 \frac{\mathrm{d}t}{\dfrac{1+p}{1-p}+t^2} = \frac{2}{1-p}\sqrt{\frac{1-p}{1+p}}\arctan\left(t\sqrt{\frac{1-p}{1+p}}\right)\Big|_{t=0}^{t=1}$$
$$= \frac{2}{\sqrt{1-p^2}}\arctan\frac{\sqrt{1-p^2}}{1+p}.$$

To see that the last expression simplifies to the right-hand side of (2), without appealing to elaborate trigonometric identities, write

$$\alpha = \arctan\frac{\sqrt{1-p^2}}{1+p} \quad \text{and} \quad \beta = \arccos p.$$

Then $0 < \alpha < \frac{\pi}{2}$, $\tan\alpha = \sqrt{1-p^2}/(1+p)$ and $0 < \beta < \pi$, $\cos\beta = p$. Using the double-angle formula for the tangent,

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{\sqrt{1-p^2}}{p} = \frac{\sin\beta}{\cos\beta} = \tan\beta,$$

which implies $2\alpha = \beta$.

Now, by integrating both sides of (2) with respect to p, from -1 to 1, we obtain

$$\int_{-1}^1\left(\int_0^{\pi/2}\frac{\mathrm{d}\theta}{1+p\cos\theta}\right)\mathrm{d}p = \int_{-1}^1\frac{\arccos p}{\sqrt{1-p^2}}\mathrm{d}p = \int_0^\pi \varphi\,\mathrm{d}\varphi = \frac{\pi^2}{2}.$$

Interchanging the order of integration on the left-hand side, we then have

$$\int_0^{\pi/2}\frac{1}{\cos\theta}\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\mathrm{d}\theta = \frac{\pi^2}{2}. \tag{4}$$

Again, from the first identity in (3) we obtain $(1-\cos\theta)/(1+\cos\theta) = t^2$, thus the integral on the left-hand side in (4) equals

$$2\int_0^1\frac{1+t^2}{1-t^2}\ln\left(\frac{1}{t^2}\right)\frac{\mathrm{d}t}{1+t^2} = 4\int_0^1\frac{\ln t}{t^2-1}\mathrm{d}t.$$

Combining this with (4), we obtain

$$\int_0^1\frac{\ln x}{x^2-1}\mathrm{d}x = \frac{\pi^2}{8}. \tag{5}$$

People familiar with integral approaches to the Basel problem are likely to recognize that the above derivation of (5) solves the Basel problem, giving the value $\pi^2/6$ to sum (1). For those not so familiar, this is one possible argument. In formula (obtained via integration by parts)

$$\int_0^1 x^{2n}\ln x\,\mathrm{d}x = -\frac{1}{(2n+1)^2}, \tag{6}$$

we sum both sides over n, on the left we interchange summation with integration, thus obtaining a geometric series, so that

$$\int_0^1 \frac{\ln x}{x^2-1}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}. \tag{7}$$

The sum on the right above is just

$$\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

This, combined with (5) and (7) gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{8}$$

This concludes our solution of the Basel problem.

Euler's identity (8) is often written in the concise form $\zeta(2) = \pi^2/6$. This notation refers to Riemann's zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, which converges for $s > 1$. Its value for $s = 3$,

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is called *Apéry's constant*, and will play some role in the next section.

3. From the Basel problem to Catalan's and Apéry's constants

The previous solution to the Basel problem was inspired by Spiegel [27], who used (2), written equivalently as

$$\int_0^{\pi/2} \frac{\mathrm{d}\theta}{1-p\cos\theta} = \frac{\pi}{2\sqrt{1-p^2}} + \frac{\arcsin p}{\sqrt{1-p^2}},$$

to derive the power series of $\arcsin p/\sqrt{1-p^2}$. To accomplish this, on the left, he expanded the denominator in geometric series and invoked Wallis integrals; on the right, he replaced the first term using the binomial series

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} x^{2n}. \tag{9}$$

As a result, Spiegel obtained

$$\frac{\arcsin x}{\sqrt{1-x^2}} = x + \sum_{n=1}^{\infty} \frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)} \frac{x^{2n+1}}{2n+1}. \tag{10}$$

This power series, although less known than other popular expansions, is classical and goes back to Euler's time. Through integration, Spiegel obtained a power series for $(\arcsin x)^2$. By inserting $x = \sin\theta$ into this series and using Wallis integrals again, Spiegel provided another solution to the Basel problem, among other things. Related arguments were presented by Choe [7] and Kimble [16]. In fact, the three solutions of the Basel problem given by Spiegel, Choe, and Kimble can be seen as modern reinterpretations of one of Euler's original approaches, published in 1741; see Lord [19] for some interesting historical comments on this. From a certain perspective, our solution above is even simpler, as it avoids the use of Wallis-type integrals.

The power series (10) has countless applications; see Lehmer's paper [17] for a small sample. Here is an application that seems not to have been noted before. In (10), put $x = \sin\theta$ to obtain

$$\frac{\theta}{\cos\theta} = \sin\theta + \sum_{n=1}^{\infty} \frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)} \frac{\sin^{2n+1}\theta}{2n+1}.$$

Now, by multiplying both sides by $\cot\theta$, then integrating with respect to θ, from 0 to some fixed $\varphi \in (0, \pi/2)$, we obtain

$$\int_0^{\varphi} \frac{\theta}{\sin\theta}\,\mathrm{d}\theta = \sin\varphi + \sum_{n=1}^{\infty} \frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)} \frac{\sin^{2n+1}\varphi}{(2n+1)^2}.$$

Next, integrate both sides over φ from 0 to $\pi/2$; then, on the left, interchange the order of integration and, on the right, use Wallis integral

$$\int_0^{\pi/2} \sin^{2n+1}\varphi\,\mathrm{d}\varphi = \frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)} \frac{1}{2n+1}.$$

As a result,

$$\int_0^{\pi/2} \Big(\frac{\pi}{2} - \theta\Big) \frac{\theta}{\sin\theta}\,\mathrm{d}\theta = 1 + \sum_{n=1}^{\infty} \Big[\frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)}\Big]^2 \frac{1}{(2n+1)^3}.$$

Now, if we interpret the term in the summation as being 1 for $n = 0$ (this would be consistent with the *central binomial coefficient* notation), then

$$\sum_{n=0}^{\infty} \Big[\frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)}\Big]^2 \frac{1}{(2n+1)^3} = \frac{\pi}{2}\int_0^{\pi/2} \frac{\theta}{\sin\theta}\,\mathrm{d}\theta - \int_0^{\pi/2} \frac{\theta^2}{\sin\theta}\,\mathrm{d}\theta.$$

Both integrals on the right are classical: the first equals $2G$, where G is *Catalan's constant*

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots,$$

whilst the second equals $2\pi G - \frac{7}{2}\zeta(3)$. The proof of both claims can be found in Bradley [4, Formulas (4), (35)]. Note that G is the sum of the alternating version of the series in (7). As a consequence,

$$\sum_{n=0}^{\infty}\Big[\frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)}\Big]^2\frac{1}{(2n+1)^3}=\frac{7}{2}\zeta(3)-\pi G.$$

To the best of our knowledge, the above sum was first established in Batir [2, Sect. 4, Application 4], where it is obtained as a special case of a sophisticated identity relating a certain power series to a double integral (depending on a parameter).

4. From Basel problem to log-sine integrals

Euler considered the so-called *log-sine integral* on at least four different occasions during his life, as reported by Lord [20], providing several proofs of the identity

$$\int_0^{\pi/2}\ln(\sin\theta)\,\mathrm{d}\theta=-\frac{\pi}{2}\ln 2. \tag{11}$$

By changing θ to $\pi/2-\theta$, the analogous identity with $\cos\theta$ in place of $\sin\theta$ holds.

In the course of his investigations on the cubic Basel problem, namely the problem of evaluating $\zeta(3)$ in terms of other known constants, Euler found

$$\int_0^{\pi/2}\theta\ln(\sin\theta)\,\mathrm{d}\theta=\frac{7}{16}\zeta(3)-\frac{\pi^2}{8}\ln 2. \tag{12}$$

See Dunham [11] for a lively description of Euler's work on this problem. Clearly, the next natural integral to consider is that of $\theta^2\ln(\sin\theta)$, and we show next that

$$\int_0^{\pi/2}\theta^2\ln(\sin\theta)\,\mathrm{d}\theta=\frac{3\pi}{16}\zeta(3)-\frac{\pi^3}{24}\ln 2. \tag{13}$$

Of course, identity (13) is classical and well known, but it seems to be usually evaluated via Fourier series. To the best of our knowledge, Euler never investigated this integral and we will evaluate it here as, we believe, Euler might have done it. By integrating (10) we find, as Spiegel did, the power series (recall our convention regarding $n=0$)

$$\begin{aligned}\frac{(\arcsin x)^2}{2} &= \int_0^x \frac{\arcsin t}{\sqrt{1-t^2}}\,\mathrm{d}t\\ &= \int_0^x \Big[\sum_{n=0}^{\infty}\frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)}\frac{t^{2n+1}}{2n+1}\Big]\,\mathrm{d}t\\ &= \sum_{n=0}^{\infty}\frac{2\cdot 4\cdots(2n)(2n+2)}{1\cdot 3\cdots(2n-1)(2n+1)}\frac{x^{2n+2}}{(2n+2)^2}\\ &= \sum_{n=1}^{\infty}\frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)}\frac{x^{2n}}{(2n)^2},\end{aligned}$$

where the last step follows after re-indexing the previous summation. For $x=\sin\theta$, this yields

$$\theta^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)}\frac{\sin^{2n}\theta}{n^2}. \tag{14}$$

Spiegel's solution to the Basel problem, mentioned previously, boils down to integrating the above with respect to θ from 0 to $\pi/2$ and using Wallis integral

$$\int_0^{\pi/2}\sin^{2n}\theta\,\mathrm{d}\theta = \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\frac{\pi}{2}. \tag{15}$$

Here, we apply the same procedure of the previous section. By multiplying (14) by $\cot\theta$, and then integrating with respect to θ, from 0 to some fixed $\varphi\in(0,\pi/2)$, we obtain

$$\int_0^{\varphi}\theta^2\cot\theta\,\mathrm{d}\theta = \frac{1}{4}\sum_{n=1}^{\infty}\frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)}\frac{\sin^{2n}\varphi}{n^3}.$$

Next, integrate both sides over φ from 0 to $\pi/2$; then, on the left, interchange the order of integration and, on the right, use Wallis integral (15). It follows that

$$\int_0^{\pi/2}\Big(\frac{\pi}{2}-\theta\Big)\theta^2\cot\theta\,\mathrm{d}\theta = \frac{\pi}{8}\sum_{n=1}^{\infty}\frac{1}{n^3}.$$

The sum on the right is $\zeta(3)$. By noting that $\cot\theta = \dfrac{\mathrm{d}}{\mathrm{d}\theta}\ln(\sin\theta)$, integration by parts yields

$$-\pi\int_0^{\pi/2}\theta\ln(\sin\theta)\,\mathrm{d}\theta + 3\int_0^{\pi/2}\theta^2\ln(\sin\theta)\,\mathrm{d}\theta = \frac{\pi}{8}\zeta(3).$$

This, combined with (12), yields (13).

Euler's first approach to the log-sine integral was to use the logarithmic series to expand

$$\ln(\cos\theta) = \frac{1}{2}\ln(1-\sin^2\theta) = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin^{2n}\theta}{n}. \tag{16}$$

Then he integrated the above identity with respect to θ from 0 to $\pi/2$, and used Wallis integral (15) combined with other arguments involving the binomial series to obtain (11); see Lord [20] for the details. However, if we multiply the above by $\cot\theta$ and integrate from 0 to $\pi/2$, we obtain

$$\int_0^{\pi/2}\cot\theta\ln(\cos\theta)\,\mathrm{d}\theta = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\pi/2}\sin^{2n-1}\theta\cos\theta\,\mathrm{d}\theta = -\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2},$$

or

$$\zeta(2) = -4\int_0^{\pi/2}\cot\theta\ln(\cos\theta)\,\mathrm{d}\theta = -4\int_0^1\frac{x\ln x}{1-x^2}\,\mathrm{d}x = \int_0^1\frac{\ln y}{y-1}\,\mathrm{d}y, \tag{17}$$

where the first equality follows from the substitution $x = \cos\theta$, and the second from $y = x^2$. All these are, in different ways, well-known integral representations of $\zeta(2)$, dating back to Euler.

On the other hand, if we apply to (16) the procedure of the previous sections, that of multiplying by $\cot\theta$, then integrating from 0 to some fixed $\varphi \in (0, \pi/2)$, then integrating again over φ from 0 to $\pi/2$, and then interchanging the order of integration on the left, we get

$$\int_0^{\pi/2}\left(\frac{\pi}{2}-\theta\right)\cot\theta\ln(\cos\theta)\,\mathrm{d}\theta = -\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\frac{1}{n^2}.$$

Boyadzhiev [3, Prop. 1.6] evaluated the series on the right as $\zeta(2) - 2\ln^2 2$. This, combined with the first equality in (17) yields

$$\int_0^{\pi/2}\theta\cot\theta\ln(\cos\theta)\,\mathrm{d}\theta = -\frac{\pi}{4}\ln^2 2.$$

Changing θ to $\pi/2 - \theta$ gives

$$\int_0^{\pi/2}\theta\tan\theta\ln(\sin\theta)\,\mathrm{d}\theta = \frac{\pi}{4}\ln^2 2 - \frac{\pi}{8}\zeta(2).$$

These last two integral evaluations are less known and may even be new.

Finally, Nahin [23, Sect. 7.4] showed that

$$\zeta(2) = \frac{4}{\pi}\int_0^{\pi/2}\ln^2(2\sin\theta)\,\mathrm{d}\theta.$$

Although the integral on the right is well-known and evaluates to $\pi^3/24$, a result obtainable without recourse to the Basel problem, this specific representation seems to be original to Nahin, who derived it by using complex variables. We argue that other derivations of this integral representation could also constitute new, alternative solutions of the Basel problem.

5. A little theorem

The method highlighted in the previous sections, that of multiplying a given expansion in powers of sines by $\cot\theta$, then integrating from 0 to some fixed $\varphi \in (0, \pi/2)$, then integrating again over φ from 0 to $\pi/2$, and finally interchanging the order of integration "on the left" can be formalised in the following theorem. Note that, in assertion (a) below, the term corresponding to $n = 0$ is simply a_0, in accordance with our previous convention.

Theorem 1. *Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series.*

(a) *The function f represented by $f(\theta) = \sum_{n=0}^{\infty} a_n \sin^{2n+1}\theta$ is continuous on $[0, \frac{\pi}{2}]$ and*

$$\int_0^{\pi/2} \left(\frac{\pi}{2} - \theta\right) \cot\theta\, f(\theta)\, \mathrm{d}\theta = \sum_{n=0}^{\infty} \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)} \frac{a_n}{(2n+1)^2}.$$

(b) *The function f represented by $f(\theta) = \sum_{n=1}^{\infty} a_n \sin^{2n}\theta$ is continuous on $[0, \frac{\pi}{2}]$ and*

$$\int_0^{\pi/2} \left(\frac{\pi}{2} - \theta\right) \cot\theta\, f(\theta)\, \mathrm{d}\theta = \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \frac{a_n}{n}.$$

Proof. By Weierstrass M-test, the convergence of $\sum |a_n|$ guarantees that, in both cases, the series defining f converges uniformly, so that f is continuous. Moreover, the series can be integrated termwise and both identities follow from Wallis formulas. □

Clearly, analogous conclusions hold when f has expansions in powers of cosines, with $\tan\theta$ replacing $\cot\theta$ in the integral. More interesting is the fact that both identities in (a) or (b) may hold even in cases where the series of coefficients diverge. In this case, the integrals are improper, which corresponds to the power series $\sum a_n x^n$ having radius of convergence 1 and not converging at $x = 1$. Moreover, the integral on the left is not always easy to evaluate. To illustrate both phenomena, consider the expansion of the inverse hyperbolic tangent,

$$\operatorname{arctanh}(x) = \frac{1}{2} \ln \frac{1+x}{1-x} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}.$$

Then,

$$\operatorname{arctanh}(\sin\theta) = \sum_{n=0}^{\infty} \frac{\sin^{2n+1}\theta}{2n+1}$$

and $\sum_{n=1}^{\infty}(2n+1)^{-1}$ diverges. Yet, Theorem 1(a) yields (formally, at least)

$$\sum_{n=0}^{\infty} \frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)}\frac{1}{(2n+1)^3} = \int_0^{\pi/2}\left(\frac{\pi}{2}-\theta\right)\cot\theta\,\operatorname{arctanh}(\sin\theta)\,\mathrm{d}\theta.$$

It is not difficult to see that the series on the left converges, for instance by Raabe's test. The integral on the right is improper, and does not appear to be classical; we will not try to evaluate it here. Another integral expression for the sum on the left is

$$\sum_{n=0}^{\infty} \frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)}\frac{1}{(2n+1)^3} = -2\int_0^{\pi/2}\frac{\phi}{\sin\phi}\ln(\sin\phi)\,\mathrm{d}\phi.$$

WolframAlpha evaluates both integrals to 1.12269. This representation can be obtained as the special case $x=2$ and $k=3$ in Batir [2, Theorem 2.1], namely,

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\frac{x^{2n+1}}{(2n+1)^k} = \frac{2(-1)^{k-2}}{(k-2)!}\int_0^{\pi/2}\frac{\phi}{\sin\phi}\ln^{k-2}\left(\frac{2\sin\phi}{x}\right)\mathrm{d}\phi,$$

valid for $|x|\le 2$ and $k=2,3,\ldots$. (The factor 2 on the right-hand side is missing in Batir's paper.)

6. Further comments

It has been some time since I first carried out the computations presented in Section 2, and I was always a bit reluctant to submit them for publication, as I felt they were merely a reinterpretation of Spiegel's article [27]. Later, I noticed that essentially the same integral appears as the first problem in Vălean's book [28], which presents it in the form (with y in place of p)

$$\int_0^1 \frac{1}{(1+px)\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\arccos p}{\sqrt{1-p^2}}. \tag{18}$$

The first step of Vălean's solution is to substitute $x=\sin\theta$, which leads to the integral in (2) with $\sin\theta$ in place of $\cos\theta$; both integrals are essentially the same, as we can see by replacing θ with $\frac{\pi}{2}-\theta$. Vălean develops (18) from scratch (almost as we did), except that he does not use the rational parametrization explicitly, which makes his solution a bit more cumbersome than the one presented here. In the end, I decided it would be reasonable to make this version available to the public as a more economical alternative.

By now, the integral representation (2) most likely belongs to the folklore of classical mathematical analysis. For instance, Duistermaat and Kolk [9, Exer. 2.79] asks for the evaluation of the analogous integral from 0 to π, in which case one obtains the right-hand side in (2) with π in place of $\arccos p$; this is, except for the factor π, the arcsine differential. It goes without saying that this allows for a solution to the Basel problem analogous to the one in

Section 2. From this version of (2), Duistermaat and Kolk [10, Exer. 6.50] further guide the reader to a derivation of the arcsine power series

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \frac{x^{2n+1}}{2n+1}.$$

We invite the reader to apply Theorem 1(a) to the above expansion and thereby recover, after some simple computations, an interesting result that appeared earlier in this article without proof, namely Euler's identity (12).

Finally, the solution of the Basel problem presented in Section 2 entails interchanging the order of integration in a double integral, namely,

$$\int_{-1}^{1} \int_{0}^{\pi/2} \frac{1}{1 + p\cos\theta}\, \mathrm{d}p\, \mathrm{d}\theta.$$

In this respect, it joins the family of approaches presented in works such as Apostol [1], Lord [18], Harper [12], Ivan [13], Ritelli [26], Jameson [14], Jameson and Lord [15], Novac [24], Murty [22] and Markov [21].

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20th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2026

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Abstract. The 20th South Eastern European Mathematical Olympiad for University Students (SEEMOUS 2026) was held from March 3 to March 8, 2026, in Paphos, Cyprus. This paper presents the competition problems along with their solutions as provided by the authors. Additionally, alternative solutions contributed by jury members or contestants are included.

Keywords: Chebyshev polynomials of the first kind, Frobenius's inequality, Gauss's lemma, Jacobson's lemma, Newton's identities, nilpotent matrices, Niven's theorem.

MSC: 15A03, 15A15, 15A18, 15A45, 26A24, 39B62.

The 20th South Eastern European Mathematical Olympiad for University Students with International Participation (SEEMOUS 2026) was hosted between March 3 and March 8 by the Cyprus Mathematical Society, with the support of the Mathematical Society of South Eastern Europe (MASSEE). Given the geopolitical context, the competition was held in a hybrid format. This competition is addressed to students in the first or second year of undergraduate studies, from universities in countries that are members of the MASSEE, or from invited countries that are not affiliated to MASSEE.

A number of 94 students participated in the contest, representing 29 universities from Albania, Bulgaria, Cyprus, Greece, North Macedonia, Romania, Serbia, and Turkmenistan. The jury awarded 12 gold medals, 23 silver medals, and 35 bronze medals. The National University of Science and Technology Politehnica Bucharest won the title of Best University.

We present the competition problems and their solutions as given by the corresponding authors, together with alternative solutions provided by members of the jury or by the contestants.

Problem 1. Determine all differentiable functions $f : \mathbb{R} \to [0, \infty)$ with continuous derivative such that $f^2(x) \leq f'(x)$, for all $x \in \mathbb{R}$.

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Author's solution. Clearly, the zero function is a solution. We prove that there are no other solutions.

Assume, by contradiction, that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) > 0$. Since $f'(x) \geq f^2(x) \geq 0$, for every $x \in \mathbb{R}$, the function f is increasing.

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Hence, for every $x \geq x_0$, we have $f(x) \geq f(x_0) > 0$. Thus the function $\frac{f'}{f^2}$ is well-defined and continuous on $[x_0, \infty)$, and $\frac{f'(x)}{f^2(x)} \geq 1$, for every $x \geq x_0$.

Therefore,

$$x - x_0 = \int_{x_0}^{x} 1 \, \mathrm{d}t \leq \int_{x_0}^{x} \frac{f'(t)}{f^2(t)} \, \mathrm{d}t = -\frac{1}{f(x)} + \frac{1}{f(x_0)},$$

for every $x \geq x_0$, hence

$$0 < \frac{1}{f(x)} \leq \frac{1}{f(x_0)} + x_0 - x.$$

This is impossible for sufficiently large x, since the right-hand side becomes negative.

Therefore no such x_0 exists, so $f(x) = 0$, for every $x \in \mathbb{R}$.

Second solution. As in the first solution, assume by contradiction that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) > 0$. Since f is increasing, we have $f(x) > 0$, for every $x \geq x_0$.

Define $h(x) = x + \frac{1}{f(x)}$, for $x \geq x_0$. Then h is continuous on $[x_0, \infty)$ and differentiable on (x_0, ∞). Moreover, $h'(x) = 1 - \frac{f'(x)}{f^2(x)} \leq 0$, so h is decreasing on $[x_0, \infty)$. Hence,

$$x + \frac{1}{f(x)} = h(x) \leq h(x_0) = x_0 + \frac{1}{f(x_0)},$$

for every $x \geq x_0$. This gives exactly the same contradiction as in the first solution, namely $0 < \frac{1}{f(x)} \leq \frac{1}{f(x_0)} + x_0 - x$, for every $x \geq x_0$. Hence f is the zero function.

Third solution. As in the first solution, assume, by contradiction, that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) > 0$. Since, as already shown, f is increasing, we have $f(x) \geq f(x_0) > 0$, for every $x \geq x_0$.

The first step is to establish the following estimate: *for every integer $m \geq 0$ and $x \geq x_0$,*

$$f(x) \geq 2^{m+2} \left(\frac{f(x_0)}{4} \right)^{2^m} (x - x_0)^{2^m - 1}.$$

This will be proved by induction on m.

For $m = 0$, the estimate simply says that $f(x) \geq f(x_0)$, which follows from the monotonicity of f.

Assume now that the inequality holds for some $m \geq 0$. Then

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(u) \, \mathrm{d}u \geq \int_{x_0}^{x} f^2(u) \, \mathrm{d}u,$$

for every $x \geq x_0$. Using the induction hypothesis, we get

$$f(x) \geq 2^{2m+4} \left(\frac{f(x_0)}{4}\right)^{2^{m+1}} \int_{x_0}^{x} (u - x_0)^{2^{m+1}-2} \, \mathrm{d}u.$$

Hence

$$f(x) \geq \frac{2^{2m+4}}{2^{m+1}-1} \left(\frac{f(x_0)}{4}\right)^{2^{m+1}} (x - x_0)^{2^{m+1}-1}.$$

Since $\frac{2^{2m+4}}{2^{m+1}-1} > 2^{m+3}$, we obtain

$$f(x) \geq 2^{m+3} \left(\frac{f(x_0)}{4}\right)^{2^{m+1}} (x - x_0)^{2^{m+1}-1}.$$

Thus the induction step is complete.

Having proved the estimate, we now return to the proof of the claim. Choose $x = x_0 + \frac{4}{f(x_0)}$. By the estimate above, for every $m \geq 0$, we have

$$f\left(x_0 + \frac{4}{f(x_0)}\right) \geq 2^{m+2} \left(\frac{f(x_0)}{4}\right)^{2^m} \left(\frac{4}{f(x_0)}\right)^{2^m-1} = 2^m f(x_0).$$

Letting $m \to \infty$, we get a contradiction, since the left-hand side is fixed and finite.

Therefore there is no x_0 such that $f(x_0) > 0$. Since f takes only non-negative values, it follows that f is the zero function.

This problem had 21 complete solutions in the contest. Based on the total number of points scored by the contestants, this problem ranked as the easiest problem in the contest.

Problem 2. Let A and B be $n \times n$ matrices with complex entries for which

$$A^2 + B^2 = 2\,(AB - BA).$$

Prove that $A^2 + B^2$ is a nilpotent matrix.

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Author's solution. Let $C = AB - BA$. Then the given relation becomes $A^2 + B^2 = 2C$.

Define $L = A - iB$ and $M = A + iB$. We have:

$$LM = (A - iB)(A + iB) = A^2 + B^2 + i\,(AB - BA) = (2 + i)\,C$$

and

$$ML = (A + iB)(A - iB) = A^2 + B^2 - i\,(AB - BA) = (2 - i)\,C.$$

It follows that:

$$ML = \frac{2 - i}{2 + i} LM = \frac{3 - 4i}{5} LM.$$

Let $\alpha = \frac{3-4i}{5}$. Since $|\alpha| = 1$, we write $\alpha = e^{i\nu}$, with $\cos\nu = \frac{3}{5}$. By Niven's theorem (see [2, Corollary 3.12.]), since $\cos\nu \in \mathbb{Q} \setminus \left\{0, \pm\frac{1}{2}, \pm 1\right\}$, the number ν is not a rational multiple of π. Therefore, α is not a root of unity, so $\alpha^k \neq 1$, for every $k \in \mathbb{N}^*$.

Now fix $k \geqslant 1$. From $ML = \alpha LM$, we get $(ML)^k = \alpha^k (LM)^k$. Taking traces and using the cyclic property of the trace, we obtain

$$\operatorname{Tr}\left((LM)^k\right) = \operatorname{Tr}\left((ML)^k\right) = \alpha^k \operatorname{Tr}\left((LM)^k\right).$$

Thus

$$\left(1-\alpha^k\right) \operatorname{Tr}\left((LM)^k\right) = 0.$$

Since $\alpha^k \neq 1$, it follows that $\operatorname{Tr}\left((LM)^k\right) = 0$, for every $k \geqslant 1$. In particular, this holds for $k = 1, 2, \ldots, n$. By Newton's identities, all coefficients of the characteristic polynomial of LM vanish, except the leading one. Thus LM is nilpotent.

Finally, since $A^2 + B^2 = 2C = \frac{2}{2+i} LM$, we conclude that $A^2 + B^2$ is also nilpotent.

Second solution. *The following solution was given by Vasile Pop, Technical University of Cluj–Napoca, Romania (deputy leader).*

As in the first solution, let $C = AB - BA$, $L = A - iB$ and $M = A + iB$. Then $A^2 + B^2 = 2C$, and we have $ML = \alpha LM$, where $\alpha = \frac{3-4i}{5}$. Recall also that α is not a root of unity, that is, $\alpha^p \neq 1$, for every $p \in \mathbb{N}^*$.

We shall use the fact that, for any $X, Y \in \mathcal{M}_n(\mathbb{C})$, the matrices XY and YX have the same spectrum.

Let $\lambda \in \sigma(LM)$. Since $ML = \alpha LM$, we have $\alpha\lambda \in \sigma(\alpha LM) = \sigma(ML) = \sigma(LM)$. Repeating the same argument, we get $\alpha^k \lambda \in \sigma(LM)$, for every $k \geqslant 0$. But $\sigma(LM)$ is a finite set. Hence there exist integers $r < s$ such that $\alpha^r \lambda = \alpha^s \lambda$. Therefore, $\lambda\left(\alpha^{s-r} - 1\right) = 0$. Since α is not a root of unity, we have $\alpha^{s-r} \neq 1$, and so $\lambda = 0$.

Thus the only eigenvalue of LM is 0, hence LM is nilpotent. Finally, since $A^2 + B^2 = 2C = \frac{2}{2+i} LM$, we conclude that $A^2 + B^2$ is nilpotent.

Remarks. (i) Niven's theorem states (for more details, see [2, Corollary 3.12]):

$$\sin(\pi\mathbb{Q}) \cap \mathbb{Q} = \cos(\pi\mathbb{Q}) \cap \mathbb{Q} = \left\{0, \pm 1, \pm\frac{1}{2}\right\}.$$

An elementary proof of this result can be found in [3].

(ii) There are several other ways to see that $\alpha = \frac{3}{5} - \frac{4}{5} i$ is not a root of unity.

(a) *The following approach was given by Petre–Luca Mirea Bulubașa, from National University of Science and Technology Politehnica Bucharest, Romania (contestant).*

We show that $(2+i)^k \neq (2-i)^k$, for every $k \in \mathbb{N}^*$. Suppose, by contradiction, that $(2+i)^k = (2-i)^k$. Then $(2+i)^k$ is real. Write $2 + i = \sqrt{5}(\cos t + i \sin t)$, where $\cos t = \frac{2}{\sqrt{5}}$ and $\sin t = \frac{1}{\sqrt{5}}$. Since $(2+i)^k$ is real, we have $\sin kt = 0$, hence $\cos kt = \pm 1$.

If k is odd, then $(2+i)^k$ is real and belongs to $\mathbb{Z}[i]$, hence $(2+i)^k \in \mathbb{Z}$. Taking norms, the square of this integer must be 5^k, which is impossible for odd k. Thus k is even.

Let $k = 2m$. Since $\cos kt = T_k(\cos t)$, where T_k is the k-th Chebyshev polynomial of the first kind, we get $T_k\left(\frac{2}{\sqrt{5}}\right) = \pm 1$. But $2T_k(x)$ has integer coefficients, leading term $2^k x^k$, and, since k is even, it contains only even powers of x. Therefore, after evaluating at $x = \frac{2}{\sqrt{5}}$ and writing the result with denominator 5^m, we obtain $2T_k\left(\frac{2}{\sqrt{5}}\right) = \frac{N}{5^m}$, where N is not divisible by 5. Indeed, the leading term contributes 2^{2k} to N, while all the other terms contribute multiples of 5.

Thus $2T_k\left(\frac{2}{\sqrt{5}}\right)$ is not an integer, contradicting $2T_k\left(\frac{2}{\sqrt{5}}\right) = \pm 2$. Hence $(2+i)^k \neq (2-i)^k$, for every $k \in \mathbb{N}^*$. Equivalently, $\frac{2+i}{2-i}$ is not a root of unity.

(b) The numbers α and $\overline{\alpha}$ are the roots of $p(x) = 5x^2 - 6x + 5$. This polynomial is primitive and irreducible in $\mathbb{Z}[x]$. If α were a root of unity, then $\alpha^m = 1$, for some $m \in \mathbb{N}^*$, and hence $p(x)$ would divide $x^m - 1$ in $\mathbb{Q}[x]$. By Gauss's Lemma, since p is primitive, it would actually divide $x^m - 1$ in $\mathbb{Z}[x]$. This is impossible, because $x^m - 1$ is monic, while the leading coefficient of p is 5.

(c) Another short argument is obtained by working in $\mathbb{Z}[i]$ modulo 5. We have $(3-4i)^2 = -7 - 24i \equiv 3 - 4i \pmod 5$. Therefore, by induction, $(3-4i)^k \equiv 3 - 4i \pmod 5$, for every $k \in \mathbb{N}^*$. In particular, $(3-4i)^k$ cannot be equal to 5^k, because $5^k \equiv 0 \pmod 5$. Thus $\alpha^k \neq 1$, for every $k \in \mathbb{N}^*$.

Generalization. A natural generalization of the problem is the following.

Let $A, B \in \mathcal{M}_n(\mathbb{C})$, $n \geqslant 2$, be two matrices satisfying

$$A^2 + B^2 = \cot\alpha\,(AB - BA),$$

for some $\alpha \in (0, \pi)$. Prove that:

(a) *if $\det(AB - BA) \neq 0$, and $\frac{\alpha}{\pi} = \frac{p}{q}$, where $p, q \in \mathbb{N}^*$ with $\gcd(p, q) = 1$, then n is a multiple of q;*

(b) *if $\frac{\alpha}{\pi}$ is irrational, then $A^2 + B^2$ is nilpotent.*

Part (a) can be found in [4], p. 43.

This problem had 3 complete solutions in the contest.

Editor's note. This type of problems, that reduces to the matrix equation $XY = \omega YX$ is well-known; see, e.g. `https://math.stackexchange.com/questions/2677706`. For a comprehensive study of this matrix equation the reader is referred to the paper [1].

Problem 3. Let A and B be $n \times n$ matrices with complex entries for which $A^2 = AB - BA$ and $\operatorname{rank}(A - B) = 1$. Prove that $ABA = O_n$.

Despite the jury's efforts to select only original problems, this one was later found to be closely related to Problem 546, published in *Gazeta Matematică, Seria A*, vol. XLI (CXX), nr. 3–4, 2023, p. 39, author Florin Stănescu, Romania.

Author's solution. Since $AB - BA = A^2$, the commutator $AB - BA$ commutes with A. Hence, by Jacobson's Lemma, $AB - BA$ is nilpotent. Thus A^2 is nilpotent, and therefore A is nilpotent.

Let $K = A - B$ and assume that $\operatorname{Im}(K) = \langle \mathbf{x} \rangle$, with $\mathbf{x} \neq \mathbf{0}$. Since $A(A - B) = -BA$, we have $AK = -BA$, and hence $ABA = -A^2 K$. Therefore, it is enough to show that $A^2\mathbf{x} = \mathbf{0}$.

We have

$$KA - AK = (A - B)\,A - A\,(A - B) = AB - BA = A^2.$$

Since A is nilpotent, there exists a minimal $k \in \mathbb{N}^*$ such that $A^k\mathbf{x} = \mathbf{0}$. Then

$$A^{k+1} = A^{k-1}A^2 = A^{k-1}\,(KA - AK) = A^{k-1}KA - A^kK.$$

Applying this relation to $\mathbf{x}$, and using $K\mathbf{x} \in \langle \mathbf{x} \rangle$, we get

$$A^{k-1}KA\mathbf{x} = A^{k+1}\mathbf{x} + A^kK\mathbf{x} = \mathbf{0}.$$

But $KA\mathbf{x} \in \operatorname{Im}(K) = \langle \mathbf{x} \rangle$. Hence $KA\mathbf{x} = \mu\mathbf{x}$, for some $\mu \in \mathbb{C}$. Since $A^{k-1}\,(KA\mathbf{x}) = \mathbf{0}$, we get $\mu A^{k-1}\mathbf{x} = \mathbf{0}$. By the minimality of k, we have $A^{k-1}\mathbf{x} \neq \mathbf{0}$, and therefore $\mu = 0$. Thus $KA\mathbf{x} = \mathbf{0}$.

Also, since $K\mathbf{x} \in \langle \mathbf{x} \rangle$, there exists $\lambda \in \mathbb{C}$ such that $K\mathbf{x} = \lambda\mathbf{x}$. Therefore

$$A^2\mathbf{x} = (KA - AK)\,\mathbf{x} = -AK\mathbf{x} = -\lambda A\mathbf{x}.$$

If $A\mathbf{x} = \mathbf{0}$, then immediately $A^2\mathbf{x} = \mathbf{0}$. Otherwise, $A\mathbf{x}$ is a non-zero eigenvector of A with eigenvalue $-\lambda$. Since A is nilpotent, all its eigenvalues are zero, hence $\lambda = 0$. Thus again $A^2\mathbf{x} = \mathbf{0}$.

Consequently $A^2K = O_n$, because $\operatorname{Im}(K) = \langle \mathbf{x} \rangle$. Finally, from $ABA = -A^2K$, we conclude that $ABA = O_n$.

Second solution. We use the same reduction, but replace the minimality argument by a direct computation involving $\mathbf{x}$ and $A\mathbf{x}$.

Let $K = A - B$ and assume that $\operatorname{Im}(K) = \langle \mathbf{x} \rangle$, with $\mathbf{x} \neq \mathbf{0}$. As seen above, it is enough to show that $A^2\mathbf{x} = \mathbf{0}$.

Let $A\mathbf{x} = \mathbf{y}$ and choose $\lambda, \mu \in \mathbb{C}$ such that $K\mathbf{x} = \lambda\mathbf{x}$ and $K\mathbf{y} = \mu\mathbf{x}$.

Since $AK\mathbf{x} = -BA\mathbf{x}$, we get $\lambda\mathbf{y} = -B\mathbf{y}$, hence $B\mathbf{y} = -\lambda\mathbf{y}$. Thus $A\mathbf{y} = K\mathbf{y} + B\mathbf{y} = \mu\mathbf{x} - \lambda\mathbf{y}$.

Also, applying $AK = -BA$ to $\mathbf{y}$, we obtain $\mu\mathbf{y} = -B\,(\mu\mathbf{x} - \lambda\mathbf{y})$. Since $B\mathbf{x} = \mathbf{y} - \lambda\mathbf{x}$ and $B\mathbf{y} = -\lambda\mathbf{y}$, this gives $\mu\mathbf{y} = -\mu\,(\mathbf{y} - \lambda\mathbf{x}) - \lambda^2\mathbf{y}$. Hence $\left(\lambda^2 + 2\mu\right)\mathbf{y} = \lambda\mu\mathbf{x}$.

So either $\mu = -\frac{\lambda^2}{2}$, or $\mathbf{y} = \alpha\mathbf{x}$, for some $\alpha \in \mathbb{C}$.

In the first case, the previous relation gives $-\frac{\lambda^3}{2}\mathbf{x} = \mathbf{0}$. Since $\mathbf{x} \neq \mathbf{0}$, we get $\lambda = 0$, and therefore $\mu = 0$. Thus $A\mathbf{x} = B\mathbf{x} = \mathbf{y}$, and hence $A^2\mathbf{x} = (AB - BA)\,\mathbf{x} = K\mathbf{y} = \mu\mathbf{x} = \mathbf{0}$.

In the second case, we have $A\mathbf{x} = \mathbf{y} = \alpha\mathbf{x}$ and $B\mathbf{x} = \mathbf{y} - \lambda\mathbf{x} = (\alpha - \lambda)\,\mathbf{x}$. Therefore $A^2\mathbf{x} = (AB - BA)\,\mathbf{x} = A\,(\alpha - \lambda)\,\mathbf{x} - B\alpha\mathbf{x} = (\alpha - \lambda)\,\alpha\mathbf{x} - \alpha\,(\alpha - \lambda)\,\mathbf{x} = \mathbf{0}$.

Thus $A^2\mathbf{x} = \mathbf{0}$. Since $\operatorname{Im}(K) = \langle \mathbf{x} \rangle$, it follows that $A^2K = O_n$. Finally, from $ABA = -A^2K$, we conclude that $ABA = O_n$.

Third solution. As shown in the first solution, A is nilpotent. Let $K = A - B$. Then $A^2 = KA - AK$, and hence A^2 is the sum of two matrices of rank at most 1. Thus $\operatorname{rank}\left(A^2\right) \leqslant 2$.

Since A is nilpotent, the ranks of its non-zero powers strictly decrease. Therefore $\operatorname{rank}\left(A^3\right) \leqslant 1$, and consequently $A^4 = O_n$.

Also, $ABA = A\,(A - K)\,A = A^3 - AKA$, while $A^3 = A\,(KA - AK) = AKA - A^2K$. Thus $ABA = -A^2K$. It remains to prove that $A^2K = O_n$.

Since $\operatorname{rank}(K) = 1$, we can write $K = \mathbf{u}\mathbf{v}^*$, for some non-zero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$.

Suppose that $A^3 = O_n$. Then $O_n = A^4 = A^2(KA - AK) = A^2KA = A^2\mathbf{u}\mathbf{v}^*A$. If $\mathbf{v}^*A = \mathbf{0}$, then $KA = O_n$, so $A^2 = -AK$. Hence $A^3 = -A^2K$, and since $A^3 = O_n$, we get $A^2K = O_n$. If $\mathbf{v}^*A \neq \mathbf{0}$, then from $A^2\mathbf{u}\mathbf{v}^*A = O_n$ we obtain $A^2\mathbf{u} = \mathbf{0}$. Hence $A^2K = A^2\mathbf{u}\mathbf{v}^* = O_n$.

Suppose that $A^3 \neq O_n$. Since $A^4 = O_n$, we may choose $\mathbf{w}$ such that $A^3\mathbf{w} \neq \mathbf{0}$. We have $\mathbf{0} = A^5\mathbf{w} = A^2\left(A^3\mathbf{w}\right) = (KA - AK)A^3\mathbf{w} = -A\mathbf{u}\left(\mathbf{v}^*A^3\mathbf{w}\right)$. If $A\mathbf{u} = \mathbf{0}$, then immediately $A^2K = O_n$. Hence we may assume that $A\mathbf{u} \neq \mathbf{0}$, and therefore $\mathbf{v}^*A^3\mathbf{w} = \mathbf{0}$. Next, $\mathbf{0} = A^4\mathbf{w} = (KA - AK)A^2\mathbf{w} = \mathbf{u}\mathbf{v}^*A^3\mathbf{w} - A\mathbf{u}\mathbf{v}^*A^2\mathbf{w} = -A\mathbf{u}\left(\mathbf{v}^*A^2\mathbf{w}\right)$. Since $A\mathbf{u} \neq \mathbf{0}$, it follows that $\mathbf{v}^*A^2\mathbf{w} = 0$. Now $\mathbf{0} \neq A^3\mathbf{w} = (KA - AK)A\mathbf{w} = \mathbf{u}\mathbf{v}^*A^2\mathbf{w} - A\mathbf{u}\mathbf{v}^*A\mathbf{w} = -A\mathbf{u}(\mathbf{v}^*A\mathbf{w})$. Therefore $\mathbf{v}^*A\mathbf{w} \neq 0$. Finally, $\mathbf{0} = A^4\mathbf{w} = A\left(A^3\mathbf{w}\right) = -A^2\mathbf{u}(\mathbf{v}^*A\mathbf{w})$. Since $\mathbf{v}^*A\mathbf{w} \neq 0$, we get $A^2\mathbf{u} = \mathbf{0}$. Thus $A^2K = A^2\mathbf{u}\mathbf{v}^* = O_n$.

In both cases, $A^2K = O_n$. Since $ABA = -A^2K$, we conclude that $ABA = O_n$.

Fourth solution. *The following solution was given by Robert Pop, from Babeș–Bolyai University of Cluj-Napoca, Romania (leader).*

We keep the notation $K = A - B$ and use the conclusions obtained in the previous solution: $A^4 = O_n$, $ABA = -A^2K$, and, in the case $A^3 = O_n$, one has $A^2K = O_n$.

If $A^2K = O_n$, then we are done. Assume, by contradiction, that $A^2K \neq O_n$. Since $\operatorname{rank}(K) = 1$, we get $\operatorname{rank}\left(A^2K\right) = 1$.

We will use Frobenius's rank inequality: $\operatorname{rank}(XY) + \operatorname{rank}(YZ) \leqslant \operatorname{rank}(Y) + \operatorname{rank}(XYZ)$.

Applying it to $X = A^2$, $Y = K$, and $Z = A^3$, we get

$$\operatorname{rank}\left(A^2K\right) + \operatorname{rank}\left(KA^3\right) \leqslant \operatorname{rank}(K) + \operatorname{rank}\left(A^2KA^3\right).$$

But $A^2KA^3 = (-ABA)A^3 = -ABA^4 = O_n$. Hence $KA^3 = O_n$.

Thus $(A - B)A^3 = O_n$, and since $A^4 = O_n$, it follows that $BA^3 = O_n$. Consequently $A^2KA^2 = (-ABA)A^2 = -ABA^3 = O_n$.

Applying Frobenius's inequality again, now with $X = A^2$, $Y = K$, and $Z = A^2$, we obtain $\operatorname{rank}(A^2K) + \operatorname{rank}(KA^2) \leq \operatorname{rank}(K) + \operatorname{rank}(A^2KA^2)$. Since $\operatorname{rank}\left(A^2K\right) = \operatorname{rank}(K) = 1$ and $A^2KA^2 = O_n$, we get $KA^2 = O_n$.

Therefore $(A - B)A^2 = O_n$, so $A^3 = BA^2$. Hence $A^2KA = (-ABA)A = -ABA^2 = -A^4 = O_n$.

Applying Frobenius's inequality one last time, with $X = A^2$, $Y = K$, and $Z = A$, we get $\operatorname{rank}\left(A^2K\right) + \operatorname{rank}(KA) \leqslant \operatorname{rank}(K) + \operatorname{rank}\left(A^2KA\right)$. Thus $KA = O_n$.

From $A^2 = KA - AK$, we now get $A^2 = -AK$, and therefore $A^3 = -AKA = O_n$. By the case $A^3 = O_n$, already treated in the previous solution, it follows that $A^2K = O_n$, contradicting our assumption.

Thus $A^2K = O_n$. Since $ABA = -A^2K$, we conclude that $ABA = O_n$.

Fifth solution. *The following solution was given by David-Mihai Rucăreanu, from National University of Science and Technology Politehnica Bucharest, Romania (contestant).*

We keep the notation $K = A - B$. Then

$$KA - AK = A^2. \qquad (*)$$

Thus $[A, KA - AK] = \left[A, A^2\right] = O_n$. Hence, by Jacobson's Lemma, $KA - AK$ is nilpotent. Since $KA - AK = A^2$, it follows that A is nilpotent.

For $m \geqslant 2$ and $0 \leq i \leq m-2$, multiplying $(*)$ on the left by A^i and on the right by A^{m-2-i} gives $A^m = A^i KA^{m-1-i} - A^{i+1} KA^{m-2-i}$. Summing these identities, we obtain

$$(m-1)\, A^m = KA^{m-1} - A^{m-1}K, \text{ for } m \geqslant 2. \qquad (**)$$

Assume, by contradiction, that both KA and AK are non-zero. Since $\operatorname{rank}(K) = 1$, both of them have rank 1.

We will prove that $A^m = O_n$, for every $2 \leqslant m \leqslant n$. Since A is nilpotent, we have $A^n = O_n$. Suppose that $A^m = O_n$, for some $3 \leqslant m \leqslant n$.

By $(**)$, we get $KA^{m-1} = A^{m-1}K$. Multiplying this relation on the left and on the right by A, we obtain $AKA^{m-1} = O_n$ and $A^{m-1}KA = O_n$.

By Frobenius's rank inequality, $\operatorname{rank}(AK) + \operatorname{rank}\left(KA^{m-1}\right) \leqslant \operatorname{rank}(K) + \operatorname{rank}\left(AKA^{m-1}\right) = 1$. Since $AK \neq O_n$, it follows that $KA^{m-1} = O_n$. Similarly, $\operatorname{rank}\left(A^{m-1}K\right) + \operatorname{rank}(KA) \leqslant \operatorname{rank}(K) + \operatorname{rank}\left(A^{m-1}KA\right) = 1$, and, since $KA \neq O_n$, we get $A^{m-1}K = O_n$.

Now, applying $(**)$ to $m-1$, we get $(m-2)\, A^{m-1} = KA^{m-2} - A^{m-2}K$. Multiplying this identity on the left and on the right by A, and using $KA^{m-1} = A^{m-1}K = O_n$, we obtain $AKA^{m-2} = O_n$ and $A^{m-2}KA = O_n$.

Applying Frobenius's rank inequality once again, we get that $\operatorname{rank}(AK) + \operatorname{rank}\left(KA^{m-2}\right) \leqslant \operatorname{rank}(K) + \operatorname{rank}\left(AKA^{m-2}\right) = 1$, hence $KA^{m-2} = O_n$. Similarly, $A^{m-2}K = O_n$.

Returning to $(m-2)\, A^{m-1} = KA^{m-2} - A^{m-2}K$, we get $A^{m-1} = O_n$. Thus the recursive descent is complete, and therefore $A^2 = O_n$.

From $(*)$ we now get $KA = AK$. Hence $AKA = O_n$, and Frobenius's rank inequality gives $\operatorname{rank}(AK) + \operatorname{rank}(KA) \leqslant \operatorname{rank}(K) + \operatorname{rank}(AKA) = 1$. This contradicts the assumption that both KA and AK are non-zero.

Therefore, at least one of KA and AK is O_n.

If $KA = O_n$, then $AKA = O_n$. Also, from $A^2 = KA - AK$, we get $A^2 = -AK$, hence $A^3 = -AKA = O_n$. Since $AKA = A\,(A-B)\,A = A^3 - ABA$, we get $A^3 = ABA$, and therefore $ABA = O_n$.

If $AK = O_n$, then again $AKA = O_n$. Also, from $A^2 = KA - AK$, we get $A^2 = KA$, hence $A^3 = AKA = O_n$. As before, $AKA = A\,(A-B)\,A = A^3 - ABA$, so $A^3 = ABA$, and therefore $ABA = O_n$.

In both cases, $ABA = O_n$.

Sixth solution. *This solution is based on an idea proposed during the contest by Iosua-Cristian Vasilioglu, from National University of Science and Technology Politehnica Bucharest, Romania (contestant).*

We keep the notation $K = A - B$. Then $\operatorname{rank}(K) = 1$ and $A^2 = KA - AK$.

As before, A is nilpotent, so $\operatorname{Tr}\left(A^j\right) = 0$, for every $j \geqslant 1$. Also, $BA = (A-K)\,A = A^2 - KA = (KA - AK) - KA = -AK$, and therefore $\operatorname{rank}(BA) \leqslant 1$.

We shall use the fact that, if M has rank at most 1, then $M^2 = \operatorname{Tr}(M)\, M$. In particular, $K^2 = \operatorname{Tr}(K)\, K$.

Now $A^4 = (AB - BA)^2 = (AB)^2 - AB^2A - BA^2B + (BA)^2$. Taking traces and using cyclicity, we get $0 = \operatorname{Tr}\left(A^4\right) = 2\operatorname{Tr}\left((BA)^2\right) - 2\operatorname{Tr}\left(B^2A^2\right)$.

But $B^2A^2 = B^2(AB - BA) = B^2AB - B^3A$, and hence

$$\operatorname{Tr}\left(B^2A^2\right) = \operatorname{Tr}\left(B^2AB\right) - \operatorname{Tr}\left(B^3A\right) = 0.$$

Thus $\operatorname{Tr}\left((BA)^2\right) = 0$.

Since $\operatorname{rank}(BA) \leq 1$, we have $(BA)^2 = \operatorname{Tr}(BA)\, BA$. Therefore $0 = \operatorname{Tr}\left((BA)^2\right) = (\operatorname{Tr}(BA))^2$, so $\operatorname{Tr}(BA) = 0$. Hence $(BA)^2 = O_n$. Since $BA = -AK$, it follows that $(AK)^2 = O_n$.

Next, $A^3 = A(KA - AK) = AKA - A^2K$, and also $A^3 = (KA - AK)A = KA^2 - AKA$. Adding the two resulting expressions for AKA, we obtain $2AKA = KA^2 + A^2K$.

On the other hand, $A^2K = (KA - AK)K = KAK - AK^2$ and $KA^2 = K(KA - AK) = K^2A - KAK$. Hence $A^2K + KA^2 = K^2A - AK^2$. Using $K^2 = \operatorname{Tr}(K)\, K$, we get $A^2K + KA^2 = \operatorname{Tr}(K)\, KA - \operatorname{Tr}(K)\, AK = \operatorname{Tr}(K)\, A^2$.

Therefore $2AKA = \operatorname{Tr}(K)\, A^2$. Multiplying this relation on the right by K and using $(AK)^2 = O_n$, we obtain $O_n = 2AKAK = \operatorname{Tr}(K)\, A^2K$.

If $\operatorname{Tr}(K) \neq 0$, then $A^2K = O_n$. From $A^3 = AKA - A^2K$, we get $A^3 = AKA$. Thus $2A^3 = \operatorname{Tr}(K)\, A^2$, or equivalently $A^2(2A - \operatorname{Tr}(K)\, I_n) = O_n$. Since A is nilpotent, the matrix $2A - \operatorname{Tr}(K)\, I_n$ is invertible, hence $A^2 = O_n$. Then $AKA = O_n$, and therefore $ABA = A(A - K)A = A^3 - AKA = O_n$.

It remains to consider the case $\operatorname{Tr}(K) = 0$. Then $K^2 = O_n$, and from $2AKA = \operatorname{Tr}(K)\, A^2$ we get $AKA = O_n$.

Since $\operatorname{rank}(K) = 1$, we can write $K = \mathbf{u}\mathbf{v}^*$, for some non-zero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$. From $K^2 = O_n$, we get $\mathbf{v}^*\mathbf{u} = 0$. Also, $AKA = A\mathbf{u}\mathbf{v}^*A = O_n$, so either $A\mathbf{u} = \mathbf{0}$, or $\mathbf{v}^*A = 0$. In both cases, $KAK = \mathbf{u}\mathbf{v}^*A\mathbf{u}\mathbf{v}^* = O_n$.

Hence $A^2K = (KA - AK)K = KAK - AK^2 = O_n$. Therefore $A^3 = AKA - A^2K = O_n$. Since $AKA = O_n$, we conclude again that $ABA = A(A - K)A = A^3 - AKA = O_n$.

In all cases, $ABA = O_n$.

Seventh solution. *The following solution was given by Ștefan Alexe, from National University of Science and Technology Politehnica Bucharest, Romania (contestant).*

We keep the notation $K = A - B$. Then $\operatorname{rank}(K) = 1$ and $A^2 = KA - AK$.

The case $\operatorname{Tr}(K) = 0$ was already treated in the previous solution. Thus we may assume that $\operatorname{Tr}(K) \neq 0$.

We first prove the following lemma: *Suppose that* $X, Y \in \mathcal{M}_n(\mathbb{C})$, $\operatorname{rank}(Y) = 1$, $\operatorname{Tr}(Y) = 1$, $Y^2 = Y$, *and* $X^2 = YX - XY$. *Then* $X^3 = XYX$.

Indeed, since $X^2 = YX - XY$, we have $X(YX - XY) = X^3 = (YX - XY)X$. Hence $XYX - X^2Y = YX^2 - XYX$, and therefore $2XYX = X^2Y + YX^2$.

But $X^2Y = (YX - XY)Y = YXY - XY$ and $YX^2 = Y(YX - XY) = YX - YXY$. Thus $2XYX = YX - XY = X^2$.

Multiplying this relation on the right by Y and on the left by Y, we get $2(XY)^2 = X^2Y$ and $2(YX)^2 = YX^2$.

Since $\operatorname{rank}(XY) \leqslant 1$ and $\operatorname{rank}(YX) \leqslant 1$, we have $(XY)^2 = \operatorname{Tr}(XY)\,XY$ and $(YX)^2 = \operatorname{Tr}(YX)\,YX$. Also, using $Y^2 = Y$ and cyclicity of the trace, we get $\operatorname{Tr}\left(X^2Y\right) = \operatorname{Tr}\left(YX^2\right) = 0$.

Taking traces, it follows that $\operatorname{Tr}(XY) = \operatorname{Tr}(YX) = 0$. Hence $X^2Y = YX^2 = O_n$.

Thus $(YX - XY)\,Y = O_n$ and $Y\,(YX - XY) = O_n$, so $YXY = XY$ and $YXY = YX$. Therefore $XY = YX$, and consequently $X^2 = O_n$. From $2XYX = X^2$, we also get $XYX = O_n$. Hence $X^3 = O_n = XYX$, and the lemma is proved.

Now let $X = \frac{1}{\operatorname{Tr}(K)}A$ and $Y = \frac{1}{\operatorname{Tr}(K)}K$. Since $\operatorname{rank}(K) = 1$, we have $K^2 = \operatorname{Tr}(K)\,K$. Hence $\operatorname{rank}(Y) = 1$, $\operatorname{Tr}(Y) = 1$, and $Y^2 = Y$.

Moreover, $X^2 = \frac{1}{\operatorname{Tr}(K)^2}A^2 = \frac{1}{\operatorname{Tr}(K)^2}\left(KA - AK\right) = YX - XY$. By the lemma, $X^3 = XYX$. Multiplying by $\operatorname{Tr}(K)^3$, we obtain $A^3 = AKA$.

Finally, $ABA = A\,(A - K)\,A = A^3 - AKA = O_n$.

Eighth solution. *The following solution was given by Marian Panțiruc, from Gheorghe Asachi Technical University of Iași, Romania (deputy leader).*

First of all, it is clear that $\operatorname{Tr}\left(A^2\right) = 0$ and, for every $k \geq 2$, we have $\operatorname{Tr}\left(A^{k+1}\right) = \operatorname{Tr}\left(A^kB - A^{k-1}BA\right) = \operatorname{Tr}\left(A^kB\right) - \operatorname{Tr}\left(A^{k-1}BA\right) = \operatorname{Tr}\left(A^kB\right) - \operatorname{Tr}\left(A^kB\right) = 0$, so A is nilpotent.

Because $\operatorname{rank}(A - B) = 1$, there are two non-zero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ such that $A - B = \mathbf{x}\mathbf{y}^* = \mathbf{x}\overline{\mathbf{y}}^T$. But then $BA = -A(A-B) = -A\mathbf{x}\mathbf{y}^*$ and $ABA = -\left(A^2\mathbf{x}\right)\mathbf{y}^*$. It becomes obvious that $ABA = O_n$ if and only if $A^2\mathbf{x} = \mathbf{0}$.

A well-known result in linear algebra states that if $M^k\mathbf{u} \neq \mathbf{0}$ and $M^{k+1}\mathbf{u} = \mathbf{0}$, for a square matrix M and a non-zero vector $\mathbf{u}$, then the vectors $\mathbf{u}, M\mathbf{u}, M^2\mathbf{u}, \ldots, M^k\mathbf{u}$ are linearly independent.

Because A is nilpotent, such a k exists for the matrix A and the vector $\mathbf{x}$.

Multiplying with $A\mathbf{x}$ to the right in the initial relation it follows that $A^3\mathbf{x} = A\,(BA\mathbf{x}) - BA\,(A\mathbf{x}) = -A^2\mathbf{x} \cdot (\mathbf{y}^* \cdot \mathbf{x}) + A\mathbf{x} \cdot (\mathbf{y}^* \cdot A\mathbf{x}) = -\alpha A^2\mathbf{x} + \beta A\mathbf{x}$, where $\alpha = \mathbf{y}^*\mathbf{x} = \langle \mathbf{x}, \mathbf{y} \rangle \in \mathbb{C}$, $\beta = \mathbf{y}^*A\mathbf{x} = \langle A\mathbf{x}, \mathbf{y} \rangle \in \mathbb{C}$. But then $A^3\mathbf{x} = 0$, otherwise $A\mathbf{x}$, $A^2\mathbf{x}$, and $A^3\mathbf{x}$ are linearly dependent, contradicting the above cited result. It follows that $\alpha A^2\mathbf{x} = \beta A\mathbf{x}$ and one can separate a few cases.

If $\alpha = 0$, then $\beta = 0$ or $A\mathbf{x} = 0$. But, if $A\mathbf{x} = 0$, then $A^2\mathbf{x} = 0$ and, if $\alpha = \beta = 0$, then, from $A = B + \mathbf{x}\mathbf{y}^*$, it follows $A^2\mathbf{x} = A \cdot A\mathbf{x} = BA\mathbf{x} + \mathbf{x}\mathbf{y}^*A\mathbf{x} = -A\mathbf{x}\mathbf{y}^*\mathbf{x} + \mathbf{x}\mathbf{y}^*A\mathbf{x} = -\alpha A\mathbf{x} + \beta\mathbf{x} = 0$.

If $\alpha \neq 0$, then $A^2\mathbf{x} = 0$, otherwise the vectors $A^2\mathbf{x}$ and $A\mathbf{x}$ would be linearly dependent.

Thus, in any situation, $A^2\mathbf{x} = 0$ and the proof is complete.

This problem had 6 complete solutions in the contest. Based on the total number of points scored by the contestants, this problem ranked as the second most difficult.

Editor's note. Problem 546, published in *Gazeta Matematică, Seria A*, vol. **XLI (CXX)** (2023), no. 3–4, p. 39, by Florin Stănescu, Romania is similar to Problem 12173, published in *Amer. Math. Monthly*, Vol. **127** (2020), no. 3, p. 275, by the same author.

Problem 4. Find all bijective functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$x^n f(x) + y^n f(y) \geq 2x^n f(y), \tag{1}$$
for all $x, y \in \mathbb{R}$, where n is a fixed positive integer.

Vasile Pop, Technical University of Cluj-Napoca, Romania

Author's solution. Setting $x = 0$ in (1), we get $y^n f(y) \geq 0$, for every $y \in \mathbb{R}$.

If n is even, then $f(y) \geq 0$, for every $y \neq 0$. Hence the image of f contains at most one negative value, namely possibly $f(0)$, which contradicts the surjectivity of f. Therefore, there are no solutions for even n.

Assume from now on that n is odd. Then $y^n f(y) \geq 0$ implies that $f(y) \geq 0$, for $y > 0$, and $f(y) \leq 0$, for $y < 0$.

We first prove that $f(0) = 0$. Suppose, by contradiction, that $f(0) = c \neq 0$. Setting $y = 0$ in (1), we obtain $x^n f(x) \geq 2x^n c$. If $c > 0$, then, for every $x > 0$, we get $f(x) \geq 2c$. Since $f(x) \leq 0$, for $x < 0$, and $f(0) = c$, no value from $(0, 2c) \setminus \{c\}$ belongs to the image of f, a contradiction. Similarly, if $c < 0$, then, for every $x < 0$, we get $f(x) \leq 2c$. Since $f(x) \geq 0$, for $x > 0$, and $f(0) = c$, no value from $(2c, 0) \setminus \{c\}$ belongs to the image of f, again a contradiction. Hence, $f(0) = 0$.

Interchanging x and y in (1), we also have
$$x^n f(x) + y^n f(y) \geq 2y^n f(x).$$
Adding this inequality to (1), we get
$$(x^n - y^n)(f(x) - f(y)) \geq 0,$$
for all $x, y \in \mathbb{R}$.

Since n is odd, the function $x \mapsto x^n$ is strictly increasing. Therefore f is increasing. As f is injective, it is in fact strictly increasing.

A strictly increasing surjective function on $\mathbb{R}$ is continuous. Indeed, if it had a jump at some point, then an interval of real values would be omitted from its image.

Since $f(0) = 0$ and f is strictly increasing, we have $f(x) > 0$, for $x > 0$, and $f(x) < 0$, for $x < 0$. Thus the restrictions $f_1 = f_{|(0,\infty)}$ and $f_2 = f_{|(-\infty,0)}$ are continuous bijections.

Let $0 < y < x$. From (1) and from its symmetric form, we get
$$\frac{f_1(y)}{x^n} \leq \frac{f_1(x) - f_1(y)}{x^n - y^n} \leq \frac{f_1(x)}{y^n}.$$
Similarly, for $0 < x < y$, we obtain
$$\frac{f_1(x)}{y^n} \leq \frac{f_1(x) - f_1(y)}{x^n - y^n} \leq \frac{f_1(y)}{x^n}.$$

Letting $y \to x$, the squeeze theorem gives
$$\lim_{y \to x} \frac{f_1(x) - f_1(y)}{x^n - y^n} = \frac{f_1(x)}{x^n}.$$
Since $\lim\limits_{y \to x} \frac{x^n - y^n}{x - y} = nx^{n-1}$, we obtain
$$f_1'(x) = \lim_{y \to x} \frac{f_1(x) - f_1(y)}{x - y} = \frac{n}{x} f_1(x).$$

Thus
$$\frac{f_1'(x)}{f_1(x)} = \frac{n}{x},$$
and so
$$\log f_1(x) = n \log x + c_1.$$
Hence $f_1(x) = ax^n$, for every $x > 0$, where $a > 0$.

The same argument on $(-\infty, 0)$ gives $f_2(x) = bx^n$, for every $x < 0$, where $b > 0$.

Therefore, every solution must be of the form
$$f(x) = \begin{cases} ax^n, & x > 0, \\ 0, & x = 0, \\ bx^n, & x < 0, \end{cases}$$
where $a, b > 0$.

Conversely, it is immediate to check that every function of this form is bijective and satisfies (1).

Second solution. We use the facts already proved in the previous solution: if n is even, then there are no solutions; if n is odd, then $f(0) = 0$ and f preserves signs. Thus it remains only to determine f on $(0, \infty)$ and on $(-\infty, 0)$.

Assume that n is odd. For $x, y > 0$, from (1) we get
$$\frac{f(x)}{f(y)} \geq 2 - \left(\frac{y}{x}\right)^n.$$

Fix $r \in \left(0, \sqrt[n]{2}\right)$. Then, for every $k \in \mathbb{N}^*$ and $x > 0$, we have
$$\frac{f(x)}{f(rx)} = \prod_{j=0}^{k-1} \frac{f\left(r^{\frac{j}{k}} x\right)}{f\left(r^{\frac{j+1}{k}} x\right)} \geq \left(2 - r^{\frac{n}{k}}\right)^k.$$
Letting $k \to \infty$, we obtain
$$\frac{f(x)}{f(rx)} \geq r^{-n}.$$
Indeed,
$$2 - r^{\frac{n}{k}} = 1 - \frac{n \log r}{k} + O\left(k^{-2}\right),$$
and therefore $\left(2 - r^{\frac{n}{k}}\right)^k \to r^{-n}$.

Applying the same estimate to $\frac{1}{r}$ instead of r, we also get
$$\frac{f(rx)}{f(x)} \geq r^n,$$
whenever $r > \frac{1}{\sqrt[n]{2}}$. Hence, for every $r \in \left(\frac{1}{\sqrt[n]{2}}, \sqrt[n]{2}\right)$, we have
$$f(rx) = r^n f(x).$$

Now let $t > 0$. Choose $m \in \mathbb{N}^*$ such that $t^{\frac{1}{m}} \in \left(\frac{1}{\sqrt[n]{2}}, \sqrt[n]{2}\right)$. Applying the previous relation repeatedly, we get

$$f(t) = f(1)t^n.$$

Thus $f(x) = ax^n$, for every $x > 0$, where $a = f(1) > 0$.

The same argument on $(-\infty, 0)$ gives $f(x) = bx^n$, for every $x < 0$, with $b > 0$.

Consequently, for odd n, the solutions are

$$f(x) = \begin{cases} ax^n, & x \geq 0, \\ bx^n, & x < 0, \end{cases}$$

where $a, b > 0$. Conversely, it is immediate to check that every function of this form is bijective and satisfies (1).

This problem had only 4 complete solutions in the contest. Judging by the total number of points, this problem proved to be the most challenging in the contest.

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MATHEMATICAL NOTES

The sums of the subseries of the Riemann series

DAN ȘTEFAN MARINESCU[1)], EUGEN PĂLTĂNEA[2)]

Abstract. In this paper we determine the set of the sums of the subseries of the Riemann series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, with $p > 0$. In this way we extend a recent result concerning the density of the subseries of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Keywords: Subseries, Riemann zeta function.
MSC: 40A05, 11M06.

The following result was proven in [2].

Proposition 1. *Let* $1 \leq x < \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. *Then there exists a set* $A \subset \{1, 2, \dots\}$ *such that* $\displaystyle\sum_{n \in A} \frac{1}{n^2} = x$.

A basic remark is that this result is closely related to the following classical theorem.

Theorem 2 ([1]). *Let* $\{a_n\}_{n \geq 1}$ *be a sequence of positive numbers such that* $a_n > a_{n+1}$, *for all* $n \geq 1$ *and* $\lim_{n \to \infty} a_n = 0$. *Denote* $S = \displaystyle\sum_{n=1}^{\infty} a_n \leq \infty$. *The following conditions are equivalent:*

(i) *For every* $x \in (0, S]$ *there is a subsequence* $\{a_{i_n}\}_{n \geq 1}$ *of the sequence* $\{a_n\}_{n \geq 1}$ *such that* $\displaystyle\sum_{n=1}^{\infty} a_{i_n} = x$.

(ii) *For every* $n \geq 1$, *the following inequality holds* $a_n \leq \displaystyle\sum_{i=n+1}^{\infty} a_i$.

Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of positive integers. We deduce an immediate consequence of the above Kakeya's theorem.

Corollary 3. *Let* $\{a_n\}_{n \geq 1}$ *be a sequence of positive numbers, with* $\displaystyle\sum_{n=1}^{\infty} a_n = S \leq \infty$, *such that* $a_n > a_{n+1}$, *for any* $n \in \mathbb{N}$, $\lim_{n \to \infty} a_n = 0$, *and* $a_n \leq \displaystyle\sum_{i=n+1}^{\infty} a_i$, *for any* $n \in \mathbb{N}$. *Then*

$$\left\{ \sum_{n \in A} a_n \,\middle|\, A \subset \mathbb{N} \right\} = [0, S],$$

with the convention that $\displaystyle\sum_{n \in \emptyset} a_n = 0$.

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An interesting problem suggested by the paper [2] is to give an explicit description of the set

$$\mathcal{A}_p := \left\{ \sum_{n\in A} \frac{1}{n^p} \,\middle|\, A \subset \mathbb{N} \right\},$$

for a given number $p > 0$. We keep the convention $\displaystyle\sum_{n\in\emptyset} \frac{1}{n^p} = 0$. To solve this problem, we need the following elementary result.

Lemma 4. *For every $p > 1$ there exists a positive integer k such that $\displaystyle\frac{1}{n^p} \le \sum_{i=n+1}^{\infty} \frac{1}{i^p}$, for all $n \ge k$.*

Proof. For given $p > 1$, we define the function $f_p : [0, \infty) \to \mathbb{R}$,

$$f_p(x) = \frac{x^p}{(p-1)(x+1)^{p-1}},$$

for $x \ge 0$. We have $f_p'(x) = \dfrac{x^{p-1}(x+1)^{p-2}(x+p)}{(p-1)(x+1)^{2(p-1)}} > 0$, for all $x > 0$. Therefore f_p is increasing on $[0, \infty)$. Since $f_p(0) = 0$, $\lim\limits_{x\to\infty} f_p(x) = \infty$, and f_p is continuous, there is a unique $x_p \in (0, \infty)$ such that $f_p(x_p) = 1$. Let k be a positive integer from the interval $[x_p, \infty)$. For any $n \in \mathbb{N}$, $n \ge k$, we have $f_p(n) \ge f_p(k) \ge f_p(x_p) = 1$, so that $\dfrac{1}{(p-1)(n+1)^{p-1}} \ge \dfrac{1}{n^p}$. Then, due to the monotonicity of f_p, we have

$$\sum_{i=n+1}^{\infty} \frac{1}{i^p} \ge \sum_{i=n+1}^{\infty} \int_i^{i+1} \frac{1}{x^p}\,\mathrm{d}x = \int_{n+1}^{\infty} \frac{1}{x^p}\,\mathrm{d}x = \frac{1}{(p-1)(n+1)^{p-1}} \ge \frac{1}{n^p},$$

for every $n \in \mathbb{N}$, with $n \ge k$. □

The following proposition describes explicitly the set $\mathcal{A}_p$, for arbitrary $p > 0$.

Proposition 5. *Let p be a positive number.*

(i) *If $p \in (0, 1]$, then $\mathcal{A}_p = [0, \infty]$.*

(ii) *If $p \in (1, \infty)$, then $\displaystyle\mathcal{A}_p = \bigcup_{M\subset\mathbb{N}\setminus N_p} \left[\sum_{n\in M} \frac{1}{n^p}, \sum_{n\in M\cup N_p} \frac{1}{n^p} \right]$, where k_p is the smallest positive integer k such that $\displaystyle\frac{1}{n^p} \le \sum_{i=n+1}^{\infty} \frac{1}{i^p}$, for any $n \ge k$, and*

$$N_p = \{n \in \mathbb{N} |\ n \ge k_p\} = \{k_p, k_p + 1, k_p + 2, \ldots\}.$$

Proof. (i) Assume $p \in (0, 1]$. Then the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent. Hence we have $\displaystyle\frac{1}{n^p} < \sum_{i=n+1}^{\infty} \frac{1}{i^p} = \infty$, for any $n \in \mathbb{N}$. Thus, the conclusion follows from Corollary 3.

(ii) If $p > 1$, then the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent with the sum $\zeta(p)$, where ζ is the Riemann's function. Note that the existence of k_p is ensured by Lemma 4.

Assume $k_p = 1$, that is $N_p = \mathbb{N}$. From Corollary 3, we obtain

$$\mathcal{A}_p = [0, \zeta(p)] = \bigcup_{M \subset \emptyset} \left[\sum_{n \in M} \frac{1}{n^p}, \sum_{n \in M \cup \mathbb{N}} \frac{1}{n^p} \right].$$

Assume now $k_p \geq 2$. Let us prove $\mathcal{A}_p \subset \bigcup_{M \subset \mathbb{N} \setminus N_p} \left[\sum_{n \in M} \frac{1}{n^p}, \sum_{n \in M \cup N_p} \frac{1}{n^p} \right]$. If $x \in \mathcal{A}_p$, then there exists $A \subset \mathbb{N}$ such that $x = \sum_{n \in A} \frac{1}{n^p}$. We choose $M = A \setminus N_p$. Then we get $x - \sum_{n \in M} \frac{1}{n^p} = \sum_{n \in A \cap N_p} \frac{1}{n^p} \in \left[0, \sum_{n \in N_p} \frac{1}{n^p} \right]$, that is $x \in \left[\sum_{n \in M} \frac{1}{n^p}, \sum_{n \in M \cup N_p} \frac{1}{n^p} \right]$. To prove $\bigcup_{M \subset \mathbb{N} \setminus N_p} \left[\sum_{n \in M} \frac{1}{n^p}, \sum_{n \in M \cup N_p} \frac{1}{n^p} \right] \subset \mathcal{A}_p$, let us assume $x \in \bigcup_{M \subset \mathbb{N} \setminus N_p} \left[\sum_{n \in M} \frac{1}{n^p}, \sum_{n \in M \cup N_p} \frac{1}{n^p} \right]$. There exists $M \subset \mathbb{N} \setminus N_p$ such that $x \in \left[\sum_{n \in M} \frac{1}{n^p}, \sum_{n \in M \cup N_p} \frac{1}{n^p} \right]$. Therefore, we have $x - \sum_{n \in M} \frac{1}{n^p} \in \left[0, \sum_{n \in N_p} \frac{1}{n^p} \right]$. From Corollary 3, there is a set $B \subset N_p$ such that $x - \sum_{n \in M} \frac{1}{n^p} = \sum_{n \in B} \frac{1}{n^p}$. Thus, $x = \sum_{n \in A} \frac{1}{n^p}$, where $A = M \cup B$. Therefore $x \in \mathcal{A}_p$, which concludes the proof. $\square$

Comments.

(1) If $k_p > 1$, then $\mathcal{A}_p \neq [0, \zeta(p)]$. Indeed, since $\frac{1}{(k_p - 1)^p} > \sum_{n=k_p}^{\infty} \frac{1}{n^p}$, we have

$$\left(\sum_{n=k_p}^{\infty} \frac{1}{n^p}, \frac{1}{(k_p - 1)^p} \right) \subset [0, \zeta(p)] \setminus \bigcup_{M \subset \mathbb{N} \setminus N_p} \left[\sum_{n \in M} \frac{1}{n^p}, \sum_{n \in M \cup N_p} \frac{1}{n^p} \right] = [0, \zeta(p)] \setminus \mathcal{A}_p.$$

(2) Assume $p = 2$. We have $\zeta(2) = \frac{\pi^2}{6}$. Since $\pi^2 < 12$, we get

$$1 > \frac{\pi^2}{6} - 1 = \sum_{n=2}^{\infty} \frac{1}{n^2}.$$

On the other hand, the following double inequality holds for all $n \geq 2$:

$$\sum_{i=n+1}^{\infty} \frac{1}{i^2} > \sum_{i=n+1}^{\infty} \frac{1}{i(i+1)} = \sum_{i=n+1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1} \right) = \frac{1}{n+1} > \frac{1}{n^2}.$$

It results $k_2 = 2$, that is $N_2 = \{2, 3, \ldots\}$. From Proposition 1 we conclude $\mathcal{A}_2 = \left[0, \frac{\pi^2}{6} - 1\right] \cup \left[1, \frac{\pi^2}{6}\right]$, which completes the result obtained in [2].

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PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take up to two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of November 2026**.

PROPOSED PROBLEMS

585. Find all values of the positive power k such that

$$a+b+c+d+e \geq a^k+b^k+c^k+d^k+e^k$$

for any nonnegative real numbers a,b,c,d,e with $a \geq b \geq c \geq d \geq e$ and $ab+bc+cd+de+ea=5$.

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania

586. Let $n \geq 1$. Detemine all quadruples (A,B,C,D), where $A,B,C,D \in M_n(\mathbb{R})$ are symmetric matrices with all eigenvalues in the interval $[0,2]$ such that

$$A^2+B^2+C^2+D^2=ABCD.$$

Proposed by Dan Dumitrescu, International Computer High-School Bucharest (ICHB), Romania.

587. If $n \geq 3$ find the smallest constant K_n such that

$$(n-2)\sum_{i=1}^{n} a_i + K_n \prod_{i=1}^{n} a_i \geq 2 \sum_{1\leq i<j\leq n} a_i a_j$$

holds for all $a_1,\ldots,a_n \geq 0$ satisfying $\sum\limits_{i=1}^{n} a_i^2 = n-1$.

Proposed by Leonard Giugiuc, Greci Middle School, Mehedinți, Romania, and Nguyen Van Huyen, Ho Chi Minh city, Viet Nam.

588. Let $\mathcal{A}$ be the set of all functions $h:[0,1]\to[0,1]$ of class C^1 such that $h(0)=0$ and $0 \leq h'(x) \leq 1$ for all $x \in (0,1)$.

Prove that for every $f,g \in \mathcal{A}$ we have

$$\int_0^1 (f\circ g)^2(x)\mathrm{d}x + \int_0^1 (g\circ f)^2(x)\mathrm{d}x \leq \frac{2}{3}\int_0^1 (f(x)+g(x))\mathrm{d}x$$

and determine all cases of equality.

Proposed by Dan Dumitrescu, International Computer High-School Bucharest (ICHB), Romania.

589. Let $A \in \mathcal{M}_n(\mathbb{C})$ and $B \in \mathrm{GL}_n(\mathbb{C})$ such that

$$A^3B + 12\,ABA^2 = 6\,A^2BA + 8\,BA^3.$$

Prove that $A^n = O_n$.

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania

590. Let $(x_n)_{n\geq 1}$ be a sequence with $x_1 \in (0,1)$ and

$$x_{n+1} = x_n - \frac{x_n^2 \ln n}{\sqrt{n}} \quad \forall n \geq 1$$

Determine $\lim\limits_{n\to\infty} \sqrt{n}\,\ln n\, x_n$.

Proposed by Mircea Rus, Technical University of Cluj-Napoca, Romania.

591. Let $n \geq 3$. Determine all values of K such that

$$\sum_{i=1}^{n} a_i^x + K\sqrt{\sum_{i=1}^{n} a_i^2 - n} \geq n$$

holds for all $x \in (0,1)$ and all $a_1, \ldots, a_n > 0$ with $\sum\limits_{i=1}^{n} a_i = n$.

Proposed by Leonard Giugiuc, Greci Middle School, Mehedinți, Romania.

592. Let a and b be positive real numbers. Calculate

$$\lim_{n\to\infty} \int_0^{\pi/2} \sqrt[n]{a|\sin(nx)|^n + b|\cos(nx)|^n}\,\mathrm{d}x.$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania.

SOLUTIONS

570. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative such that $f(0) = f(1) = 0$ and $\int\limits_0^1 x f(x)\,\mathrm{d}x = 0$. Prove that

$$\int_0^1 (f'(x))^2\,\mathrm{d}x \geq 192 \left(\int_0^1 f(x)\,\mathrm{d}x \right)^2.$$

(This version of problem 563 from the 3-4/2024 issue of GMA states that if in the hypothesis we strengthen the condition $f(0) = f(1)$ to $f(0) = f(1) = 0$, then the constant 180 can be improved to 192.)

Proposed by Ulrich Abel and Vitaliy Kushnirevych, Technische Hochschule Mittelhessen, Germany.

Solution by the authors. By the assumptions $\int_0^1 x f(x)\,\mathrm{d}x = f(0) = f(1) = 0$, we have, for arbitrary reals a, b,

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 (2ax+1)\, f(x)\,\mathrm{d}x = -\int_0^1 \left(ax^2 + x + b\right) f'(x)\,\mathrm{d}x,$$

since $\left(ax^2 + x + b\right) f(x)\big|_0^1 = 0$. An application of the Cauchy–Schwarz inequality yields

$$\left(\int_0^1 f(x)\,\mathrm{d}x \right)^2 = \left(\int_0^1 \left(ax^2 + x + b\right) f'(x)\,\mathrm{d}x \right)^2 \leq M(a,b) \int_0^1 (f'(x))^2\,\mathrm{d}x,$$

where

$$\begin{aligned} M(a,b) &= \int_0^1 \left(ax^2 + x + b\right)^2\,\mathrm{d}x = \frac{1}{3} + \frac{a}{2} + \frac{a^2}{5} + b + \frac{2ab}{3} + b^2 \\ &= \frac{1}{192} + \frac{1}{5}\left(a + \frac{15}{16}\right)^2 + \frac{2}{3}\left(a + \frac{15}{16}\right)\left(b + \frac{3}{16}\right) + \left(b + \frac{3}{16}\right)^2. \end{aligned}$$

Since $M\left(\frac{-15}{16}, \frac{-3}{16}\right) = \frac{1}{192}$, we conclude that

$$\int_0^1 (f'(x))^2\,\mathrm{d}x \geq \frac{1}{M\left(\frac{-15}{16}, \frac{-3}{16}\right)} \left(\int_0^1 f(x)\,\mathrm{d}x \right)^2 = 192 \left(\int_0^1 f(x)\,\mathrm{d}x \right)^2.$$

Editor's note. We received solutions from Marian Cucoaneș, Valeriu D. Cotea Technical College, Focșani and Alexandru Pîrvulescu, Babeș–Bolyai University, Cluj–Napoca. They follow the same idea from the authors' solution — the integral Cauchy–Schwarz inequality for the functions f' and P, where P is a certain second degree polynomial. Namely, $P(X) = 180X^2 - 192X + 36$ and $P(X) = -15/2\,X^2 + 8X - 3/2$, respectively. In both cases, P is a scalar multiple of the polynomial $aX^2 + X + b = -15/16\,X^2 + X - 3/16$, from the authors' solution.

571. Let $a_1, a_2, a_3, a_4 \geq 0$ be real numbers. Prove that

$$a_1 + a_2 + 3a_3 + 3a_4 + \sqrt{(a_1 - a_3)^2 + (a_2 - a_4)^2} \geq \sum_{k=1}^{4} \sqrt{a_k^2 + a_{k+1}^2},$$

where $a_5 = a_1$.

Proposed by Leonard Giugiuc, Middle School Greci, Mehedinți, Romania.

Solution by the author. We consider the complex numbers $u = a_1$, $v = a_2 i$, $w = -a_3$ and $z = -a_4 i$. We use the Adamović–Hlawka inequality

$$2\sum_{cyc}|u| + \left|\sum_{cyc} u\right| \geq \sum_{sym}|u+v| \tag{1}$$

In our case, (1) becomes

$$\begin{aligned}2\sum_{k=1}^{4} a_k + \sqrt{(a_1-a_3)^2+(a_2-a_4)^2} &\geq |a_1-a_3| + |a_2-a_4| + \sum_{k=1}^{4}\sqrt{a_k^2+a_{k+1}^2}\\ &\geq a_1 - a_3 + a_2 - a_4 + \sum_{k=1}^{4}\sqrt{a_k^2+a_{k+1}^2},\end{aligned}$$

which proves the claimed inequality.

Editor's note. We received solutions from Mihalcea Andrei and Marian Cucoaneș, Valeriu D. Cotea Technical College, Focșani. Both these solutions involve writing the inequality we want to prove as a sum of several inequalities as follows.

In Mihalcea's proof the inequalities are

$$a_2 + a_3 \geq \sqrt{a_2^2+a_3^2},\ 2(a_3+a_4) \geq 2\sqrt{a_3^2+a_4^2},\ a_4+a_1 \geq \sqrt{a_4^2+a_1^2},$$
$$\sqrt{(a_1-a_3)^2+(a_2-a_4)^2} \geq \sqrt{a_1^2+a_2^2} - \sqrt{a_3^2+a_4^2},$$

while in the proof by Cucoaneș they are

$$a_2 + a_3 \geq \sqrt{a_2^2+a_3^2},\ a_3+a_4 \geq \sqrt{a_3^2+a_4^2},\ a_4+a_1 \geq \sqrt{a_4^2+a_1^2},$$
$$a_3 + a_4 + \sqrt{(a_1-a_3)^2+(a_2-a_4)^2} \geq \sqrt{a_1^2+a_2^2}.$$

In both solutions, all but last inequality follow from the elementary observation that for every $x, y \geq 0$ we have $x^2+2xy+y^2 \geq x^2+y^2$, so $x+y \geq \sqrt{x^2+y^2}$, with equality iff $xy = 0$.

For the proof of his last inequality, which also writes as

$$\sqrt{a_3^2+a_4^2} + \sqrt{(a_1-a_3)^2+(a_2-a_4)^2} \geq \sqrt{a_1^2+a_2^2},$$

Mihalcea used a geometric method: If $O, A, B \in \mathbb{R}^2$, $O = (0,0)$, $A = (a_3, a_4)$, $B = (a_1, a_2)$, then the inequality writes as $|OA| + |AB| \geq |OB|$.

As for the proof of $a_3 + a_4 + \sqrt{(a_1-a_3)^2+(a_2-a_4)^2} \geq \sqrt{a_1^2+a_2^2}$, Cucoaneș used a lengthy, algebraic method. However, this too can be done geometrically. If $O = (0,0)$, $A = (a_3, 0)$, $B = (a_3, a_4)$, and $C = (a_1, a_2)$, then the left side of the inequality is the length of the broken line $OABC$, while the right side in the length of the segment OC.

Mihalcea also correctly determined all cases of equality. To have equality in the main equality, one must have equalities in all inequalities we used to prove it. The conditions $a_2+a_3=\sqrt{a_2^2+a_3^2}$, $2(a_3+a_4)=2\sqrt{a_3^2+a_4^2}$ and $a_4+a_1=\sqrt{a_4^2+a_1^2}$ are equivalent to $a_2a_3=a_3a_4=a_4a_1=0$. This happens iff $a_3=a_4=0$ or $a_1=a_3=0$ (when $a_4>0$) or $a_2=a_4=0$ (when $a_3>0$). If $a_3=a_4=0$, then the last condition, $\sqrt{(a_1-a_3)^2+(a_2-a_4)^2}=\sqrt{a_1^2+a_2^2}-\sqrt{a_3^2+a_4^2}$, holds unconditionally. If $a_1=a_3=0$, it writes as $|a_2-a_4|=a_2-a_4$, which is equivalent to $a_2-a_4\geq 0$. And if $a_2=a_4=0$, it writes as $|a_1-a_3|=a_1-a_3$, which is equivalent to $a_1-a_3\geq 0$.

In conclusion, we have three cases of equality: (1) $a_3=a_4=0$, $a_1,a_2\geq 0$, (2) $a_1=a_3=0$, $a_2\geq a_4>0$ and (3) $a_2=a_4=0$, $a_1\geq a_3>0$.

572. Prove that there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ the equation

$$n^2\iint\limits_{[0,1]^2} x^n y^n e^{xyz}\,\mathrm{d}x\mathrm{d}y=e$$

has an unique real solution denoted by z_n and, moreover, $\lim\limits_{n\to\infty} n\,(z_n-1)=4$.

`Proposed by Dumitru Popa, Department of Mathematics, Ovidius University of Constanța, Romania.`

Solution by the author. We use the following result:
If $f:[0,1]^2\to\mathbb{R}$ is continuous and $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ are continuous, then

$$\lim_{n\to\infty} n\left(n^2\iint\limits_{[0,1]^2} x^n y^n f(x,y)\mathrm{d}x\mathrm{d}y-f(1,1)\right)$$
$$=-2f(1,1)-\frac{\partial f}{\partial x}(1,1)-\frac{\partial f}{\partial y}(1,1). \tag{1}$$

In particular,

$$\lim_{n\to\infty} n^2\iint\limits_{[0,1]^2} x^n y^n f\left(x,y\right)\mathrm{d}x\mathrm{d}y=f\left(1,1\right), \tag{2}$$

see Problem 431, GMA Nr. 1–2, 2015, p. 45, with solution in GMA, 3–4, 2016, pp. 47–51.

For every $n\in\mathbb{N}$ let $\varphi_n:\mathbb{R}\to\mathbb{R}$, $\varphi_n(z)=n^2\iint_{[0,1]^2} x^n y^n e^{xyz}\mathrm{d}x\mathrm{d}y$ and note that φ_n is continuous and strictly increasing. Moreover,

$$\varphi_n(1)=n^2\iint\limits_{[0,1]^2} x^n y^n e^{xy}\mathrm{d}x\mathrm{d}y\leq e\cdot n^2\iint\limits_{[0,1]^2} x^n y^n\mathrm{d}x\mathrm{d}y=\frac{en^2}{(n+1)^2}<e$$

and, by (2)

$$\lim_{n\to\infty}\varphi_n(2)=\lim_{n\to\infty} n^2\iint\limits_{[0,1]^2} x^n y^n e^{2xy}\mathrm{d}x\mathrm{d}y=e^2>e.$$

Hence, there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ we have $\varphi_n(2)>e$. It follows that for all $n\geq n_0$ the equation $\varphi_n(z)=e$ has a unique real solution, denote it z_n, such that $1<z_n<2$ and $\varphi_n(z_n)=e$.

Now we use another well-known result: If $g_n : [a,b] \to \mathbb{R}$ is a sequence of increasing functions and $g : [a,b] \to \mathbb{R}$ is a continuous function such that $\lim\limits_{n\to\infty} g_n(t) = g(t)$ for all $t \in [a,b]$, then

$$\lim_{n\to\infty} g_n(t) = g(t) \text{ uniformly with respect to } t \in [a,b], \tag{3}$$

see for example, [1, Problem 43]. From (2) and (3) we deduce that $\lim\limits_{n\to\infty} \varphi_n(z) = e^z$ uniformly with respect to $z \in [1,2]$. Hence, $\lim\limits_{n\to\infty} [\varphi_n(z_n) - e^{z_n}] = 0$, so $\lim\limits_{n\to\infty} (e - e^{z_n}) = 0$, that is, $\lim\limits_{n\to\infty} z_n = 1$.

Let $n \geq n_0$. We have $\varphi_n(z_n) - \varphi_n(1) = e - \varphi_n(1)$. By the Lagrange theorem there exists $1 < c_n < z_n$ such that $\frac{\varphi_n(z_n)-\varphi_n(1)}{z_n-1} = \varphi_n'(c_n)$ and so

$$\frac{z_n - 1}{e - \varphi_n(1)} = \frac{1}{\varphi_n'(c_n)}.$$

By differentiating under the integral we get

$$\varphi_n'(z) = n^2 \iint\limits_{[0,1]^2} x^n y^n xy e^{xyz} \mathrm{d}x\mathrm{d}y$$

and by relations (2) and (3) we deduce that $\lim\limits_{n\to\infty} \varphi_n'(z) = e^z$ uniformly with respect to $z \in [1,2]$. Hence $\lim\limits_{n\to\infty} [\varphi_n'(z_n) - e^{z_n}] = 0$ or

$$\lim_{n\to\infty} \varphi_n'(z_n) = \lim_{n\to\infty} e^{z_n} = e.$$

From relation (1) we get

$$\lim_{n\to\infty} n(e - \varphi_n(1)) = \lim_{n\to\infty} n \left(e - n^2 \iint\limits_{[0,1]^2} x^n y^n e^{xy} \mathrm{d}x\mathrm{d}y \right) = 4e.$$

We deduce that $\lim\limits_{n\to\infty} n(z_n - 1) = \lim\limits_{n\to\infty} \frac{z_n-1}{e-\varphi_n(1)} \lim\limits_{n\to\infty} n(e - \varphi_n(1)) = 4.$

REFERENCES

1. D. Popa, *Exerciții de analiză matematică*, Colecția Biblioteca S.S.M.R., Editura Mira, București 2007.

573. Let $\mathcal{B} = \bigcup\limits_{n\geq 1} \mathcal{B}_n$, where

$$\mathcal{B}_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \leq x_{i+2} \text{ for all } 1 \leq i \leq n-2\}.$$

On $\mathcal{B}$ we define the relations $\leq$ and $\prec$ as follows:

Let $x, y \in \mathcal{B}$, $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$.

We say that $x \leq y$ if $m \geq n$ and for every $1 \leq i \leq n$ we have either $x_i \leq y_i$ or $1 < i < m$ and $x_i + x_{i+1} \leq y_{i-1} + y_i$. It is known that $(\mathcal{B}, \leq)$ is partially ordered.[1]

We say that $x \prec y$ if $x \leq y$, $m - n \leq 2$, and for every $1 < i < m$ we have either $x_{i+1} \geq y_{i-1}$ or $i \leq n$ and $x_i + x_{i+1} = y_{i-1} + y_i$.

[1] See problem 331 from the 1-2/2011 issue of GMA, with solution in the 1-2/2012 issue.

If $x \prec y$ we denote $\delta(x,y) = \sum_{i=1}^{n} (-1)^i (x_i - y_i)_+$.

(Here $x_+ := (x+|x|)/2$ is the positive part of x, with $x_+ = x$ if $x \geq 0$ and $x_+ = 0$ if $x \leq 0$.)

Prove that if $x, y, z \in \mathcal{B}$ such that $x \leq y \leq z$ and $x \prec z$, then $x \prec y \prec z$ and $\delta(x,y) + \delta(y,z) = \delta(x,z)$.

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Solution by the author. We consider first the case when x, y, z have the same length n, i.e., $x, y, z \in \mathcal{B}_n$ for some $n \geq 1$. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, and $z = (z_1, \ldots, z_n)$. We have $x_i \leq x_{i+2}$, $y_i \leq y_{i+2}$, and $z_i \leq z_{i+2}$ for $1 \leq i \leq i+2$. Consequently, $x_i + x_{i+1} \leq x_{i+1} + x_{i+2}$, so the sequence $(x_i + x_{i+1})$ is increasing. The same conclusion holds for for $(y_i + y_{i+1})$ and $(z_i + z_{i+1})$.

We now give some consequences following from $x \leq y$ and from $x \prec y$.

Lemma 1. *If $x \leq y$, then:*

(i) $x_i + x_{i+1} \leq y_i + y_{i+1}$ *for all* $1 \leq i \leq n-1$.

(ii) *If* $x_{i+1} \leq y_{i-1}$, *then* $x_i + x_{i+1} \leq y_{i-1} + y_i$.

Proof. (i) If $x_i \leq y_i$ and $x_{i+1} \leq y_{i+1}$, then $x_i + x_{i+1} \leq y_i + y_{i+1}$ trivially. If $x_i > y_i$, then $x \leq y$ implies $x_i + x_{i+1} \leq y_{i-1} + y_i \leq y_i + y_{i+1}$ and if $x_{i+1} > y_{i+1}$, it implies that $x_i + x_{i+1} \leq x_{i+1} + x_{i+2} \leq y_i + y_{i+1}$.

(ii) If $x_i \leq y_i$, since also $x_{i+1} \leq y_{i-1}$, we get $x_i + x_{i+1} \leq y_{i-1} + y_i$. If $x_i > y_i$, then $x \leq y$ implies $x_i + x_{i+1} \leq y_{i-1} + y_i$. □

Lemma 2. *If $x \prec y$, then:*

(i) *We have* $x_i > y_i$ *iff* $1 < i < n$ *and* $x_{i+1} < y_{i-1}$. *Consequently, if* $x_i > y_i$, *then* $x_i + x_{i+1} = y_{i-1} + y_i$.

(ii) *We have* $(x_1 - y_1)_+ = (x_n - y_n)_+ = 0$ *and for* $1 < i < n$ *we have* $(x_i - y_i)_+ = (y_{i-1} - x_{i+1})_+$.

(iii) $x_{i+1} + x_{i+2} \geq y_{i-1} + y_i$ *for all* $2 \leq i \leq n-2$.

Proof. (i) Since $x \leq y$, if $x_i > y_i$, then we also have $x_i + x_{i+1} \leq y_{i-1} + y_i$ and so $x_{i+1} < y_{i-1}$. Since $x \prec y$, if $x_{i+1} < y_{i-1}$, then we also have $x_i + x_{i+1} = y_{i-1} + y_i$ and so $x_i > y_i$.

(ii) Since $x \leq y$, we cannot have $x_1 > y_1$ or $x_n > y_n$, so $x_1 - y_1, x_n - y_n \leq 0$, which implies $(x_1 - y_1)_+ = (x_n - y_n)_+ = 0$. If $1 < i < n$, by (i), we have two cases: If $x_i > y_i$ and $x_{i+1} < y_{i-1}$, then $(x_i - y_i)_+ = x_i - y_i$ and $(y_{i-1} - x_{i+1})_+ = y_{i-1} - x_{i+1}$. But we also have $x_i + x_{i+1} = y_{i-1} + y_i$ and so $x_i - y_i = y_{i-1} - x_{i+1}$. If $x_i \leq y_i$ and $x_{i+1} \geq y_{i-1}$, then $(x_i - y_i)_+ = (y_{i-1} - x_{i+1})_+ = 0$.

(iii) Suppose that $x_{i+1} + x_{i+2} < y_{i-1} + y_i$. It follows that $x_{i+1} < y_{i-1}$ or $x_{i+2} < y_i$. Then, from $x, y \in \mathcal{B}_n$ and $x \prec y$, in the first case we get $x_{i+1} + x_{i+2} \geq x_i + x_{i+1} = y_{i-1} + y_i$ and in the second case $x_{i+1} + x_{i+2} = y_i + y_{i+1} \geq y_{i-1} + y_i$. Contradiction. Thus $x_{i+1} + x_{i+2} \geq y_{i-1} + y_i$. □

We now start our proof.

Proof of $x \prec y$. Let i such that $x_{i+1} < y_{i-1}$. We must prove that $x_i + x_{i+1} = y_{i-1} + y_i$. Since $x \leq y$, by Lemma 1(ii), $x_{i+1} < y_{i-1}$ implies $x_i + x_{i+1} \leq y_{i-1} + y_i$. We have two cases:

(a) $y_{i-1} > z_{i-1}$. Since $y \leq z$, we have $x_i + x_{i+1} \leq y_{i-1} + y_i \leq z_{i-2} + z_{i-1}$. But $x \prec z$ so, by Lemma 2(iii), $x_i + x_{i+1} \geq z_{i-2} + z_{i-1}$. Hence we must have equalities. In particular, $x_i + x_{i+1} = y_{i-1} + y_i$.

(b) $y_{i-1} \leq z_{i-1}$. We have $x_{i+1} < y_{i-1} \leq z_{i-1}$, which, since $x \prec z$, implies $x_i + x_{i+1} = z_{i-1} + z_i$. But we have $x_i + x_{i+1} \leq y_{i-1} + y_i$ and, since $y \leq z$, by Lemma 1(i), $y_{i-1} + y_i \leq z_{i-1} + z_i$. It follows that $x_i + x_{i+1} = y_{i-1} + y_i$.

Proof of $y \prec z$. Let i such that $y_{i+1} < z_{i-1}$. We must prove that $y_i + y_{i+1} = z_{i-1} + z_i$. Since $y \leq z$, by Lemma 1(ii), $y_{i+1} < z_{i-1}$ implies $y_i + y_{i+1} \leq z_{i-1} + z_i$. We have two cases:

(a) $x_{i+1} > y_{i+1}$. Since $x \leq y$, we have $x_{i+1} + x_{i+2} \leq y_i + y_{i+1} \leq z_{i-1} + z_i$. But $x \prec z$ so, by Lemma 2(iii), $x_{i+1} + x_{i+2} \geq z_{i-1} + z_i$. Hence we must have equalities. In particular, $y_i + y_{i+1} = z_{i-1} + z_i$.

(b) $x_{i+1} \leq y_{i+1}$. We have $x_{i+1} \leq y_{i+1} < z_{i-1}$, which, since $x \prec z$, implies $x_i + x_{i+1} = z_{i-1} + z_i$. But we have $y_i + y_{i+1} \leq z_{i-1} + z_i$ and, since $x \leq y$, by Lemma 1(i), $x_i + x_{i+1} \leq y_i + y_{i+1}$. It follows that $y_i + y_{i+1} = z_{i-1} + z_i$.

Proof of $\delta(x, y) + \delta(y, z) = \delta(x, z)$. We proved that together with $x \prec z$, we also have $x \prec y$ and $y \prec z$. If $I = \{1, \ldots, n\}$, then $\delta(x, y) = \sum_{i \in I} (-1)^i (x_i - y_i)_+$, $\delta(y, z) = \sum_{i \in I} (-1)^i (y_i - z_i)_+$ and $\delta(x, z) = \sum_{i \in I} (-1)^i (x_i - z_i)_+$. It follows that

$$\delta(x, y) + \delta(y, z) - \delta(x, z) = \sum_{i \in I} (-1)^i \phi(i), \quad \phi(i) := (x_i - y_i)_+ + (y_i - z_i)_+ - (x_i - z_i)_+.$$

We write I as a disjoint union $I = I_0 \cup I_1 \cup I_2$, where

$$\begin{aligned} I_0 &= \{i \in I \mid x_i \leq y_i \leq z_i \text{ or } x_i \geq y_i \geq z_i\}, \\ I_1 &= \{i \in I \mid y_i > x_i, z_i\}, \\ I_2 &= \{i \in I \mid y_i < x_i, z_i\}. \end{aligned}$$

Lemma 3. *If $i \in I_1$, then $i + 1 \in I_2$. If $i \in I_2$, then $i - 1 \in I_1$.*

Proof. Suppose that $i \in I_1$. We have $y_i > z_i$, which, by Lemma 2(i), implies that $y_i + y_{i+1} = z_{i-1} + z_i$ and $y_{i+1} < z_{i-1} \leq z_{i+1}$. (We have $y \prec z$.) To complete the proof of $i + 1 \in I_2$, we must prove that $y_{i+1} < x_{i+1}$. Assume the contrary. Then $x_{i+1} \leq y_{i+1} < z_{i-1}$, which, since $x \prec z$, implies $x_i + x_{i+1} = z_{i-1} + z_i = y_i + y_{i+1}$. But this is impossible, because $x_i < y_i$ (we have $i \in I_1$) and $x_{i+1} \leq y_{i+1}$. Hence $y_{i+1} < x_{i+1}$ and we have $i + 1 \in I_2$.

Suppose now that $i \in I_2$. We have $y_i < x_i$, which, by Lemma 2(i), implies that $y_{i-1} + y_i = x_i + x_{i+1}$ and $y_{i-1} > x_{i+1} \geq x_{i-1}$. (We have $x \prec y$.) To complete the proof of $i - 1 \in I_2$, we must prove that $y_{i-1} > z_{i-1}$. Assume the contrary. Then $z_{i-1} \geq y_{i-1} > x_{i+1}$, which, since $x \prec z$, implies $z_{i-1} + z_i = x_i + x_{i+1} = y_{i-1} + y_i$. But this is impossible, because $z_i > y_i$ (we have $i \in I_2$) and $z_{i-1} \geq y_{i-1}$. Hence $y_{i-1} > z_{i-1}$ and we have $i - 1 \in I_1$. $\square$

As a consequence of Lemma 3, we have a bijection $I_1 \to I_2$, given by $i \mapsto i+1$. Then we have

$$\begin{aligned}\delta(x,y)+\delta(y,z)-\delta(x,z) &= \sum_{i\in I}(-1)^i\phi(i)\\ &= \sum_{i\in I_0}(-1)^i\phi(i)+\sum_{i\in I_1}(-1)^i\phi(i)+\sum_{i\in I_2}(-1)^i\phi(i)\\ &= \sum_{i\in I_0}(-1)^i\phi(i)+\sum_{i\in I_1}(-1)^i\phi(i)+\sum_{i\in I_1}(-1)^{i+1}\phi(i+1)\\ &= \sum_{i\in I_0}(-1)^i\phi(i)+\sum_{i\in I_1}(-1)^i\big(\phi(i)-\phi(i+1)\big).\end{aligned}$$

To complete the proof of $\delta(x,y)+\delta(y,z)=\delta(x,z)$, i.e., of $\delta(x,y)+\delta(y,z)-\delta(x,z)=0$, we show that each term of the two sums above is zero. Namely, we prove:

Lemma 4. *We have $\phi(i)=0$ for all $i\in I_0$ and $\phi(i)-\phi(i+1)=0$ for all $i\in I_1$.*

Proof. Let $i\in I_0$. If $x_i\le y_i\le z_i$, then $(x_i-y_i)_+=(y_i-z_i)_+=(x_i-z_i)_+=0$, so $\phi(i)=0+0-0=0$. If $x_i\ge y_i\ge z_i$, then $(x_i-y_i)_+=x_i-y_i$, $(y_i-z_i)_+=y_i-z_i$, and $(x_i-z_i)_+=x_i-z_i$, so $\phi(i)=(x_i-y_i)+(y_i-z_i)-(x_i-z_i)=0$.

Suppose now that $i\in I_1$ and so, by Lemma 3, $i+1\in I_2$. We have $y_i>\max\{x_i,z_i\}$, so $(x_i-y_i)_+=0$ and $(y_i-z_i)_+=y_i-z_i$. Thus,

$$\phi(i)=0+(y_i-z_i)-(x_i-z_i)_+=y_i-z_i-(z_{i-1}-x_{i+1})_+.$$

We also have $y_{i+1}<\min\{x_{i+1},z_{i+1}\}$, so $(x_{i+1}-y_{i+1})_+=x_{i+1}-y_{i+1}$ and $(y_{i+1}-z_{i+1})_+=0$. Thus

$$\phi(i+1)=(x_{i+1}-y_{i+1})+0-(x_{i+1}-z_{i+1})_+=x_{i+1}-y_{i+1}-(z_i-x_{i+2})_+.$$

(We have $x\prec z$, so, by Lemma 2(ii), $(x_i-z_i)_+=(z_{i-1}-x_{i+1})_+$ and $(x_{i+1}-z_{i+1})_+=(z_i-x_{i+2})_+$.) Consequently,

$$\phi(i)-\phi(i+1)=y_i+y_{i+1}-x_{i+1}-z_i+(z_i-x_{i+2})_+-(z_{i-1}-x_{i+1})_+.$$

Since $x_{i+1}>y_{i+1}$ and $y_i>z_i$, by Lemma 2(ii), we get $x_{i+1}+x_{i+2}=y_i+y_{i+1}=z_{i-1}+z_i$. (We have $x\prec y$ and $y\prec z$.) Then $z_{i-1}-x_{i+1}=-(z_i-x_{i+2})$, which implies that $(z_i-x_{i+2})_+-(z_{i-1}-x_{i+1})_+=z_i-x_{i+2}$. (For every $x\in\mathbb{R}$ we have $x_+-(-x)_+=(x+|x|)/2-(-x+|x|)/2=x$.) It follows that $\phi(i)-\phi(i+1)=y_i+y_{i+1}-x_{i+1}-z_i+z_i-x_{i+2}=(y_i+y_{i+1})-(x_{i+1}+x_{i+2})=0$, which concludes the solution in case x, y, $z\in\mathcal{B}_n$. □

We now show that we can always reduce to the case when x, y, z have the same length. The idea is to extend them to longer tuples by adding a large number M to the right.

If $\ell\ge 1$ is an integer and $M\in\mathbb{R}$ we denote by $\mathcal{B}(\ell,M)$ the set of all elements of $\mathcal{B}$ of length $\le\ell$ and with entries $<M$, i.e.,

$$\mathcal{B}(\ell,M)=\mathcal{B}\cap\left(\cup_{n\le\ell}(-\infty,M)^n\right).$$

Lemma 5. (i) *We have a map* $\mathcal{B}(\ell, M) \to \mathcal{B}_\ell$ *given by* $x = (x_1, \ldots, x_n) \mapsto \bar{x} = (x_1, \ldots, x_n, M, \ldots, M)$.

Let $x, y \in \mathcal{B}(\ell, M)$. *Then:*

(ii) *We have* $x \le y$ *iff* $\bar{x} \le \bar{y}$ *and* $x \prec y$ *iff* $\bar{x} \prec \bar{y}$.

(iii) *If* $x \prec y$, *then* $\delta(x, y) = \delta(\bar{x}, \bar{y})$.

Proof. (i) First note that $\bar{x}_i \le M$ for $1 \le i \le \ell$ ($\bar{x}_i = x_i$ if $i \le n$, $\bar{x}_i = M$ if $i > n$.) If $1 \le i \le n-2$, the relation $\bar{x}_i \le \bar{x}_{i+2}$ is the same as $x_i \le x_{i+2}$, which follows from $x \in \mathcal{B}_n$, and for $n-1 \le i \le \ell-2$ we have $\bar{x}_i \le M = \bar{x}_{i+2}$. Thus $\bar{x} \in \mathcal{B}_\ell$.

For (ii) and (iii) let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$.

(ii) If $m < n$ then $x \le y$ fails and so does $\bar{x} \le \bar{y}$, since the corresponding condition fails at $i = m+1$: We cannot have $M = \bar{x}_{m+1} \le \bar{y}_{m+1} = y_{m+1}$ or $2M = \bar{x}_{m+1} + \bar{x}_{m+2} \le \bar{y}_m + \bar{y}_{m+1} = y_m + y_{m+1}$. (For $1 \le i \le n$ we have $\bar{y}_i = y_i < M$.) Suppose now that $m \ge n$. If $n < i \le \ell$, then the condition for $\bar{x} \le \bar{y}$ at index i is satisfied unconditionally, as $\bar{x}_i < M = \bar{y}_i$. Suppose now that $i \le n$. If $i < m$, then the condition for both $x \le y$ and $\bar{x} \le \bar{y}$ at i is the same, i.e., either $x_i \le y_i$ or $i > 1$ and $x_i + x_{i+1} \le y_{i-1} + y_i$. We are left with the case when $m \le i \le n$, i.e., when $i = m = n$. Here the condition for $x \le y$ is $x_m \le y_m$ and for $\bar{x} \le \bar{y}$ it is that either $x_m = \bar{x}_m \le \bar{y}_m = y_m$ or $x_m + M = \bar{x}_m + \bar{x}_{m+1} \le \bar{y}_{m-1} + \bar{y}_m = y_{m-1} + y_m$. But in the latter case, since $M > y_{m-1}$, we have $x_m < y_m$. So again we have equivalence. Thus $x \le y$ iff $\bar{x} \le \bar{y}$.

Suppose now that $x \le y$ and $\bar{x} \le \bar{y}$. We prove that $x \prec y$ iff $\bar{x} \prec \bar{y}$. If $m - n > 2$, then $x \prec y$ fails and so does $\bar{x} \prec \bar{y}$: We cannot have $M = \bar{y}_{n+1} \le \bar{x}_{n+3} = x_{n+3}$ or $2M = \bar{y}_{n+1} + \bar{y}_{n+1} = \bar{x}_{n+2} + \bar{x}_{n+3} = x_{n+2} + x_{n+3}$. (We have $x_{n+2}, x_{n+3} < M$.) Thus we may assume that $m - n \le 2$. If $m \le i < \ell$, then the condition for $x \prec y$ at i is satisfied unconditionally: We have $\bar{x}_{i+1} = M \ge \bar{y}_{i-1}$. Assume now that $1 < i < m$. If $i \le n$, the condition at index i for both $x \le y$ and $\bar{x} \le \bar{y}$ is the same: either $x_{i+1} \ge y_{i-1}$ or $x_i + x_{i+1} = y_{i-1} + y_i$. We are left with the case when $n < i < m$. Since $m - n \le 2$, this means $i = n+1 = m-1$. At this index the condition for $x \le y$ is $x_{n+2} \ge y_n$ and for $\bar{x} \le \bar{y}$ it states that either $x_{n+2} = \bar{x}_{n+2} \ge \bar{y}_n = y_n$ or $x_{n+1} + x_{n+2} = \bar{x}_{n+1} + \bar{x}_{n+2} \ge \bar{y}_n + \bar{y}_{n+1} = y_n + M$. But, since $x_{n+1} < M$, in the latter case it also holds $x_{n+2} > y_n$. So again we have equivalence. Thus $x \prec y$ iff $\bar{x} \prec \bar{y}$.

(iii) For $1 \le i \le n$ we have $(\bar{x}_i - \bar{y}_i)_+ = (x_i - y_i)_+$ and for $n < i \le \ell$ we have $\bar{x}_i \le M = \bar{y}_i$, so $(\bar{x}_i - \bar{y}_i)_+ = 0$. It follows that $\delta(\bar{x}, \bar{y}) = \sum_{i=1}^{\ell} (\bar{x}_i - \bar{y}_i)_+ = \sum_{i=1}^{n} (x_i - y_i)_+ = \delta(x, y)$. $\square$

Suppose now that $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_k)$ are arbitrary elements of $\mathcal{B}$. We choose an integer $\ell \ge \max\{m, n, k\}$ and $M \in \mathbb{R}$ which is greater than all x_i, y_i, z_i, so that $x, y, z \in \mathcal{B}(\ell, M)$. By Lemma 5, $\bar{x}, \bar{y}, \bar{z} \in \mathcal{B}_\ell$ and, since $x \le y \le z$ and $x \prec z$, we also have $\bar{x} \le \bar{y} \le \bar{z}$ and $\bar{x} \prec \bar{z}$. Since $\bar{x}, \bar{y}, \bar{z}$ have the same length, this implies that $\bar{x} \prec \bar{y} \prec \bar{z}$ and $\delta(\bar{x}, \bar{y}) + \delta(\bar{y}, \bar{z}) = \delta(\bar{x}, \bar{z})$. This, again by Lemma 5, implies that $x \prec y \prec z$ and $\delta(x, y) + \delta(y, z) = \delta(x, z)$.

Remark. Our result can also be proved directly, without reducing to the case when x, y, z have the same length. But this would imply some extra complications, which are particularly annoying for the proof of $\delta(x, y) + \delta(y, z) = \delta(x, z)$. Again if $I = \{1, \ldots, n\}$, then the relation $\delta(x, y) + \delta(y, z) - \delta(x, z) = \sum_{i \in I} (-1)^i \phi(i)$ and

Lemmas 3 and 4 still hold, but with $\phi(i)$ and I_1, I_2, I_3 defined slightly differently. Namely,

$$\phi(i) = (x_i - y_i)_+ + (y_i - z_i)_+ - (x_i - z_i)_+$$

for $i \leq k$, but $\phi(i) = (x_i - y_i)_+$ for $i > k$. Also

$$I_0 = \{i \in I \mid i \leq k,\ x_i \leq y_i \leq z_i \text{ or } x_i \geq y_i \geq z_i\} \cup \{i \in I : i > k,\ x_i \leq y_i\},$$
$$I_1 = \{i \in I \mid i \leq k,\ y_i > x_i, z_i\},$$
$$I_2 = \{i \in I \mid i \leq k,\ y_i < x_i, z_i\} \cup \{i \in I : i > k,\ y_i < x_i\}.$$

574. The matrices $A, B \in \mathcal{M}_n(\mathbb{C}), n \geq 2$, satisfy the relations:

$$(1)\ A^2 = B^2; \quad (2)\ A^3 + BAB = 2I_n.$$

(a) Prove that $A^3 = I_n$;
(b) Prove that if A and B have no common eigenvalues, then $A + B = O_n$.

`Proposed by Vasile Pop and Constantin-Cosmin Todea, Technical University of Cluj-Napoca, Romania.`

Solution by the authors. (a) We have

$$\begin{aligned} B^6 = A^6 = (A^3)^2 = (2I_n - BAB)^2 &= 4I_n - 4BAB + BABBAB \\ &= 4I_n - 4BAB + BAA^2AB = 4I_n - 4BAB + B^6, \end{aligned}$$

so

$$BAB = I_n. \tag{3}$$

From (2) and (3) we get $A^3 = I_n$.

(b) Since $A^2 = B^2$ and $A^3 = I_n$, we have $B^6 = A^6 = I_n$. By (3), the matrices A and B are invertibile and

$$B^{-1} = AB = BA,$$

i.e., they commute. Since $A^3 = I_n$, $B^6 = I_n$,, the matrices A and B are diagonalizable. (We have $g_A(A) = O_n$ and $g_B(B) = O_n$, where $g_A = X^3 - 1$ and $g_B = X^6 - 1$ have only simple roots.). Since also they commute, they are simultaneously diagonalizable, so there is $P, J_A, J_B \in \mathcal{M}_n(\mathbb{C})$, with P invertible and J_A, J_B diagonal, such that

$$A = P \cdot J_A \cdot P^{-1}, B = P \cdot J_B \cdot P^{-1}.$$

The eigenvalues of A satisfy the relation $\lambda_A^3 = 1$, whence $\lambda_A \in \{1, \varepsilon, \varepsilon^2\}$, where $\varepsilon = \frac{-1+i\sqrt{3}}{2}$. Then the eigenvalues of A^2 belong to the set

$$\{1, \varepsilon^2, \varepsilon^4\} = \{1, \varepsilon, \varepsilon^2\},$$

so

$$\operatorname{Spec}(A^2) = \operatorname{Spec}(B^2) \subseteq \{1, \varepsilon, \varepsilon^2\}.$$

Morever, since J_A and J_B have no common eigenvalues, it follows that $J_A - J_B$ is invertible. Then from

$$O_n = J_{A^2} - J_{B^2} = (J_A - J_B)(J_A + J_B)$$

we get $J_A + J_B = O_n$, which is equivalent to $A + B = O_n$.

Solution by Robert Rogojan, Baia Mare, Romania. Rewrite (2) as $B^2A + BAB = 2I_n$, or $B(BA + AB) = 2I_n$. From here we conclude that B is invertible.

Multiplying (2) to the right by B, we get $A^3B+BAB^2=2B$ and, since $B^2=A^2$, this gives us $A^3B+BA^3=2B$, which rewrites as $(A^3-I_n)B+B(A^3-I_n)=O_n$. Denote A^3-I_n by C, and thus $BC+CB=O_n$.

Since B and C anticommute, it follows that B and $C^2=A^6-2A^3+I_n$ commute. But B commutes with both $A^6=B^6$ and I_n. It follows that B also commutes with A^3. Then $A^3B+BA^3=2B$ rewrites as $2A^3B=2B$. Since B is invertible, we get that $A^3=I_n$.

For the (b) part, recall Sylvester's equation $MX-XN=P$. It is known that for $M,N,P\in\mathcal{M}_n(\mathbb{C})$ this equation has a unique solution $X\in\mathcal{M}_n(\mathbb{C})$ iff $\operatorname{Spec}(M)\cap\operatorname{Spec}(N)=\emptyset$. We apply this result to $M:=A$, $N:=B$, and $P:=O_n$. Since $\operatorname{Spec}(A)\cap\operatorname{Spec}(B)=\emptyset$, it follows that the equation $AX-XB=O_n$ has a unique solution. We have the obvious solution $X=O_n$, but also $X=A+B$, since $A(A+B)-(A+B)B=A^2+AB-AB-B^2=O_n$. By the unicity, we must have $A+B=O_n$.

Second proof of (b). During the proof of (a) we showed that B copmmutes with A^3. But it also commutes with B^{-2} (recall that B is invertible), so it commutes with $A^3B^{-2}=A$. (We have $B^{-2}=(B^2)^{-1}=(A^2)^{-1}$). Since $AB=BA$, by a result due to Frobenius, the eigenvalues of $A-B$ have the form $a-b$, where $a\in\operatorname{Spec}(A)$ and $b\in\operatorname{Spec}(B)$. But, since $\operatorname{Spec}(A)\cap\operatorname{Spec}(B)=\emptyset$, the eigenvalues of $A-B$ are $\neq 0$, so $A-B$ is invertible. Then, since $AB=BA$, (1) can be written as $(A-B)(A+B)=O_n$, which implies that $A+B=O_n$.

Solution by Daniel Văcaru, Pitești, Romania, only for the (a) part. Since $A^2=B^2$, we have $2I_n=A^3+BAB=AB^2+BAB=(AB+BA)\cdot B$, so B is invertible and $B^{-1}=\frac{1}{2}(AB+BA)$. It follows that

$$A^{-2}=B^{-2}=\left[\frac{1}{2}(AB+BA)\right]^2=\frac{1}{4}\left((AB)^2+ABBA+BAAB+(BA)^2\right).$$

But, since $A^2=B^2$, we have $AB^2A=AA^2A=A^4$ and $BA^2B=BB^2B=B^4=A^4$. And, by mutiplying the relation $A^3+BAB=2I_n$ to the left and to the right by A, we get $A^4+(AB)^2=2A$ and $A^4+(BA)^2=2A$. Hence $(AB)^2=(BA)^2=2A-A^4$. In conclusion,

$$A^{-2}=\frac{1}{4}(2A-A^2+A^4+A^4+2A-A^4)=A,$$

which implies $A^3=I_n$.

575. Let n be a positive integer with $n\geq 3$. Prove that $\dfrac{n}{4}$ is the least positive value of the constant k such that the inequality

$$a_1+a_2+\cdots+a_n+\frac{k(a_1-a_n)^2}{a_1+a_n}\geq\sqrt{n(a_1^2+a_2^2+\cdots+a_n^2)}$$

holds whenever $a_1\geq a_2\geq\cdots\geq a_n\geq 0$.

`Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania, and Leonard Giugiuc, Middle School Greci, Mehedinți, Romania.`

Solution by the authors. First we show that the claimed property holds for $k = \frac{n}{4}$. We need to prove that

$$a_1 + a_2 + \cdots + a_n + \frac{n(a_1 - a_n)^2}{4(a_1 + a_n)} \geq \sqrt{n(a_1^2 + a_2^2 + \cdots + a_n^2)}.$$

Due to homogeneity, we may set $a_1 = 1$. Then $a_n \leq 1$. If $a_n = 1$, so $a_1 = \cdots = a_n = 1$, then our inequality holds trivially (with equality). Therefore we may assume that $a_n := \alpha < 1$. Suppose that α is fixed. We need to show that $f(a_2, \ldots, a_{n-1}) \geq 0$ for $a_2, \ldots, a_{n-1} \in [\alpha, 1]$, where

$$f(a_2, \ldots, a_{n-1}) = 1 + a_2 + \cdots + a_{n-1} + \alpha + \frac{n(1-\alpha)^2}{4(1+\alpha)} - \sqrt{n(1 + a_2^2 + \cdots + a_{n-1}^2 + \alpha^2)}.$$

(Note that f is symmetric, so the constraint that $a_2 \geq \cdots \geq a_{n-1}$ is not necessary.)

Since f is continuous and defined on the compact set $[\alpha, 1]^{n-2}$, it has a minimum. Since f is strictly concave with respect to each variable, the minimum value is reached at points where $a_2, \ldots, a_{n-1} \in \{\alpha, 1\}$. We may assume that

$$a_2 = \cdots = a_j = 1, \; a_{j+1} = \cdots = a_{n-1} = \alpha,$$

where $j \in \{1, 2, \ldots, n-1\}$. So, it suffices to show that for every such j we have

$$j + (n-j)\alpha + \frac{n(1-\alpha)^2}{4(1+\alpha)} \geq \sqrt{nB},$$

where $B = j + (n-j)\alpha^2$. We write the inequality as

$$\frac{n(1-\alpha)^2}{4(1+\alpha)} \geq \sqrt{nB} - j - (n-j)\alpha.$$

But, since $nB - (j + (n-j)\alpha)^2 = j(n-j)(1-\alpha)^2$, this also writes as

$$\frac{n(1-\alpha)^2}{4(1+\alpha)} \geq \frac{j(n-j)(1-\alpha)^2}{\sqrt{nB} + j + (n-j)\alpha},$$

which is equivalent to

$$\frac{n}{4(1+\alpha)} \geq \frac{j(n-j)}{\sqrt{nB} + j + (n-j)\alpha},$$

i.e.,

$$n\sqrt{nB} \geq j(3n - 4j) + (n-j)(4j - n)\alpha. \tag{*}$$

It suffices to show that

$$n^3 B \geq \left(j(3n-4j) + (n-j)(4j-n)\alpha\right)^2,$$

which, after some calculations, is equivalent to the obvious inequality

$$j(n-j)\left((3n-4j)\alpha + n - 4j\right)^2 \geq 0.$$

For $a_1 = 1$, the equality occurs when $a_1 = a_2 = \cdots = a_n = 1$, and also when $a_1 = \cdots = a_j = 1$ and $a_{j+1} = \cdots = a_n = \alpha := \frac{4j - n}{3n - 4j}$. In order that $0 \leq \alpha < 1$, we need that j is an integer such that $\frac{n}{4} \leq j < \frac{n}{2}$.

Let $a_1, \dots, a_n$ be any of the numbers above, with $a_1 > a_n$, for which we have equality, i.e.,

$$a_1 + a_2 + \cdots + a_n + \frac{n/4 \cdot (a_1 - a_n)^2}{a_1 + a_n} = \sqrt{n(a_1^2 + a_2^2 + \cdots + a_n^2)}.$$

Then for every $k < \dfrac{n}{4}$ we have

$$a_1 + a_2 + \cdots + a_n + \frac{k(a_1 - a_n)^2}{a_1 + a_n} < \sqrt{n(a_1^2 + a_2^2 + \cdots + a_n^2)},$$

so k doesn't have the required property. This proves the minimality of $\dfrac{n}{4}$.

A more difficult version. Let n be a positive integer with $n \geq 3$. Find the least positive value of the constant k_n such that the inequality

$$a_1 + a_2 + \cdots + a_n + \frac{k_n(a_1 - a_n)^2}{a_1 + a_n} \geq \sqrt{n(a_1^2 + a_2^2 + \cdots + a_n^2)}$$

holds whenever $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$.

Editor's note. We also received a solution from Mihalcea Andrei, based on authors' main idea.

576. We say that $x \in \mathbb{R}$ is a *nice number* if $\displaystyle\sum_{k=1}^{\infty} k^n x^k$ is an integer, for all $n \in \mathbb{N}$, $n \geq 1$.

(a) Find all the nice numbers.

(b) If x is a nice number, show that $\displaystyle\sum_{k=1}^{\infty} k^n x^k$ is an even natural number, for all $n \in \mathbb{N}$, $n \geq 1$.

Proposed by Mircea Rus, Technical University of Cluj-Napoca, Romania.

Solution by the author. For every $n \in \mathbb{N}$, denote $S_n(x) = \displaystyle\sum_{k=1}^{\infty} k^n x^k$. By an elementary argument, this power series is convergent if and only if $x \in (-1, 1)$, so every nice number lies in $(-1, 1)$.

We search for a recurrence for the sequence of functions $(S_n)_{n \geq 0}$, so let $n \in \mathbb{N}$ and $x \in (-1, 1)$. Then

$$S_n(x) = x + \sum_{k=2}^{\infty} k^n x^k = x + \sum_{k=1}^{\infty} (k+1)^n x^{k+1} = x + x \sum_{k=1}^{\infty} \left(\sum_{p=0}^{n} \binom{n}{p} k^p \right) x^k.$$

Since each series $S_p(x)$ is absolutely convergent, we may change the order of summation and obtain

$$S_n(x) = x + x\sum_{p=0}^{n}\binom{n}{p}\sum_{k=1}^{\infty}k^p x^k = x\left(1+\sum_{p=0}^{n}\binom{n}{p}S_p(x)\right)$$
$$= xS_n(x) + x\left(1+\sum_{p=0}^{n-1}\binom{n}{p}S_p(x)\right),$$

which gives the recurrence

$$S_n(x) = \frac{x}{1-x}\left(1+\sum_{p=0}^{n-1}\binom{n}{p}S_p(x)\right), \quad \text{for all } n\in\mathbb{N} \text{ and } x\in(-1,1).$$

In particular, $S_0(x) = \dfrac{x}{1-x}$. Denoting $y = y(x) = \dfrac{x}{1-x} \in \left(-\dfrac{1}{2},\infty\right)$, it follows from the recurrence that

$$\begin{aligned}
&S_0(x) = y,\\
&S_1(x) = y(y+1),\\
&S_2(x) = y(1+y+2y(y+1)) = y(y+1)(2y+1),\\
&\quad \dots
\end{aligned}$$

Let x be a nice number. Obviously, 0 is a nice number, so we may assume that $x \neq 0$, hence $y \neq 0$. Then $S_1(x) \neq 0$, $S_1(x)\in\mathbb{Z}$, and $S_2(x)\in\mathbb{Z}$, so $2y+1 = \dfrac{S_2(x)}{S_1(x)} \in \mathbb{Q}$, whence $y\in\mathbb{Q}$. We prove that $y\in\mathbb{N}$. Writing $y = \dfrac{a}{b}$, with $a\in\mathbb{Z}$, $b\in\mathbb{N}$, $a,b\neq 0$, a and b relatively prime, it follows that $y\,(y+1) = \dfrac{a(a+b)}{b^2} \in \mathbb{N}$, so $b^2 \mid a(a+b)$. Since a and b are relatively prime, the only possibility is $b=1$, so $y\in\mathbb{N}$.

So far, we proved that if x is a nice number, then $\dfrac{x}{1-x}\in\mathbb{N}$ ($x=0$ is also included here), which means that $x = \dfrac{y}{y+1}$, with $y\in\mathbb{N}$.

Conversely, we must show that if $x = \dfrac{y}{y+1}$, with $y\in\mathbb{N}$, then $S_n(x)$ is an even natural number, for all $n\in\mathbb{N}$, $n\geq 1$. The result is obvious if $y=0$, so we assume that $y\geq 1$.

For $n\geq 1$, we rewrite the recurrence as

$$S_n(x) = y\left((1+y)+\sum_{p=1}^{n-1}\binom{n}{p}S_p(x)\right). \tag{1}$$

Since $S_1(x)\in\mathbb{N}$ and $(y+1)\mid S_1(x)$, it follows by induction using (1) that $S_n(x)\in\mathbb{N}$ and $(y+1)\mid S_n(x)$, for all $n\geq 1$. Next, (1) implies that $y \mid S_n(x)$, for all $n\geq 1$. Concluding, $S_n(x)\in\mathbb{N}$ and $y(y+1)\mid S_n(x)$, for all $n\geq 1$, hence $S_n(x)$ is an even natural number, for all $n\in\mathbb{N}$, $n\geq 1$.

Remark. It is also possible to find the explicit form of $S_n(x)$, written in terms of the powers of $y = \dfrac{x}{1-x}$ as

$$S_n(x) = (y+1)\sum_{k=1}^{n} k^{\{n\}} y^k, \quad n \geq 1,$$

where

$$k^{\{n\}} = \sum_{p=1}^{k} (-1)^{k-p}\binom{k}{p} p^n = k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \cdots$$

is a natural number that represents the number of surjective functions from a set with n elements to a set with k elements. Equivalently, $k^{\{n\}} = \Delta^k x^n(0)$, where Δ is the forward difference operator that maps a function f to the difference function Δf, defined by $\Delta f(x) = f(x+1) - f(x)$. Also, $k^{\{n\}}$ are the coefficients of the expansion $x^n = \sum_{k=1}^{n} k^{\{n\}}\binom{x}{k}$, which expresses the monomial x^n in terms of $\binom{x}{k} = \dfrac{x(x-1)\cdots(x-k+1)}{k!}$, with $1 \leq k \leq n$.

577. Let $a \in \mathbb{R}\setminus\{0\}$ and $f : \mathbb{R} \to \mathbb{R}$ a differentiable function with the property

$$|f'(x) + af(x)| \leq 1, \quad \text{for all } x \in \mathbb{R}.$$

Prove that there exists a unique differentiable function $g : \mathbb{R} \to \mathbb{R}$ that satisfies

$$g'(x) + ag(x) = 0 \quad \text{and} \quad |f(x) - g(x)| \leq \frac{1}{|a|}, \quad \text{for all } x \in \mathbb{R}.$$

`Proposed by Dorian Popa, Technical University of Cluj-Napoca, Romania.`

Solution by the author. We proceed in two steps.

Existence. Let $h(x) = f'(x) + af(x)$, $x \in \mathbb{R}$. We can rewrite this equality as $(\mathrm{e}^{ax}f(x))' = \mathrm{e}^{ax}h(x)$, for all $x \in \mathbb{R}$. Integration on $[0,x]$ leads to

$$f(x) = \mathrm{e}^{-ax}\left(f(0) + \int_0^x \mathrm{e}^{at}h(t)\,\mathrm{d}t\right), \quad \text{for all } x \in \mathbb{R}.$$

In the same way, we obtain that the solutions g of the equation $g'(x) + ag(x) = 0$ verify $g(x) = g(0)\mathrm{e}^{-ax}$, for all $x \in \mathbb{R}$. It is enough to provide an appropriate value for $g(0)$ such that $|g(x) - f(x)| \leq \dfrac{1}{|a|}$, for all $x \in \mathbb{R}$. This condition is equivalent to

$$\left|g(0) - f(0) - \int_0^x \mathrm{e}^{at}h(t)\,\mathrm{d}t\right| \leq \frac{\mathrm{e}^{ax}}{|a|}, \quad \text{for all } x \in \mathbb{R}. \tag{1}$$

If $a < 0$, then we choose $g(0) = f(0) + \displaystyle\int_0^\infty \mathrm{e}^{at}h(t)\,\mathrm{d}t$. The convergence of the integral is assured by the fact that $|h(t)| \leq 1$ for all $t \in \mathbb{R}$ and the convegence of

$\displaystyle\int_0^{\infty} \mathrm{e}^{at}\,\mathrm{d}t$. Also, (1) becomes

$$\left|\int_x^{\infty} \mathrm{e}^{at} h(t)\,\mathrm{d}t\right| \leq \frac{\mathrm{e}^{ax}}{|a|}, \quad \text{for all } x \in \mathbb{R},$$

which is true, since

$$\left|\int_x^{\infty} \mathrm{e}^{at} h(t)\,\mathrm{d}t\right| \leq \int_x^{\infty} \mathrm{e}^{at}\,|h(t)|\,\mathrm{d}t \leq \int_x^{\infty} \mathrm{e}^{at}\,\mathrm{d}t = \frac{\mathrm{e}^{-\infty} - \mathrm{e}^{ax}}{a} = \frac{\mathrm{e}^{ax}}{|a|}.$$

If $a > 0$, then we choose $g(0) = f(0) + \displaystyle\int_{-\infty}^{0} \mathrm{e}^{at} h(t)\,\mathrm{d}t$ and the proof follows the same steps as for $a < 0$.

Uniqueness. Let two functions g_1, g_2 satisfy the properties

$$g_k'(x) + a g_k(x) = 0 \quad \text{and} \quad |f(x) - g_k(x)| \leq \frac{1}{|a|}, \quad \text{for all } x \in \mathbb{R}, \quad k \in \{1, 2\}.$$

As in the study of the existence, it follows that there exist two constants $C_1, C_2 \in \mathbb{R}$ such that $g_k(x) = C_k \mathrm{e}^{-ax}$ for all $x \in \mathbb{R}$, $k \in \{1, 2\}$. Also,

$$|C_1 - C_2|\,\mathrm{e}^{-ax} = |g_1(x) - g_2(x)| \leq |f(x) - g_1(x)| + |f(x) - g_2(x)| \leq \frac{2}{|a|},$$

hence $|C_1 - C_2| \leq \dfrac{2}{|a|}\mathrm{e}^{ax}$, for all $x \in \mathbb{R}$. Letting $x \to \infty$ if $a < 0$, or $x \to -\infty$ if $a > 0$, it follows that $|C_1 - C_2| = 0$, so $g_1 = g_2$.