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ARTICLES

The limit, continuity, and Fréchet differentiability of some bivariate functions

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Abstract. Let $k \in \mathbb{N}$, $0 < p_1, q_1, \ldots, p_k,\ q_k < \infty$, and $f : \mathbb{R} \to \mathbb{R}$ be such that $f(0) = 0$. We find the necessary and sufficient conditions for the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x,y) = \begin{cases} \frac{f(xy)}{(|x|^{p_1}+|y|^{p_1})^{q_1} \cdots (|x|^{p_k}+|y|^{p_k})^{q_k}} & \text{if} \quad (x,y) \neq (0,0), \\ 0 & \text{if} \quad (x,y) = (0,0), \end{cases}$$

to be continuous, respectively Fréchet differentiable at $(0,0)$. Moreover, we show that these results can be extended to the case of real normed spaces. Various examples are given.

Keywords: limit of a function, continuous function, bivariate function, Fréchet differentiable.

MSC: Primary 26B05; Secondary 54C30.

1. Preliminaries

In the study of the Fréchet differentiability, the standard example is the following: The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if} \quad (x,y) \neq (0,0), \\ 0 & \text{if} \quad (x,y) = (0,0), \end{cases} \tag{1}$$

is continuous at $(0,0)$, there exist $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0,0)$, but is not Fréchet differentiable at $(0,0)$. In this note we study the existence of the limit, continuity, and Fréchet differentiability of some functions which extend the example (1), see Theorem 2. Moreover, we show in Theorem 7 that these results can be extended to the case of normed spaces. Various examples are given. For

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results of different type, the reader can consult [6]. The notation and notions used in this paper are standard, see any of the books [1, 2, 3, 4].

2. A technical lemma

In the following lemma we find the necessary and sufficient condition for a bivariate function to have the limit 0 at the point $(0,0)$. It is the natural analogue of [5] and [6, Proposition 1].

Lemma 1. *Let $m \in \mathbb{N}$, $0 < p_1, q_1, \ldots, p_m, q_m < \infty$, and $f : \mathbb{R} \to \mathbb{R}$ be such that $f(0) = 0$ and $g : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ defined by*

$$g(x,y) = \frac{f(xy)}{(|x|^{p_1} + |y|^{p_1})^{q_1} \cdot \cdots \cdot (|x|^{p_m} + |y|^{p_m})^{q_m}}.$$

Then the following assertions are equivalent:
(i) $\lim\limits_{(x,y)\to(0,0)} g(x,y) = 0.$
(ii) $\lim\limits_{x\to 0} \frac{f(x)}{|x|^{\frac{p_1q_1+\cdots+p_mq_m}{2}}} = 0.$

Proof. *(i)* $\Rightarrow$ *(ii)*. Let $x_n \to 0$ (in $\mathbb{R}$) be such that $x_n > 0$ for all $n \in \mathbb{N}$. Then $\left(\sqrt{x_n}, \sqrt{x_n}\right) \to (0,0)$ (in $\mathbb{R}^2$) and $\left(\sqrt{x_n}, \sqrt{x_n}\right) \neq (0,0)$ for all $n \in \mathbb{N}$. From *(i)* $\lim\limits_{n\to\infty} g\left(\sqrt{x_n}, \sqrt{x_n}\right) = 0$. Since for all $x \neq 0$, $g(x,x) = \frac{f(x^2)}{2^{q_1+\cdots+q_m}|x|^{p_1q_1+\cdots+p_mq_m}}$, we deduce that $\lim\limits_{n\to\infty} \frac{f(x_n)}{|x_n|^{\frac{p_1q_1+\cdots+p_mq_m}{2}}} = 0$. It follows that

$$\lim_{x\to 0,\, x>0} \frac{f(x)}{|x|^{\frac{p_1q_1+\cdots+p_mq_m}{2}}} = 0. \tag{2}$$

Now let $x_n \to 0$ (in $\mathbb{R}$) be such that $x_n < 0$ for all $n \in \mathbb{N}$. Then $(\sqrt{-x_n}, -\sqrt{-x_n}) \to (0,0)$ (in $\mathbb{R}^2$) and $(\sqrt{-x_n}, -\sqrt{-x_n}) \neq (0,0)$ for all $n \in \mathbb{N}$. The hypothesis *(i)* gives us that $\lim\limits_{n\to\infty} g(\sqrt{-x_n}, -\sqrt{-x_n}) = 0$. Since for all $x \neq 0$ we have $g(x,-x) = \frac{f(-x^2)}{2^{q_1+\cdots+q_m}|x|^{p_1q_1+\cdots+p_mq_m}}$, we deduce that $\lim\limits_{n\to\infty} \frac{f(x_n)}{|x_n|^{\frac{p_1q_1+\cdots+p_mq_m}{2}}} = 0$. It follows that

$$\lim_{x\to 0,\, x<0} \frac{f(x)}{|x|^{\frac{p_1q_1+\cdots+p_mq_m}{2}}} = 0. \tag{3}$$

From (2) and (3) we get *(ii)*.

(ii) $\Rightarrow$ *(i)*. Let $\varepsilon > 0$. From *(ii)* there exists $\delta_\varepsilon > 0$ such that for all $0 < |x| < \delta_\varepsilon$ the following inequality holds

$$\frac{|f(x)|}{|x|^{\frac{p_1q_1+\cdots+p_mq_m}{2}}} < 2^{q_1+\cdots+q_m}\varepsilon. \tag{4}$$

Now notice that if $x \neq 0$ and $y \neq 0$, then

$$|g(x,y)| = \frac{|f(xy)|}{(|x|^{p_1}+|y|^{p_1})^{q_1}\cdots(|x|^{p_m}+|y|^{p_m})^{q_m}}$$

$$= \frac{|f(xy)|}{|xy|^{\frac{p_1q_1+\cdots+p_mq_m}{2}}} \cdot \left(\frac{\sqrt{|x|^{p_1}|y|^{p_1}}}{|x|^{p_1}+|y|^{p_1}}\right)^{q_1} \cdots \left(\frac{\sqrt{|x|^{p_m}|y|^{p_m}}}{|x|^{p_m}+|y|^{p_m}}\right)^{q_m}$$

$$\leq \frac{1}{2^{q_1+\cdots+q_m}} \cdot \frac{|f(xy)|}{|xy|^{\frac{p_1q_1+\cdots+p_mq_m}{2}}}. \tag{5}$$

We have used that $\frac{\sqrt{ab}}{a+b} \leq \frac{1}{2}$ for all $a>0$, $b>0$. Let $(x,y) \neq (0,0)$ be such that $\max(|x|,|y|) < \sqrt{\delta_\varepsilon}$. We can have the situations:
a) $x=0$. In this case $|g(x,y)| = |g(0,y)| = 0 < \varepsilon$.
b) $y=0$. In this case $|g(x,y)| = |g(x,0)| = 0 < \varepsilon$.
c) $x \neq 0$ and $y \neq 0$. In this case, since $\max(|x|,|y|) < \sqrt{\delta_\varepsilon}$, it follows that $0<|x|<\sqrt{\delta_\varepsilon}$ and $0<|y|<\sqrt{\delta_\varepsilon}$. We deduce that $0<|xy|<\delta_\varepsilon$ and from (1), $\frac{|f(xy)|}{|xy|^{\frac{p_1q_1+\cdots+p_mq_m}{2}}} < 2^{q_1+\cdots+q_m}\varepsilon$. From (5) we get $|g(x,y)|<\varepsilon$. Hence, $\lim_{(x,y)\to(0,0)} g(x,y) = 0$. □

3. The main result

Theorem 2. *Let $k \in \mathbb{N}$, $0 < p_1, q_1, \ldots, p_k, q_k < \infty$, and $f : \mathbb{R} \to \mathbb{R}$ be such that $f(0)=0$ and $g : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$g(x,y) = \begin{cases} \frac{f(xy)}{(|x|^{p_1}+|y|^{p_1})^{q_1}\cdots(|x|^{p_k}+|y|^{p_k})^{q_k}} & if \quad (x,y) \neq (0,0), \\ 0 & if \quad (x,y) = (0,0). \end{cases}$$

Then:
(i) g is continuous at $(0,0)$ if and only if $\lim_{x\to 0} \frac{f(x)}{|x|^{\frac{p_1q_1+\cdots+p_kq_k}{2}}} = 0$.
(ii) g is Fréchet differentiable at $(0,0)$ if and only if $\lim_{x\to 0} \frac{f(x)}{|x|^{\frac{p_1q_1+\cdots+p_kq_k}{2}+\frac{1}{2}}} = 0$.

Proof. (i) g is continuous at $(0,0)$ if and only if $\lim_{(x,y)\to(0,0)} g(x,y) = g(0,0) = 0$. By Lemma 1 for $m=k$, this is equivalent to the stated limit.

(ii) Let us note that $g(x,0)=0$ for all $x \in \mathbb{R}$ and hence $\frac{\partial g}{\partial x}(0,0)=0$. From $g(0,y)=0$ for all $y\in\mathbb{R}$ we deduce $\frac{\partial g}{\partial y}(0,0)=0$. Hence g is Fréchet differentiable at $(0,0)$ if and only if $\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{\sqrt{x^2+y^2}} = 0$, that is the function $h : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ defined by

$$h(x,y) = \frac{f(xy)}{(|x|^{p_1}+|y|^{p_1})^{q_1}\cdots(|x|^{p_k}+|y|^{p_k})^{q_k}\left(|x|^2+|y|^2\right)^{\frac{1}{2}}}$$

has the property that $\lim\limits_{(x,y)\to(0,0)} h(x,y) = 0$. By Lemma 1 for $m = k+1$ and $p_{k+1} = 2$, $q_{k+1} = \frac{1}{2}$, this is equivalent to the stated limit. □

4. Some examples

Corollary 3. *Let $0 < p < \infty$, $f : \mathbb{R} \to \mathbb{R}$ be such that $f(0) = 0$ and $g : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$g(x,y) = \begin{cases} \frac{f(xy)}{(|x|^p+|y|^p)^{\frac{1}{p}}} & if \quad (x,y) \neq (0,0), \\ 0 & if \quad (x,y) = (0,0). \end{cases}$$

Then:
(i) g is continuous at $(0,0)$ if and only if $\lim\limits_{x\to 0} \frac{f(x)}{\sqrt{|x|}} = 0$.
(ii) g is Fréchet differentiable at $(0,0)$ if and only if f is derivable at 0 and $f'(0) = 0$.

Proof. It follows from Theorem 2 for $k = 1$, $p_1 = p$, $q_1 = \frac{1}{p}$. □

Let us note that for $p = 2$ and $f(x) = x$ in Corollary 3 we get the standard example (1).

Corollary 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(0) = 0$ and $g : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$g(x,y) = \begin{cases} \frac{f(xy)}{\sqrt{x^2+y^2}\,\sqrt[4]{x^4+y^4}} & if \quad (x,y) \neq (0,0), \\ 0 & if \quad (x,y) = (0,0). \end{cases}$$

Then:
(i) g is continuous at $(0,0)$ if and only if f is derivable at 0 and $f'(0) = 0$.
(ii) g is Fréchet differentiable at $(0,0)$ if and only if $\lim\limits_{x\to 0} \frac{f(x)}{x\sqrt{|x|}} = 0$.

Proof. It follows from Theorem 2 for $k = 2$, $p_1 = 2$, $q_1 = \frac{1}{2}$, $p_2 = 4$, and $q_2 = \frac{1}{4}$. □

Corollary 5. *Let $\alpha \in \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by*

$$g(x,y) = \begin{cases} \frac{|x|^\alpha |y|^\alpha}{\sqrt{x^2+y^2}\,\sqrt[4]{x^4+y^4}\,\sqrt[6]{x^6+y^6}} & if \quad (x,y) \neq (0,0), \\ 0 & if \quad (x,y) = (0,0). \end{cases}$$

Then:
(i) g is continuous at $(0,0)$ if and only $\alpha > \frac{3}{2}$.
(ii) g is Fréchet differentiable at $(0,0)$ if and only if $\alpha > 2$.

Proof. We take $k = 3$, $p_1 = 2$, $q_1 = \frac{1}{2}$, $p_2 = 4$, $q_2 = \frac{1}{4}$, $p_3 = 6$, $q_3 = \frac{1}{6}$, and $f(x) = |x|^{\alpha}$ in Theorem 2. By *(i)*, g is continuous at $(0,0)$ if and only if $\lim\limits_{x\to 0} \frac{|x|^{\alpha}}{|x|^{\frac{3}{2}}} = 0$, or equivalently, $\alpha > \frac{3}{2}$. By part *(ii)* of Theorem 2, g is Fréchet differentiable at $(0,0)$ if and only if $\lim\limits_{x\to 0} \frac{|x|^{\alpha}}{x^2} = 0$, or equivalently, $\alpha > 2$. □

5. The case of real normed spaces

In the sequel we show that the above results can be extended to the context of real normed spaces. If $(X, \|\ \|)$, $(Y, \|\ \|)$ are two real normed spaces, on the cartezian product $X \times Y$ we consider, as is usual, the norm $\|(x,y)\| = \max(\|x\|, \|y\|)$. For the definition of the Fréchet differentiable function in the case of the real normed spaces we refer the reader to consult the books [2, 3, 4]; in particular, we will use that $\Psi : X \times Y \to \mathbb{R}$ is Fréchet differentiable at $(0,0)$ if and only if there exist $\frac{\partial \Psi}{\partial x}(0,0)$, $\frac{\partial \Psi}{\partial y}(0,0)$, and $\lim\limits_{(x,y)\to(0,0)} \frac{\Psi(x,y) - \Psi(0,0) - \frac{\partial \Psi}{\partial x}(0,0)(x) - \frac{\partial \Psi}{\partial y}(0,0)(y)}{\sqrt{\|x\|^2 + \|y\|^2}} = 0$.

Lemma 6. *Let $m \in \mathbb{N}$, $0 < p_1, q_1, \ldots, p_m, q_m < \infty$, and $\varphi : [0,\infty) \to \mathbb{R}$ be such that $\varphi(0) = 0$. Let X, Y be non-null real normed spaces and $\Psi : (X \times Y) \setminus \{(0,0)\} \to \mathbb{R}$ defined by*

$$\Psi(x,y) = \frac{\varphi(\|x\| \, \|y\|)}{(\|x\|^{p_1} + \|y\|^{p_1})^{q_1} \cdots (\|x\|^{p_m} + \|y\|^{p_m})^{q_m}}.$$

Then the following assertions are equivalent:
(i) $\lim\limits_{(x,y)\to(0,0)} \Psi(x,y) = 0$.
(ii) $\lim\limits_{t\to 0,\, t>0} \frac{\varphi(t)}{t^{\frac{p_1 q_1 + \cdots + p_m q_m}{2}}} = 0$.

Proof. $(i) \Rightarrow (ii)$. Since X and Y are non-null, there exist $a \in X$ with $\|a\| = 1$ and $b \in Y$ with $\|b\| = 1$. Let $\lambda_n \to 0$ (in $\mathbb{R}$) be such that $\lambda_n > 0$ for all $n \in \mathbb{N}$. Then $\left(\sqrt{\lambda_n} a, \sqrt{\lambda_n} b\right) \to (0,0)$ (in $X \times Y$) and $\left(\sqrt{\lambda_n} a, \sqrt{\lambda_n} b\right) \neq (0,0)$ for all $n \in \mathbb{N}$. From *(i)*, $\lim\limits_{n\to\infty} \Psi\left(\sqrt{\lambda_n} a, \sqrt{\lambda_n} b\right) = 0$. Since $\|a\| = 1$, $\|b\| = 1$, $\Psi\left(\sqrt{\lambda_n} a, \sqrt{\lambda_n} b\right) = \frac{\varphi(\lambda_n)}{2^{q_1 + \cdots + q_m} \lambda_n^{\frac{p_1 q_1 + \cdots + p_m q_m}{2}}}$ and hence $\lim\limits_{n\to\infty} \frac{\varphi(\lambda_n)}{\lambda_n^{\frac{p_1 q_1 + \cdots + p_m q_m}{2}}} = 0$. It follows that $\lim\limits_{t\to 0,\, t>0} \frac{\varphi(t)}{t^{\frac{p_1 q_1 + \cdots + p_m q_m}{2}}} = 0$.

$(ii) \Rightarrow (i)$. Let $\varepsilon > 0$. From (ii), there exists $\delta_{\varepsilon} > 0$ such that for all $0 < t < \delta_{\varepsilon}$ the following inequality holds

$$\frac{|\varphi(t)|}{t^{\frac{p_1 q_1 + \cdots + p_m q_m}{2}}} < 2^{q_1 + \cdots + q_m} \varepsilon. \tag{6}$$

Now notice that if $x \neq 0$ and $y \neq 0$, then

$$|\Psi(x,y)| = \frac{|\varphi(\|x\|\,\|y\|)|}{(\|x\|^{p_1}+\|y\|^{p_1})^{q_1}\cdots(\|x\|^{p_m}+\|y\|^{p_m})^{q_m}}$$

$$= \frac{|\varphi(\|x\|\,\|y\|)|}{(\|x\|\,\|y\|)^{\frac{p_1q_1+\cdots+p_mq_m}{2}}} \cdot \left(\frac{\sqrt{\|x\|^{p_1}\,\|y\|^{p_1}}}{\|x\|^{p_1}+\|y\|^{p_1}}\right)^{q_1} \cdots \left(\frac{\sqrt{\|x\|^{p_m}\,\|y\|^{p_m}}}{\|x\|^{p_m}+\|y\|^{p_m}}\right)^{q_m}$$

$$\leq \frac{1}{2^{q_1+\cdots+q_m}} \cdot \frac{|\varphi(\|x\|\,\|y\|)|}{(\|x\|\,\|y\|)^{\frac{p_1q_1+\cdots+p_mq_m}{2}}}. \tag{7}$$

Again we have used that $\frac{\sqrt{ab}}{a+b} \leq \frac{1}{2}$ when $a>0$, $b>0$. Let $(x,y) \neq (0,0)$ be such that $\max(\|x\|,\|y\|) < \sqrt{\delta_\varepsilon}$. We can have the situations:
a) $x = 0$. In this case $|\Psi(x,y)| = |\Psi(0,y)| = 0 < \varepsilon$.
b) $y = 0$. In this case $|\Psi(x,y)| = |\Psi(x,0)| = 0 < \varepsilon$.
c) $x \neq 0$ and $y \neq 0$. In this case, since $\max(\|x\|,\|y\|) < \sqrt{\delta_\varepsilon}$, it follows that $0 < \|x\| < \sqrt{\delta_\varepsilon}$ and $0 < \|y\| < \sqrt{\delta_\varepsilon}$. We deduce that $0 < \|x\|\,\|y\| < \delta_\varepsilon$ and from (6)

$$\frac{|\varphi(\|x\|\,\|y\|)|}{(\|x\|\,\|y\|)^{\frac{p_1q_1+\cdots+p_mq_m}{2}}} < 2^{q_1+\cdots+q_m}\varepsilon.$$

From (7) we deduce that $|\Psi(x,y)| < \varepsilon$. Hence, $\lim_{(x,y)\to(0,0)} \Psi(x,y) = 0$. □

Theorem 7. *Let $k \in \mathbb{N}$, $0 < p_1, q_1, \ldots, p_k$, $q_k < \infty$, and $\varphi : [0,\infty) \to \mathbb{R}$ be such that $\varphi(0) = 0$. Let X, Y be non-null real normed spaces and $\Psi : X \times Y \to \mathbb{R}$ defined by*

$$\Psi(x,y) = \begin{cases} \frac{\varphi(\|x\|\|y\|)}{(\|x\|^{p_1}+\|y\|^{p_1})^{q_1}\cdots(\|x\|^{p_k}+\|y\|^{p_k})^{q_k}} & if \quad (x,y) \neq (0,0), \\ 0 & if \quad (x,y) = (0,0). \end{cases}$$

Then:
(i) Ψ is continuous at $(0,0)$ if and only if $\lim_{t\to0,\,t>0} \frac{\varphi(t)}{t^{\frac{p_1q_1+\cdots+p_kq_k}{2}}} = 0$.
(ii) Ψ is Fréchet differentiable at $(0,0)$ if and only if $\lim_{t\to0,\,t>0} \frac{\varphi(t)}{t^{\frac{p_1q_1+\cdots+p_kq_k}{2}+\frac{1}{2}}} = 0$.

Proof. *(i)* Ψ is continuous at $(0,0)$ if and only if $\lim_{(x,y)\to(0,0)} \Psi(x,y) = \Psi(0,0) = 0$. By Lemma 6 for $m = k$, this is equivalent to the stated limit.

(ii) Let us note that $\Psi(x,0) = 0$ for all $x \in X$ and hence $\frac{\partial\Psi}{\partial x}(0,0) = 0$, and from $\Psi(0,y) = 0$ for all $y \in Y$ we deduce $\frac{\partial\Psi}{\partial y}(0,0) = 0$. Hence Ψ is Fréchet differentiable at $(0,0)$ if and only if $\lim_{(x,y)\to(0,0)} \frac{\Psi(x,y)}{\sqrt{\|x\|^2+\|y\|^2}} = 0$, that is

the function $H : (X \times Y) \setminus \{(0,0)\} \to \mathbb{R}$ defined by

$$H(x,y) = \frac{\varphi(\|x\| \, \|y\|)}{(\|x\|^{p_1} + \|y\|^{p_1})^{q_1} \cdots (\|x\|^{p_k} + \|y\|^{p_k})^{q_k} \left(\|x\|^2 + \|y\|^2\right)^{\frac{1}{2}}}$$

has the property that $\lim\limits_{(x,y)\to(0,0)} H(x,y) = 0$. By Lemma 6 for $m = k+1$ and $p_{k+1} = 2$, $q_{k+1} = \frac{1}{2}$, this is equivalent to the stated limit. □

Corollary 8. *Let $0 < p < \infty$, $\varphi : [0,\infty) \to \mathbb{R}$ be such that $\varphi(0) = 0$. Let X, Y be non-null real normed spaces and $\Psi : X \times Y \to \mathbb{R}$ defined by*

$$\Psi(x,y) = \begin{cases} \frac{\varphi(\|x\|\|y\|)}{(\|x\|^p + \|y\|^p)^{\frac{1}{p}}} & if \quad (x,y) \neq (0,0), \\ 0 & if \quad (x,y) = (0,0). \end{cases}$$

Then:
(i) Ψ is continuous at $(0,0)$ if and only if $\lim\limits_{t\to 0,\, t>0} \frac{\varphi(t)}{\sqrt{t}} = 0$.
(ii) Ψ is Fréchet differentiable at $(0,0)$ if and only if φ is derivable at 0 and $\varphi'(0) = 0$.

Proof. It follows from Theorem 7 for $k = 1$, $p_1 = p$, $q_1 = \frac{1}{p}$. □

Let us give some concrete applications of these general results.

Corollary 9. *Let $\varphi : [0,\infty) \to \mathbb{R}$ be such that $\varphi(0) = 0$. Let $\Psi : \mathbb{R}^3 \to \mathbb{R}$ be defined by*

$$\Psi(x,y,z) = \begin{cases} \frac{\varphi\left(|x|\sqrt{y^2+z^2}\right)}{\sqrt{x^2+y^2+z^2}} & if \quad (x,y,z) \neq (0,0,0), \\ 0 & if \quad (x,y,z) = (0,0,0). \end{cases}$$

Then:
(i) Ψ is continuous at $(0,0,0)$ if and only if $\lim\limits_{t\to 0,\, t>0} \frac{\varphi(t)}{\sqrt{t}} = 0$.
(ii) Ψ is Fréchet differentiable at $(0,0,0)$ if and only if φ is derivable at 0 and $\varphi'(0) = 0$.

Proof. Let us take in Corollary 8 $X = \mathbb{R}$, $Y = \mathbb{R}^2$, and recall that the norm in $\mathbb{R}$ is $|x|$ and the norm in $\mathbb{R}^2$ is $\|(y,z)\| = \sqrt{y^2+z^2}$. □

In the next examples we write $C([0,1])$ to denote the real linear space of all continuous functions $f : [0,1] \to \mathbb{R}$.

Corollary 10. *Let* $\Psi : C([0,1]) \times C([0,1]) \to \mathbb{R}$ *be defined by*

$$\Psi(f,g) = \begin{cases} \dfrac{\left(\sup\limits_{x\in[0,1]} |f(x)|\right)\left(\sup\limits_{x\in[0,1]} |g(x)|\right)}{\sqrt{\sup\limits_{x\in[0,1]} |f(x)|^2 + \sup\limits_{x\in[0,1]} |g(x)|^2}} & if \quad (f,g) \neq (0,0), \\ 0 & if \quad (f,g) = (0,0). \end{cases}$$

Then: Ψ *is continuous at* $(0,0)$*, there exist* $\frac{\partial\Psi}{\partial x}(0,0)$ *and* $\frac{\partial\Psi}{\partial y}(0,0)$*, but* Ψ *is not Fréchet differentiable at* $(0,0)$*.*

Proof. Let us note that, as is well known, $\|f\| = \sup\limits_{x\in[0,1]} |f(x)|$ is a norm on $C([0,1])$. We apply Corollary 8 for $p = 2$ and $\varphi(t) = t$. □

Corollary 11. *Let* $\Psi : C([0,1]) \times C([0,1]) \to \mathbb{R}$ *be defined by*

$$\Psi(f,g) = \begin{cases} \dfrac{\sqrt{\left(\int_0^1 |f(x)|^2 \,\mathrm{d}x\right)\left(\int_0^1 |g(x)|^2 \,\mathrm{d}x\right)}}{\sqrt{\int_0^1 \left(|f(x)|^2 + |g(x)|^2\right) \mathrm{d}x}} & if \quad (f,g) \neq (0,0), \\ 0 & if \quad (f,g) = (0,0). \end{cases}.$$

Then: Ψ *is continuous at* $(0,0)$*, there exist* $\frac{\partial\Psi}{\partial x}(0,0)$ *and* $\frac{\partial\Psi}{\partial y}(0,0)$*, but* Ψ *is not Fréchet differentiable at* $(0,0)$*.*

Proof. In this case we consider on $C([0,1])$ the norm $\|f\| = \sqrt{\int_0^1 |f(x)|^2 \,\mathrm{d}x}$. We apply Corollary 8 for $p = 2$ and $\varphi(t) = t$. □

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Isogonal conjugation in isosceles tetrahedron

Saro Harutyunyan[1)]

Abstract. In this article we investigate the properties of isogonal conjugation in isosceles tetrahedron. Particularly we reveal three hyperbolic parabolas each of which is formed by pairs of isogonal conjugate points symmetric in the respective bimedian, as well as we prove that the circumsphere of an isosceles tetrahedron is invariant under isogonal conjugation in that tetrahedron.

Keywords: Isosceles tetrahedron, isogonal conjugate, inversion, angle between circles/spheres.

MSC: 51M04, 51M20.

1. Introduction

Tetrahedron $ABCD$ is called *isosceles* (or *equihedral*) if its opposite edges are equal, i.e., $AB = CD$, $BC = AD$, $AC = BD$. This type of tetrahedron is already well-investigated. One may refer to [3] for a list of its known properties. For the purposes of this paper we will need only a few of these properties which will be discussed in Section 2.

One may note that the isosceles tetrahedron is kind of a generalization of the equilateral triangle and is inherently "symmetric" so it should have a "center" that will coincide with its circumcenter, incenter, and centroid. Actually this is true for the high-dimensional analogue of isosceles tetrahedron too, see [1] and [2].

We will need the notion of *isogonal conjugation* with respect to a polyhedron. This is the natural generalization of this transformation for polygons. First, for a given dihedron $\mathcal{D}$ with edgeline e and a point P define the *isogonal plane* of Pe in $\mathcal{D}$ as the plane symmetric to Pe with repect to the bisector plane of $\mathcal{D}$ (if P lies on e then the plane Pe, as well as its isogonal, can be any plane through e). Then, for a given polydron $\mathcal{P}$ and a point P define the *isogonal conjugate* of P in $\mathcal{P}$ as the point Q (in case of existence) so that P and Q lie on isogonal planes in each dihedron of $\mathcal{P}$. Obviously if P is the isogonal conjugate of Q, then Q is the isogonal conjugate of P.

Isogonal conjugation is well-defined for an arbitrary tetrahedron, that is, any point of the space has an isogonal conjugate with respect to the tetrahedron. On the other hand this is not the case with other polyhedra, there might be only a few or not even a single point which have isogonal conjugates. Anyway we will only work with tetrahedron and will give a proof of the first sentence of this paragraph in Section 3.

Section 4 will address several auxiliary facts, mainly concerning circles and spheres, which will be leveraged later.

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One of the main results of this paper will be presented in Section 5 where we consider isogonal conjugate pairs symmetric in bimedians of isosceles tetrahedron. We prove that they lie on hyperbolic paraboloids. This is the only section where we will utilize coordinate chasing opposed to the geometric techniques used elsewhere.

The other major result of this paper is that the circumsphere (except for the vertices) of isosceles tetrahedron is invariant with respect to isogonal conjugation. In other words, the isogonal conjugate of any point of the circumsphere of an isosceles tetrahedron other than a vertex lies on its circumsphere. This will be proved in Section 6.

Interestingly, the two-dimensional case of isosceles tetrahedron, namely the equilateral triangle, does not have any similar property. Moreover, the isogonal conjugate of any point $P \notin \{A, B, C\}$ of the circumcircle of *any* triangle ABC is "infinite", i.e., the isogonals of AP, BP, CP are parallel.

2. Properties of isosceles tetrahedron

Proposition 2.1. *Isosceles tetrahedron has the following properties:*

(i) *any of its edges is seen in equal angles from the other two vertices and its faces are congruent triangles (hence the name equihedral);*
(ii) *its circumcenter and incenter coincide;*
(iii) *its faces are acute triangles.*

Proof. (i) Obvious by the equality of opposite edges.

(ii) Let O be the center of circumsphere Ω of $ABCD$. The locus of points $Z \in \Omega$ such that $\angle AZB = \angle ACB$ is a union of two circular arcs which are symmetric with respect to the plane ABO. This means that the planes ADB and ACB are symmetric with respect to ABO for $\angle ADB = \angle ACB$, which follows from (i). So O lies on the bisector plane of dihedron AB. Similarly O lies on the bisector planes of the other dihedrons and thus coincides with the incenter.

(iii) Assume for the sake of contradiction that $\angle ABC \geq 90°$, for example. Choose the point D' in such a way that $ABCD'$ is a parallelogram. Then if M is the midpoint of AC, we have $\triangle ADC = \triangle AD'C$ and

$$BD < BM + MD = BM + MD' = BD' \leq AC$$

which is obviously false (the last inequality follows from the fact that BD' lies inside the circle with diameter AC). Thus the faces of $ABCD$ are acute angled. □

3. Isogonal conjugation in tetrahedron

Here we will work out a proof of the correctness of isogonal conjugation in tetrahedron. Note that by definition if P is a vertex of the tetrahedron, then, for any Q lying in the opposite faceplane, P and Q are isogonal conjugates. Note as well that any two points on the opposite edgelines of the tetrahedron are isogonal conjugates too.

Theorem 1. *For a given tetrahedron $ABCD$ and a point P not lying on its surface, its isogonal conjugate Q with respect to $ABCD$ exists.*

Proof. Let P_A, P_B, P_C, P_D be the reflections of P in the respective faces of $ABCD$. Since $DP_A = DP_B = DP_C$, the line through D perpendicular to $P_AP_BP_C$ passes through the circumcenter Q of $P_AP_BP_CP_D$. The lines through A, B, C defined similarly pass through Q too.

Now let's show that, for example, the planes ABP and ABQ are isogonal in dihedron AB, i.e., they make equal angles with its bisector plane; see Figure 1 for a perspective in the direction of edge AB. Choose a positive direction of rotation around AB and define $\angle(ABX, ABY)$ for X and Y not lying on AB as the minimal angle of rotation in that positive direction that sends the plane ABX to ABY. Then

$$\begin{aligned}\angle(ABP, ABD) &= \frac{\angle(ABP, ABP_C)}{2}\\ &= \frac{\angle(ABP_D, ABP_C) - \angle(ABP_D, ABP)}{2}\\ &= \angle(ABP_D, ABQ) - \angle(ABP_D, ABC)\\ &= \angle(ABC, ABQ).\end{aligned}$$

Similarly, planes eP and eQ are isogonal in dihedron e for any other edge e of $ABCD$. Hence P and Q are isogonal conjugates.

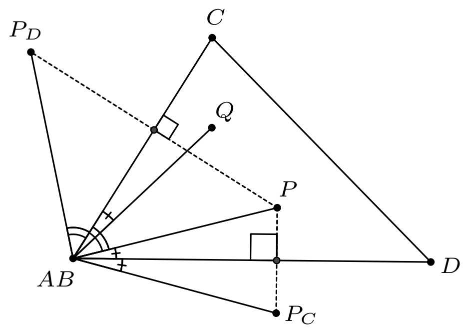

Figure 1.

□

Recall that for a given tetrahedron $ABCD$ and point P the sphere passing through the projections of P onto the faces of $ABCD$ is called the *pedal sphere* of P. For degenerated cases, i.e., when the set of projections contains less than four points, the pedal sphere can be defined via limit.

Remark 1. Note that in the proof of Theorem 1 the homothety with coefficient $1/2$ centered at P sends the vertices of $P_A P_B P_C P_D$ and its circumcenter Q to the projections of P onto the faces of $ABCD$ and the midpoint M_{PQ} of PQ, respectively. So the resulting sphere centered at M_{PQ} passes through the projections of P. Similarly, it passes through the projections of Q too. Hence we may formulate the following proposition which is the generalization of the respective result for the triangle.

Corollary 2. *The eight projections of two isogonal conjugate points in a tetrahedron onto its faces lie on a (pedal) sphere centered at the midpoint of the segment joining the two isogonal conjugate points. Moreover, the projections onto the same face are diametrically opposite in the intersection circle of the sphere and the face.*

It is not difficult to see that the reasonings in the proof of Theorem 1 and Remark 1 are reversible. This lets us formulate the following corollary.

Corollary 3. *Points P and Q are isogonal conjugates in a tetrahedron iff their pedal spheres coincide.*

4. Auxiliary facts

When working with spheres it is useful to study the analogous configurations (if existing) for circles on the plane. Many properties of circles on the plane are valid for spheres too. Particularly, this concerns to inversion.

Define the *angle between two intersecting spheres* as the angle between their tangent planes in a point of their intersection. This definition can be extended for a sphere and a plane too. Indeed, one may think of the plane as a special case of a sphere whose center is at infinity.

It is well-known that angles between circles are preserved under inversion. Naturally, this is the case with spheres too.

Proposition 4.1. *Angles between spheres are preserved under inversion.*

Proof. Let us be given two intersecting spheres γ_1, γ_2 and an inversion sphere Ω. Consider their section with the plane π through their centers. Then the angle between γ_1 and γ_2 is equal to the angle between the circles $\gamma_1 \cap \pi$ and $\gamma_2 \cap \pi$. Also note that for these two circles inversion in Ω is equivalent to inversion in $\Omega \cap \pi$. Thus, since the angles between circles are preserved under inversion on plane, the angles between the spheres constructed on these circles having the same center and radius are also preserved. □

The second fact that will come handy for the proof of the main result is as well a generalization of a plane construction.

Proposition 4.2. *Let us be given a sphere Ω and a circle σ on it, as well as a sphere Γ passing through σ. Suppose that Γ makes equal angles with Ω and the plane of σ. Then the center of Γ lies on Ω.*

Proof. Let Q be the center of Γ. Consider a section of the construction with a plane through the centers of Γ and Ω. Let the sections of these spheres be the circles γ and ω, respectively, and let S_1, S_2 be the points of intersection of σ with the secant plane; see Figure 2. Let also UV be the diameter in ω perpendicular to S_1S_2 and t be the tangent at S_2 to ω. Without loss of generality we may assume that Q and V lie in the same side of S_1S_2.

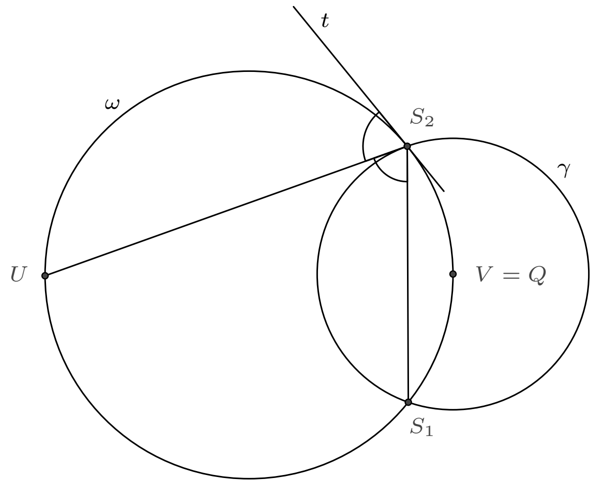

Figure 2.

By simple angle chasing one may check that US_2 bisects the angle between t and S_1S_2. This means that γ should touch US_2. Similarly, γ should touch US_1, too, so the center of γ coincides with V and thus lies on Ω. □

We will utilize the following property of isogonal conjugate points in a triangle as well.

Proposition 4.3. *Let X and Y be isogonal conjugate points in the triangle ABC. Let M and N be the midpoints of arcs AB not containing and containing C, respectively. Let γ_M and γ_N be the circles centered at M and N, respectively, passing through A, B. Then:*

(i) *each of the circles γ_M and γ_N makes equal angles with (bisects) the circles ABX and ABY;*
(ii) *N and M are respectively the external and internal homothety centers of the circles ABX and ABY.*

Proof. Let I be the incenter of ABC, and S, T be the circumcenters of ABX, ABY, respectively; see Figure 3. Recall that I lies on γ_M.

Since AI bisects $\angle XAY$ and BI bisects $\angle XBY$, easy angle chasing yields that AM bisects $\angle TAS$. Hence γ_M bisects the circles ABX and ABY. But $AN \perp AM$, so AN bisects $\angle TAS$ too and γ_N also bisects the circles ABX and ABY. Thus part (i) is proved.

By the bisector property, $\frac{SM}{MT} = \frac{SA}{AT} = \frac{SN}{NT}$. This proves part (ii). □

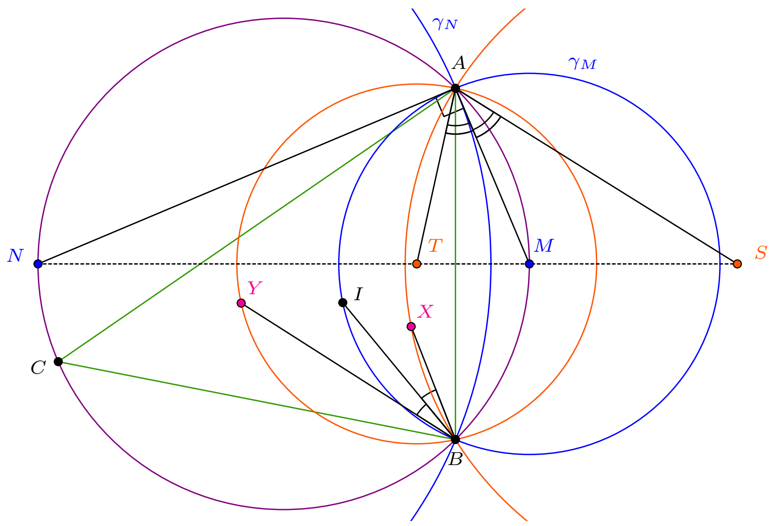

FIGURE 3.

Proposition 4.4. *Let us be given two spheres Ω_1 and Ω_2 intersecting each other through the circle σ. Let S be the center of their external homothety. Then the sphere Γ centered at S and passing through σ bisects the spheres Ω_1 and Ω_2.*

Proof. Consider a section of the construction by any plane π passing through the centers of Ω_1 and Ω_2; see Figure 4. Let $\omega_i = \Omega_i \cap \pi, i \in \{1, 2\}$, and $\gamma = \Gamma \cap \pi$. Let A be one of the points of intersection of ω_1 and ω_2. Let SA intersect ω_1 second time at B.

By known properties of homothety, the tangents of ω_1 at B and of ω_2 at A are parallel. On the other hand, SA makes equal angles with the tangents of ω_1 at A and B. Thus SA bisects the angle between the tangents of ω_1 and ω_2 at A. Equivalently, this angle is bisected by the tangent at A to γ, as needed. □

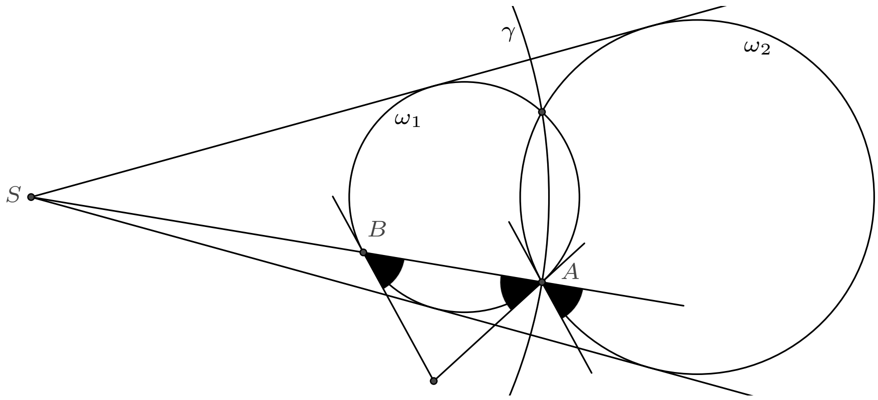

Figure 4.

5. Isogonal conjugates symmetric in bimedians

From here on we will denote by $ABCD$ our isosceles tetrahedron, by Ω its circumsphere, and by O its center.

In this section we will heavily rely on coordinate chasing (though incorporating it with some crucial geometric reasonings) and will use several quantitative characteristics of the tetrahedron in terms of its coordinates. Namely, embed $ABCD$ into the Cartesian coordinate system $Oxyz$ so that

$$A = (-a, b, c),\ B = (a, -b, c),\ C = (a, b, -c),\ D = (-a, -b, -c) \qquad (1)$$

for some $a, b, c \in \mathbb{R} \setminus \{0\}$.

The following proposition defines pretty much all the values that we need.

Proposition 5.1. *In $ABCD$*

(i) *if S is the area of a face*

$$S = 2\sqrt{a^2b^2 + b^2c^2 + c^2a^2};$$

(ii) *if d is the distance from O to a face*

$$d = \frac{2|abc|}{S};$$

(iii) *if θ is half the angle of dihedron CD (or equivalently AB)*

$$\sin\theta = \frac{d}{|c|}.$$

Proof. (i) Taking into account that the sides of ABC are equal to $AB = 2\sqrt{a^2 + b^2}$ etc. and utilizing Heron's formula, after some manipulations we find the presented formula for S.

(ii) Note that the volume of $ABCD$ is $[ABCD] = \dfrac{4dS}{3}$. On the other hand, $[ABCD]$ is one third of the volume of the circumscribed parallelepiped, which is rectangular in case of isosceles tetrahedron, i.e., $[ABCD] = \dfrac{8|abc|}{3}$. Hence we find the presented value for d.

(iii) Let M be the midpoint of CD and Q be the circumcenter of ACD. Then $\theta = \angle OMQ$, so

$$\sin\theta = \frac{OQ}{OM} = \frac{d}{|c|}.$$

□

Recall that a *bimedian* of tetrahedron is a line joining midpoints of opposite edges. We will denote by ℓ_A, ℓ_B, ℓ_C the bimedians of $ABCD$ joining the midpoints of edges DA and BC, DB and AC, DC and AB, respectively.

We will need the following fact too to prove the upcoming theorem.

Lemma 4. *Let X be any point of the space. Let P and Q be its projections on ACD and BCD, respectively. Also let R be its projection on ℓ_C. Then $PR = RQ$.*

Proof. Let S be the projection of X on the bisector plane of dihedron CD and let T be the projection of S on CD; see Figure 5. Then X, P, S, Q, T lie on a circle with diameter XT. Since TS bisects $\angle PTQ$, we get that $PS = SQ$. Hence, from $RS \perp PQR$ we deduce that $PR = RQ$. □

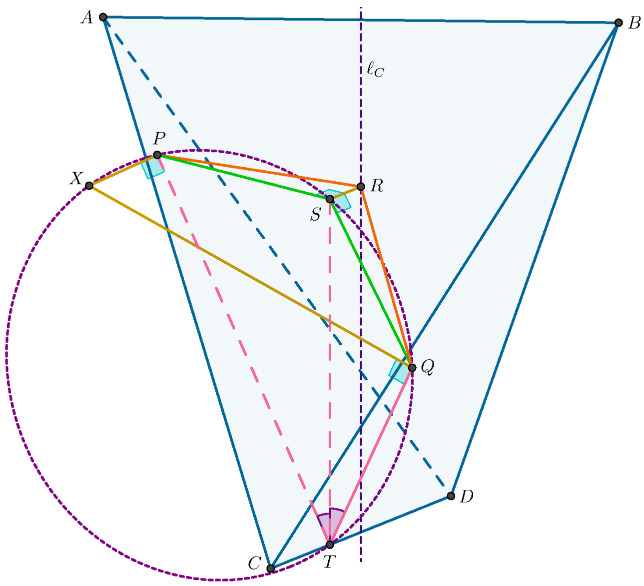

FIGURE 5.

Theorem 5. *Let ℓ be a bimedian of $ABCD$. Then the pairs of isogonal conjugate points in $ABCD$ which are symmetric with respect to ℓ form a hyperbolic paraboloid.*

Proof. Without loss of generality we may assume that $\ell = \ell_C$. Let P and Q be isogonal conjugate points symmetric in ℓ. Then their midpoint M lies on ℓ. According to Corollary 2, the pedal spheres of P and Q coincide and have the center M. Denote this sphere by Γ.

Since M lies on the bisector of dihedron CD, the circles γ_A and γ_B cut from Γ by the planes BCD and ACD, respectively, are symmetric in the bisector of dihedron CD.

The projections of P and Q on BCD lie on γ_A, so P and Q lie on the straight cylinder $\mathfrak{C}_A$ based on γ_A. Similarly, P and Q lie on the straight cylinder $\mathfrak{C}_B$ based on γ_B. On the other hand, P and Q lie on a plane π perpendicular to ℓ. Thus P and Q lie on the ellipses $\pi \cap \mathfrak{C}_A$ and $\pi \cap \mathfrak{C}_B$. However these ellipses coincide since both of them have the center M and minor axes parallel to CD. Name this ellipse ε_1.

Repeating the reasoning above for the dihedron AB, we find out that P and Q lie on the ellipse ε_2 defined similarly. Hence P and Q are an opposite pair of the points of intersection $\varepsilon_1 \cap \varepsilon_2$.

Now recall the embedding (1). Then $M = (0, 0, z_0)$ for some $z_0 \in \mathbb{R}$. Note that ℓ coincides with Oz and the other two bimedians of $ABCD$ coincide with Ox and Oy.

Let d be the distance from O to the faces of $ABCD$. Then it is easy to see that the distances d_1 and d_2 from M to ACD and ABC, respectively, are $\left|\dfrac{z_0 + c}{c}\right| d$ and $\left|\dfrac{z_0 - c}{c}\right| d$.

If r_1 and r_2 are the radii of γ_B and $\gamma_C = \Gamma \cap ABD$, respectively, then

$$r_1 = \sqrt{r^2 - d_1^2}, \quad r_2 = \sqrt{r^2 - d_2^2}.$$

Clearly the minor axis of ε_1 is equal to r_1, while its major axis is $\dfrac{r_1}{\sin\theta}$, where θ is half the angle of dihedron CD (or equivalently AB).

Note that the minor axis of ε_1 is parallel to CD. Thus ε_1 can be obtained from the ellipse

$$\begin{cases} \dfrac{x^2}{r_1^2/\sin^2\theta} + \dfrac{y^2}{r_1^2} = 1, \\ z = z_0, \end{cases}$$

by rotating it around Oz in the negative direction (from y to x) in the angle $\varphi = \arctan\left(\frac{a}{b}\right)$. This gives the following equations for ε_1:

$$\begin{cases} (x\cos\varphi - y\sin\varphi)^2 \sin^2\theta + (x\sin\varphi + y\cos\varphi)^2 = r_1^2, \\ z = z_0. \end{cases}$$

Similarly, ε_2 is given by

$$\begin{cases}(x\cos\varphi + y\sin\varphi)^2\sin^2\theta + (-x\sin\varphi + y\cos\varphi)^2 = r_2^2,\\ z = z_0.\end{cases}$$

Subtracting these equations we get the following equations which hold for the points of $\varepsilon_1 \cap \varepsilon_2$:

$$\begin{cases}-4xy\sin\varphi\cos\varphi\cos^2\theta = r_2^2 - r_1^2,\\ z = z_0.\end{cases} \tag{2}$$

We have

$$\sin\varphi = \frac{a}{\sqrt{a^2+b^2}}, \quad \cos\varphi = \frac{b}{\sqrt{a^2+b^2}},$$

$$\begin{aligned} r_2^2 - r_1^2 &= d_1^2 - d_2^2 \\ &= \frac{(z_0+c)^2 - (z_0-c)^2}{c^2}d^2 = \frac{4z_0}{c}d^2. \end{aligned}$$

Using these as well as our calculated values in Proposition 5.1, we get

$$\cos^2\theta = 1 - \frac{d^2}{c^2} = \frac{c^2(a^2+b^2)}{a^2b^2+b^2c^2+c^2a^2}$$

and (2) simplifies to

$$\begin{cases}-xy = \dfrac{ab}{c}z_0,\\ z = z_0.\end{cases}$$

Now letting z_0 vary we get the hyperbolic paraboloid $\mathcal{H}$ containing all the isogonal conjugate pairs P and Q symmetric in ℓ:

$$z = -\frac{c}{ab}xy.$$

However we still need to show that any pair of points P' and Q' on $\mathcal{H}$ symmetric in ℓ are isogonal conjugates. To this end we will use the equivalence of isogonal conjugate points from Corollary 3.

Let P_A, P_B, P_C, P_D be the projections of P' on BCD, CDA, DAB, ABC, respectively. Define similarly Q_A, Q_B, Q_C, Q_D. Also let M' be the projection of P' (or equivalently of Q') on ℓ.

According to Lemma 4, $P_AM' = M'P_B$. In view of symmetry in ℓ_C, we get that all the four points P_A, P_B, Q_A, Q_B are equidistant from M'. Similarly, P_C, P_D, Q_C, Q_D are equidistant from M' too. So if we prove that $P_BM' = P_CM'$, then the pedal spheres of P' and Q' will coincide and we will be done by Corollary 3.

We have the following equations for the faceplanes:

$$ACD : \frac{x}{a} - \frac{y}{b} + \frac{z}{c} + 1 = 0, \qquad ABD : \frac{x}{a} + \frac{y}{b} - \frac{z}{c} + 1 = 0.$$

If $P' = (x_0, y_0, z_0)$, then

$$M'P_B^2 = \left(x_0 - \frac{p+q}{ar}\right)^2 + \left(y_0 + \frac{p+q}{br}\right)^2 + \left(\frac{p+q}{cr}\right)^2,$$

$$M'P_C^2 = \left(x_0 - \frac{-p+q}{ar}\right)^2 + \left(y_0 - \frac{-p+q}{br}\right)^2 + \left(\frac{-p+q}{cr}\right)^2,$$

where

$$p = -\frac{y_0}{b} + \frac{z_0}{c}, \qquad q = \frac{x_0}{a} + 1, \qquad r = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Then

$$\begin{aligned} \frac{M'P_B^2 - M'P_C^2}{4} &= -\frac{p}{a}\left(x_0 - \frac{q}{ar}\right) + \frac{q}{b}\left(y_0 + \frac{p}{br}\right) + \frac{p}{c}\cdot\frac{q}{cr} \\ &= -\frac{x_0}{a}p + \frac{y_0}{b}q + pq \\ = -\frac{x_0}{a}\left(-\frac{y_0}{b} + \frac{z_0}{c}\right) + \frac{y_0}{b}\left(\frac{x_0}{a} + 1\right) &+ \left(-\frac{y_0}{b} + \frac{z_0}{c}\right)\left(\frac{x_0}{a} + 1\right) \\ &= \frac{x_0 y_0}{ab} + \frac{z_0}{c}. \end{aligned}$$

Since $P' = (x_0, y_0, z_0)$ lies on $\mathcal{H}$, the last expression is zero, as needed. □

Remark 2. Note that the other two bimedians different from ℓ lie on $\mathcal{H}$ ($z = 0$ yields $x = 0$ or $y = 0$). Also note that the vertices of $ABCD$ lie on $\mathcal{H}$ as well.

Denote by $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$ the hyperbolic paraboloids defined above corresponding to bimedians ℓ_A, ℓ_B, ℓ_C.

Proposition 5.2. *For any point $P \in \Omega$ the circles ABP and CDP touch iff $P \in \mathcal{H}_C$.*

Proof. Recall the embedding (1). Let $P = (x, y, z)$.

We need to check if the line $ABP \cap CDP$ touches Ω, i.e., is perpendicular to OP. Or, equivalently, whether (x, y, z) lies in the linear span of the normal vectors of ABP and CDP. It is not hard to check that these normals are

$$\begin{bmatrix} b(z-c) \\ a(z-c) \\ -(bx+ay) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b(z+c) \\ -a(z+c) \\ -bx+ay \end{bmatrix}.$$

Their linear span coincides with the linear span of their half-sum and half-difference:

$$\begin{bmatrix} bz \\ -ac \\ -bx \end{bmatrix}, \quad \begin{bmatrix} bc \\ -az \\ ay \end{bmatrix}.$$

We need to check the singularity of matrix

$$M = \begin{pmatrix} bz & bc & x \\ -ac & -az & y \\ -bx & ay & z \end{pmatrix}.$$

Taking into account the fact that $P \in \Omega$, i.e., $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$, we get

$$\begin{aligned} \det(M) &= -abz(x^2 + y^2 + z^2 - c^2) - xyc(a^2 + b^2) \\ &= -(a^2 + b^2)(abz + xyc), \end{aligned}$$

which is zero iff $abz + xyc = 0$, that is, when $P \in \mathcal{H}_C$. □

Remark 3. Obviously, similar results hold for $\mathcal{H}_A$ and $\mathcal{H}_B$ too.

6. Isogonal conjugation on the circumsphere

In this section too $ABCD$ is an isosceles tetrahedron. All the notations are preserved.

Theorem 6. *Let $X \notin \{A, B, C, D\}$ be a point on Ω. Then the isogonal conjugate of X with respect to $ABCD$ lies on Ω.*

Proof. Let Y be the second intersection point of Ω and the line joining D with the isogonal conjugate of X. Points X and Y lie in isogonal planes with respect to each of the dihedrons DA, DB, DC. We need to prove that they lie in isogonal planes in the dihedrons AB, BC, CA as well.

Invert with center D; see Figure 6. For any point Z of space denote its image by Z_1.

Clearly A_1, B_1, C_1, X_1, Y_1 are coplanar (they lie in the image plane of Ω). By property (i) of isosceles tetrahedron we get

$$\angle B_1A_1D = \angle ABD = \angle ACD = \angle C_1A_1D \qquad (*)$$

and two other similar equalities. Taking into account this, as well as the fact that X_1 and Y_1 lie in isogonal planes in dihedrons DA_1, DB_1, DC_1, we deduce that X_1 and Y_1 are isogonal conjugates in $A_1B_1C_1$.

Recall that by property (ii) O is also the incenter of $ABCD$. This means that the plane eO is the bisector of the dihedron e for any edge e of $ABCD$.

It is easy to see that O_1 is the reflection of D in $A_1B_1C_1$ (consider the diametrically opposite point to D). Angles between spheres are preserved under inversion (Proposition 4.1), so, since the plane ABO makes equal angles with ABD and ABC, the sphere $A_1B_1O_1D$ makes equal angles with the plane A_1B_1D and the sphere $A_1B_1C_1D$. According to Proposition 4.2, this is possible only in the case when the circumcenter N of $A_1B_1O_1D$ lies on the sphere $A_1B_1C_1D$. Tetrahedron $A_1B_1O_1D$ is symmetric with respect to the plane $A_1B_1C_1$, so N lies on the circle $A_1B_1C_1$.

Note that N is the midpoint of arc $A_1C_1B_1$. Indeed, $NA_1 = NB_1$, so N is the midpoint of either one of the two arcs A_1B_1. By property (iii), the angles in (*) are acute. This means that the projection of D on $A_1B_1C_1$ lies on the triangle $A_1B_1C_1$, so the dihedrons A_1B_1, B_1C_1 and C_1A_1 in $DA_1B_1C_1$ are acute. Thus N and C_1 lie on the same side of A_1B_1 and N is on the arc $A_1B_1C_1$.

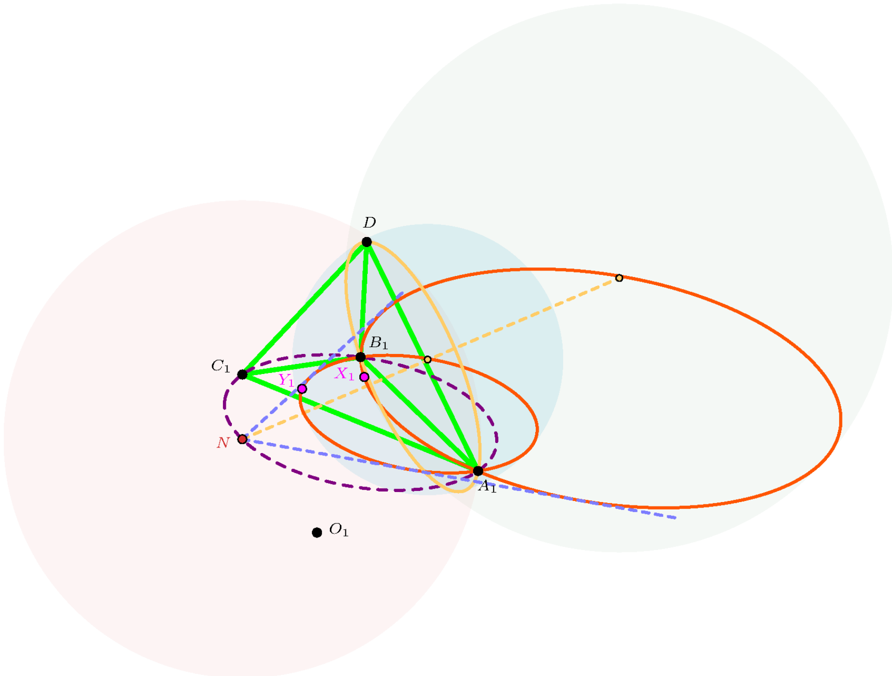

Figure 6.

According to Proposition 4.3, N is the external homothety center of circles $A_1B_1X_1$ and $A_1B_1Y_1$. On the other hand, $NA_1 = NB_1 = NC_1$, so the line passing through the circumcenters of $A_1B_1DX_1$ and $A_1B_1DY_1$ passes through N too. This and the previous argument lead to the conclusion that N is the external homothety center of the spheres $A_1B_1DX_1$ and $A_1B_1DY_1$ too.

In view of Proposition 4.4, the sphere $A_1B_1O_1D$ with center N passing through A_1 makes equal angles with the spheres $A_1B_1DX_1$ and $A_1B_1DY_1$. Therefore its preimage plane ABO also makes equal angles with the planes ABX and ABY.

Similarly, X and Y are in isogonal planes with respect to the dihedrons BC and AC too, whence the conclusion follows. □

In his proof of Theorem 6, Ilya Bogdanov, professor at Moscow Institute of Physics and Technology, found the following interesting construction of isogonal conjugate points on the circumsphere of isosceles tetrahedron (I. Bogdanov, personal communication, January 17, 2020).

Proposition 6.1. *Let $X \notin \{A, B, C, D\}$ be a point on Ω. Let X_A, X_B, X_C be the second intersection points of circles XDA and XBC, XDB and XCA, XDC and XAB, respectively. Then the reflections of X_A in ℓ_A, X_B in ℓ_B, and X_C in ℓ_C coincide with the isogonal conjugate of X.*

Proof. Let Y be the isogonal conjugate of X. Let X' and X'_A be the reflections of X and X_A in ℓ_A, respectively. We will prove that $Y = X'_A$. Similarly, the reflections of X in ℓ_B and ℓ_C will coincide with Y too.

Note that the planes $DAXX_A$ and $DAX'X'_A$, as well as $BCXX_A$ and $BCX'X'_A$, are pairs of isogonals. This means that Y lies on $X'X'_A$. According to Theorem 6, Y also lies on Ω. Hence Y should coincide with either X' or X'_A.

If $Y = X'_A$, then there is nothing to prove. Else $Y = X'$. This means that Y and X lie on $\mathcal{H}_A$. According to Proposition 5.2, $X = X_A$ and $Y = X' = X'_A$, as desired. □

Remark 4. To prove this result Bogdanov used inversion with respect to one of the points $\ell_A \cap \Omega$. Points A, B, C, D were mapped to the vertices of a parallelogram and angle chasing finished the proof. However we chose a different approach based on the results already proven.

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A new proof for a series similar to the Sandham–Yeung quadratic series

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Abstract. In both wonderful papers [1], [2], Ovidiu Furdui and Alina Sîntămărian have recently given three new proofs for the following remarkable Sandham–Yeung quadratic series formula

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\frac{17}{4}\zeta(4),$$

where $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$ denotes the n-th classical harmonic number. In this paper, we develop a new concise approach to the evaluation of the following classical quadratic harmonic series formula

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}=\frac{7}{2}\zeta(5)-\zeta(2)\zeta(3).$$

The proof utilizes a difficult definite integral formula due to Ali Shather.

Keywords: Classical harmonic numbers, linear harmonic sums, nonlinear harmonic sums, logarithmic integrals, Riemann zeta function, polylogarithm function, infinite summation formulas.

MSC: Primary 40A25; Secondary 11M06.

1. Introduction

In this paper, we give a new proof of the following classical quadratic harmonic series formula:

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}=\frac{7}{2}\zeta(5)-\zeta(2)\zeta(3). \tag{1}$$

The motivation for this paper comes from the alternative proofs of Sandham–Yeung quadratic series $\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2$, in the recent *Gazeta Matematică, Seria A* articles [1], [2].

There are many proofs of (1) of varying levels of sophistication in the literature. But most of the proofs use advanced tools. For example, the proof in [3, p. 24] invokes contour integration, while in [4] the author proves (1) using the series formula $\sum_{n\geq 1}\frac{1}{n^3}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)=-\frac{9}{2}\zeta(5)+3\zeta(2)\zeta(3)$ and the two logarithmic integrals $\int_0^1\ln(1-x)x^{n-1}\mathrm{d}x$ and $\int_0^1\ln^2(1-x)x^{n-1}\mathrm{d}x$.

Now, we construct an alternative approach to the classical quadratic harmonic series (1), which solely relies on the calculation of the following logarithmic integrals

$$\int_0^1\frac{\mathrm{Li}_3(x)\ln(x)}{x}\mathrm{d}x \quad\text{and}\quad \int_0^1\frac{\ln(x)\ln^3(1-x)}{x}\mathrm{d}x.$$

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Throughout this paper, H_n denotes the n-th classical harmonic number defined by $H_n = \sum_{k=1}^{n} \frac{1}{k}$, $\zeta(s)$ denotes the Riemann zeta function, which is defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $\Re(s) > 1$, and $\mathrm{Li}_n(x)$ denotes the polylogarithm function, which is defined for $|x| \leq 1$ by $\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}, n \in \mathbb{N}, n \geq 2$.

To evaluate (1), we shall establish some lemmas.

Lemma 1. *Let $n > 0$. The following equality holds*

$$\int_0^1 x^{n-1} \ln(x)\, \mathrm{d}x = -\frac{1}{n^2}.$$

Proof. We have, using integration by parts, that

$$\int_0^1 x^{n-1} \ln(x)\, \mathrm{d}x = \left[\frac{x^n \ln(x)}{n}\right]_0^1 - \int_0^1 \frac{x^{n-1}}{n} \mathrm{d}x = -\frac{1}{n^2}.$$

□

Lemma 2. *Let $n > -1$. The following equality holds*

$$\int_0^1 x^n \ln^3(x)\, \mathrm{d}x = -\frac{6}{(n+1)^4}.$$

Proof. We have, using integration by parts, that

$$\begin{aligned}
\int_0^1 x^n \ln^3(x)\, \mathrm{d}x &= \left.\frac{x^{n+1} \ln^3(x)}{n+1}\right|_0^1 - \frac{3}{n+1} \int_0^1 \ln^2(x) x^n \mathrm{d}x \\
&= \left.\frac{x^{n+1} \ln^3(x)}{n+1}\right|_0^1 \\
&\quad - \frac{3}{n+1}\left(\left.\frac{x^{n+1} \ln^2(x)}{n+1}\right|_0^1 - \frac{2}{n+1} \int_0^1 \ln(x) x^n \mathrm{d}x\right) \\
&= -\frac{6}{(n+1)^4}.
\end{aligned}$$

□

Lemma 3. *The following equality holds*

$$\int_0^1 \ln^3(1-x)(1-x)^n\, \mathrm{d}x = -\frac{6}{(n+1)^4}.$$

Proof. We make the substitution $u = 1 - x$ and it follows that

$$\int_0^1 \ln^3(1-x)(1-x)^n\, \mathrm{d}x = \int_0^1 \ln^3(u)\, u^n \mathrm{d}u = -\frac{6}{(n+1)^4}.$$

□

Lemma 4. *The following identity holds*

$$\sum_{n=1}^{\infty} \frac{H_n^2 x^{n-1}}{n} = \frac{\mathrm{Li}_3(x)}{x} - \frac{\mathrm{Li}_2(x)\ln(1-x)}{x} - \frac{\ln^3(1-x)}{3x}, \quad x \in [-1,1).$$

Proof. In [5, Lemma 2.2], Seán Mark Stewart had evaluated the generating function for the sequence $\{H_n^2/n\}_{n\geq 1}$ as follows:

$$\sum_{n=1}^{\infty} \frac{H_n^2 x^n}{n} = \mathrm{Li}_3(x) - \mathrm{Li}_2(x)\ln(1-x) - \frac{1}{3}\ln^3(1-x), \quad x \in [-1,1). \tag{2}$$

Dividing by x on both sides of (2), gives us the desired result.

□

Lemma 5. *The following equality holds*

$$\int_0^1 \frac{\mathrm{Li}_3(x)\ln(x)}{x}\mathrm{d}x = -\zeta(5).$$

Proof.

$$\begin{aligned}
\int_0^1 \frac{\mathrm{Li}_3(x)\ln(x)}{x}\mathrm{d}x &= \int_0^1 \frac{\ln(x)}{x}\mathrm{Li}_3(x)\,\mathrm{d}x \\
&= \int_0^1 \frac{\ln(x)}{x}\left(\sum_{k=1}^{\infty}\frac{x^k}{k^3}\right)\mathrm{d}x \\
&= \sum_{k=1}^{\infty}\frac{1}{k^3}\left(\int_0^1 x^{k-1}\ln(x)\,\mathrm{d}x\right) \\
&= \sum_{k=1}^{\infty}\frac{1}{k^3}\left(-\frac{1}{k^2}\right) \\
&= -\sum_{k=1}^{\infty}\frac{1}{k^5} \\
&= -\zeta(5).
\end{aligned}$$

□

Lemma 6. *The following equality holds*

$$\int_0^1 \frac{\ln(x)\ln^3(1-x)}{x}\mathrm{d}x = 12\zeta(5) - 6\zeta(2)\zeta(3).$$

Proof. Using the well-known generating function for the classical harmonic number

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x}, \quad -1 \leq x < 1, \tag{3}$$

and changing x by $1-x$ on both sides of (3), gives

$$\sum_{n=1}^{\infty} H_n (1-x)^n = -\frac{\ln(x)}{x}.$$

Then,

$$\begin{aligned}
\int_0^1 \frac{\ln(x)\ln^3(1-x)}{x}\mathrm{d}x &= \int_0^1 \ln^3(1-x)\left(-\sum_{n=1}^{\infty} H_n (1-x)^n\right)\mathrm{d}x \\
&= -\sum_{n=1}^{\infty} H_n \left(\int_0^1 \ln^3(1-x)(1-x)^n\,\mathrm{d}x\right) \\
&= -\sum_{n=1}^{\infty} H_n \left(-\frac{6}{(n+1)^4}\right) \\
&= 6\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^4} \\
&= 6\left(\sum_{n=1}^{\infty} \frac{H_{n+1}}{(n+1)^4} - \sum_{n=1}^{\infty} \frac{1}{(n+1)^5}\right) \\
&= 6\left(\sum_{i=1}^{\infty} \frac{H_i}{i^4} - \sum_{i=1}^{\infty} \frac{1}{i^5}\right) \\
&= 6\left(\sum_{i=1}^{\infty} \frac{H_i}{i^4} - \zeta(5)\right).
\end{aligned}$$

In [3, p. 16], Flajolet and Salvy have listed the following linear harmonic sum

$$\sum_{i=1}^{\infty} \frac{H_i}{i^4} = 3\zeta(5) - \zeta(2)\zeta(3).$$

It follows that

$$\begin{aligned}
\int_0^1 \frac{\ln(x)\ln^3(1-x)}{x}\mathrm{d}x &= 6\left(3\zeta(5) - \zeta(2)\zeta(3) - \zeta(5)\right) \\
&= 12\zeta(5) - 6\zeta(2)\zeta(3).
\end{aligned}$$

□

Theorem 7. *The following identity holds*

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} = \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3).$$

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} &= \sum_{n=1}^{\infty} \frac{H_n^2}{n}\left(-\int_0^1 x^{n-1}\ln(x)\,\mathrm{d}x\right) \\
&= -\int_0^1 \ln(x)\left(\sum_{n=1}^{\infty} \frac{H_n^2\,x^{n-1}}{n}\right)\mathrm{d}x \\
&= -\int_0^1 \frac{\mathrm{Li}_3(x)\ln(x)}{x}\mathrm{d}x + \int_0^1 \frac{\mathrm{Li}_2(x)\ln(x)\ln(1-x)}{x}\mathrm{d}x \\
&\quad + \frac{1}{3}\int_0^1 \frac{\ln(x)\ln^3(1-x)}{x}\mathrm{d}x \\
&= \zeta(5) + 4\zeta(5) - 2\zeta(2)\zeta(3) + \int_0^1 \frac{\mathrm{Li}_2(x)\ln(x)\ln(1-x)}{x}\mathrm{d}x \\
&= 5\zeta(5) - 2\zeta(2)\zeta(3) + \int_0^1 \frac{\mathrm{Li}_2(x)\ln(x)\ln(1-x)}{x}\mathrm{d}x.
\end{aligned}$$

Very recently, in an issue of the Romanian Mathematical Magazine [6], Ali Shather has obtained the following definite integral formula

$$\int_0^1 \frac{\mathrm{Li}_2(x)\ln(x)\ln(1-x)}{x}\mathrm{d}x = \zeta(2)\zeta(3) - \frac{3}{2}\zeta(5),$$

which shows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} &= 5\zeta(5) - 2\zeta(2)\zeta(3) + \zeta(2)\zeta(3) - \frac{3}{2}\zeta(5) \\
&= \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3),
\end{aligned}$$

and Theorem 7 is proved. □

We used in the proofs of Lemma 5, Lemma 6 and Theorem 7, Bernstein's theorem [7, Thm. 9.30, p. 243] which justifies interchanging the order of integration and summation because of the positivity of the coefficients.

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19th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2025

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Abstract. The 19th South Eastern European Mathematical Olympiad for University Students (SEEMOUS 2025) was held from March 4 to March 9, 2025, in Korçë, Albania. This paper presents the competition problems along with their solutions as provided by the authors. Additionally, alternative solutions contributed by jury members or contestants are included.

Keywords: Frobenius norm, Gamma function, improper integrals, Lebesgue dominated convergence theorem, normal matrices, Perron–Frobenius theorem, positive matrices, series, summation by parts, Taylor's formula.

MSC: 15A21, 40A05, 26A24, 26A42.

The 19th South Eastern European Mathematical Olympiad for University Students with International Participation (SEEMOUS 2025) was hosted between 4th and 9th of March by the Fan S. Noli University of Korçë, Albania, with the support of the Mathematical Society of South Eastern Europe (MASSEE) and of the Albanian Mathematical Association. This competition is addressed to students in the first or second year of undergraduate studies, from universities in countries that are members of the MASSEE, or from invited countries that are not affiliated to MASSEE.

A number of 111 students participated in the contest, representing 27 universities from Albania, Bulgaria, Greece, North Macedonia, Romania, and Turkmenistan. The jury awarded 11 gold medals, 23 silver medals, and 36 bronze medals. Five contestants achieved the maximum possible score, including three Romanian students: Ștefan–George Ghinescu (National University of Science and Technology Politehnica Bucharest), Ana Negoiță and Vlad

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Perpelea (University of Bucharest). Moreover, the University of Bucharest won the title of Best University.

We present the competition problems and their solutions as given by the corresponding authors, together with alternative solutions provided by members of the jury or by contestants.

Problem 1. Let A be an $n\times n$ matrix with positive elements and two vectors $\mathbf{u}$, $\mathbf{v}\in\mathbb{R}^n$, also with positive elements, such that $A\mathbf{u}=\mathbf{v}$ and $A\mathbf{v}=\mathbf{u}$. Prove that $\mathbf{u}=\mathbf{v}$.

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Author's solution. We analyze two cases, depending on whether the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent or not.

If $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, there exists $\lambda>0$, such that $\mathbf{v}=\lambda\mathbf{u}$. In this case, we have $\lambda\mathbf{u}=\mathbf{v}=A\mathbf{u}=A(\lambda^{-1}\mathbf{v})=\lambda^{-1}A\mathbf{v}=\lambda^{-1}\mathbf{u}$. Therefore, $\lambda^2=1$ and, as $\lambda>0$, we get that $\lambda=1$, which leads to $\mathbf{u}=\mathbf{v}$.

Now, consider the case when $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, with $\mathbf{u}=(u_1,u_2,\ldots,u_n)^T$ and $\mathbf{v}=(v_1,v_2,\ldots,v_n)^T$. Let $t=\min\limits_{1\le i\le n}\frac{u_i}{v_i}$. Clearly, $u_i\ge tv_i$, for all $i\in\overline{1,n}$. Also, there exist $k,l\in\{1,2,\ldots,n\}$ such that $u_k=tv_k$ and $u_l>tv_l$, since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. We consider the vector $\mathbf{w}=\mathbf{u}-t\mathbf{v}$. Let $\mathbf{w}=(w_1,w_2,\ldots,w_n)^T$. The elements of $\mathbf{w}$ are all non-negative. In particular, $w_k=0$ and $w_l>0$. On the other hand, since $A^2\mathbf{u}=\mathbf{u}$ and $A^2\mathbf{v}=\mathbf{v}$, we have that $A^2\mathbf{w}=A^2(\mathbf{u}-t\mathbf{v})=\mathbf{u}-t\mathbf{v}=\mathbf{w}$. All the elements of the matrix A^2 are positive and, since all the elements of the vector $\mathbf{w}$ are non-negative, with at least one of them positive, we obtain that the vector $A^2\mathbf{w}$ should have only positive elements. This contradicts the fact that the vector $\mathbf{w}=A^2\mathbf{w}$ has at least one zero element.

Alternative solution. *This solution was given by Vlad–Andrei Perpelea, from University of Bucharest, Romania (contestant).*

From $A^2\mathbf{u}=\mathbf{u}$, we deduce that 1 is an eigenvalue of A^2, a matrix with positive elements, and $\mathbf{u}$ is a corresponding eigenvector, also with positive elements. Therefore, thanks to the Perron–Frobenius theorem (see [2], Theorem 8.4.4 and Problem 8.4.P15, and [1], Theorem 12.1.7), we deduce that $\lambda_r=1$ is the Perron root of the matrix A^2. Moreover, any other eigenvalue of A^2 has the absolute value strictly less than 1. It follows, also from the Perron–Frobenius theorem, that any eigenvalue of the matrix A which is not the Perron root has the absolute value less than 1.

Finally, suppose that $\mathbf{u}\neq\mathbf{v}$. Then, from $A\left(\mathbf{u}-\mathbf{v}\right)=A\mathbf{u}-A\mathbf{v}=\mathbf{v}-\mathbf{u}=-\left(\mathbf{u}-\mathbf{v}\right)$, we obtain that -1 is an eigenvalue of A, which contradicts the last statement of the previous paragraph. Therefore, we have that $\mathbf{u}=\mathbf{v}$.

This problem had 23 *complete solutions in the contest.*

Problem 2. Calculate

$$\lim_{n\to\infty} n \int_0^\infty \mathrm{e}^{-x} \sqrt[n]{\mathrm{e}^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}}\, \mathrm{d}x.$$

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For convenience, we denote by (u_n) the sequence whose limit is to be determined. We show that the limit equals e.

Authors' first solution. The starting point is *Taylor's formula* with *the Lagrange form of the remainder*, applied to the function e^x. It follows that for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$, there exists some $\theta = \theta(x, n) \in (0, 1)$ such that $\mathrm{e}^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}\mathrm{e}^{\theta x}$. This leads to the inequalities $\frac{x^{n+1}}{(n+1)!} < \mathrm{e}^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} < \frac{x^{n+1}}{(n+1)!}\mathrm{e}^x$, which in turn show that $\frac{x^{1+\frac{1}{n}}}{\sqrt[n]{(n+1)!}}\,\mathrm{e}^{-x} < \mathrm{e}^{-x} \sqrt[n]{\mathrm{e}^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}} < \frac{x^{1+\frac{1}{n}}}{\sqrt[n]{(n+1)!}}\mathrm{e}^{-\left(1-\frac{1}{n}\right)x}$. It follows, by integration, that

$$\frac{n}{\sqrt[n]{(n+1)!}} \int_0^\infty x^{1+\frac{1}{n}} \mathrm{e}^{-x}\, \mathrm{d}x < u_n < \frac{n}{\sqrt[n]{(n+1)!}} \int_0^\infty x^{1+\frac{1}{n}} \mathrm{e}^{-\left(1-\frac{1}{n}\right)x}\, \mathrm{d}x. \quad (2a)$$

By the change of variable $y = \left(1 - \frac{1}{n}\right)x$ on the right hand side integral, this becomes $\frac{1}{\left(1-\frac{1}{n}\right)^{2+\frac{1}{n}}} \int_0^\infty y^{1+\frac{1}{n}} \mathrm{e}^{-y}\, \mathrm{d}y$. Finally, using *Euler's Gamma function* Γ, the inequalities (2a) can be further written as

$$\frac{n}{\sqrt[n]{(n+1)!}}\Gamma\left(2 + \frac{1}{n}\right) < u_n < \frac{n}{\sqrt[n]{(n+1)!}} \cdot \frac{1}{\left(1-\frac{1}{n}\right)^{2+\frac{1}{n}}}\Gamma\left(2 + \frac{1}{n}\right). \quad (2b)$$

Passing to the limit, as $n \to \infty$, in (2b) and using that Γ is continuous on $(0, \infty)$, $\Gamma(2) = 1$, and that $\lim\limits_{n\to\infty} \frac{n}{\sqrt[n]{(n+1)!}} = \mathrm{e}$, the desired limit equals e.

Authors' second solution (simplified). This also uses *Taylor's formula* applied to the function e^x, but with the *integral form of the remainder*, which leads to

$$\mathrm{e}^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} = \int_0^x \frac{\mathrm{e}^t}{n!}(x - t)^n\, \mathrm{d}t = \frac{\mathrm{e}^x}{n!} \int_0^x y^n \mathrm{e}^{-y}\, \mathrm{d}y.$$

Consequently,

$$u_n = \frac{n}{\sqrt[n]{n!}} \int_0^\infty \sqrt[n]{\int_0^x y^n \mathrm{e}^{-y}\,\mathrm{d}y}\; \mathrm{e}^{-\left(1-\frac{1}{n}\right)x}\,\mathrm{d}x. \tag{2c}$$

Let $f_n(x) = \sqrt[n]{\int_0^x y^n \mathrm{e}^{-y}\,\mathrm{d}y}\; \mathrm{e}^{-\left(1-\frac{1}{n}\right)x}$, for all $x > 0$ and $n \geq 2$. The aim is to find $L = \lim_{n\to\infty} \int_0^\infty f_n(x)\,\mathrm{d}x$, using the *Lebesgue dominated convergence theorem*, which in turn will give $\lim_{n\to\infty} u_n = L \cdot \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = L\mathrm{e}$.

Let $x > 0$ and $n \geq 2$. Then

$$f_n(x) \leq \sqrt[n]{\int_0^x x^n \mathrm{e}^{-y}\,\mathrm{d}y}\; \mathrm{e}^{-\left(1-\frac{1}{n}\right)x} = x\sqrt[n]{1-\mathrm{e}^{-x}}\; \mathrm{e}^{-\left(1-\frac{1}{n}\right)x} \leq x\mathrm{e}^{-\left(1-\frac{1}{n}\right)x}. \tag{2d}$$

For $0 < \varepsilon < x$,

$$\sqrt[n]{\int_0^x y^n \mathrm{e}^{-y}\,\mathrm{d}y} \geq \sqrt[n]{\int_{x-\varepsilon}^x y^n \mathrm{e}^{-y}\,\mathrm{d}y} \geq \sqrt[n]{\int_{x-\varepsilon}^x (x-\varepsilon)^n \mathrm{e}^{-x}\,\mathrm{d}y} = \sqrt[n]{\varepsilon}(x-\varepsilon)\mathrm{e}^{-\frac{x}{n}},$$

so

$$f_n(x) \geq \sqrt[n]{\varepsilon}(x-\varepsilon)\mathrm{e}^{-x}. \tag{2e}$$

Passing to the limit superior in (2d) and to limit inferior in (2e), it follows that $(x-\varepsilon)\mathrm{e}^{-x} \leq \liminf_n f_n(x) \leq \limsup_n f_n(x) \leq x\mathrm{e}^{-x}$. Letting $\varepsilon \to 0$, leads to $\lim_{n\to\infty} f_n(x) = x\mathrm{e}^{-x}$. Also, (2d) gives $f_n(x) \leq x\mathrm{e}^{-\frac{x}{2}}$. Since $\int_0^\infty x\mathrm{e}^{-\frac{x}{2}}\,\mathrm{d}x < \infty$, it follows that the conditions in the *Lebesgue dominated convergence theorem* are met. Concluding,

$$L = \lim_{n\to\infty} \int_0^\infty f_n(x)\,\mathrm{d}x = \int_0^\infty \left(\lim_{n\to\infty} f_n(x)\right)\,\mathrm{d}x = \int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x = 1,$$

so $\lim_{n\to\infty} u_n = L\mathrm{e} \; = \mathrm{e}$.

Third solution. *This is based on several similar solutions proposed by members of the jury and contestants.*

Using Taylor's formula as in the first solution, u_n can be written as

$$u_n = n\int_0^\infty \mathrm{e}^{-x}\;\sqrt[n]{\frac{x^{n+1}}{(n+1)!}\mathrm{e}^{\theta x}}\,\mathrm{d}x = \frac{n}{\sqrt[n]{(n+1)!}} \int_0^\infty x\mathrm{e}^{-x}\sqrt[n]{x\mathrm{e}^{\theta x}}\,\mathrm{d}x,$$

where $\theta = \theta(x,n) \in (0,1)$. Consider $f_n(x) = x\mathrm{e}^{-x}\sqrt[n]{x\mathrm{e}^{\theta x}}$, for all $x > 0$ and $n \geq 2$. As in the second solution, the argument will be based on the *Lebesgue dominated convergence theorem*.

Since $\sqrt[n]{x} \le \sqrt[n]{x\mathrm{e}^{\theta x}} \le \sqrt[n]{x\mathrm{e}^{x}}$, it follows that $\lim\limits_{n\to\infty} \sqrt[n]{x\mathrm{e}^{\theta x}} = 1$, for all $x > 0$, so $\lim\limits_{n\to\infty} f_n(x) = x\mathrm{e}^{-x}$, for all $x > 0$. Also,

$$0 \le f_n(x) \le x^{1+\frac{1}{n}}\mathrm{e}^{-\left(1-\frac{1}{n}\right)x} \le x^{1+\frac{1}{n}}\mathrm{e}^{-\frac{x}{2}} \le \max\left\{1, x^2\right\}\mathrm{e}^{-\frac{x}{2}},$$

for all $x > 0$ and $n \ge 2$. It is immediate that $\int_0^\infty \max\left\{1, x^2\right\}\mathrm{e}^{-\frac{x}{2}}\,\mathrm{d}x < \infty$, so

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,\mathrm{d}x = \int_0^\infty \left(\lim_{n\to\infty} f_n(x)\right)\mathrm{d}x = \int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x = 1.$$

Finally, $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{n}{\sqrt[n]{(n+1)!}} \cdot \lim\limits_{n\to\infty} \displaystyle\int_0^\infty f_n(x)\,\mathrm{d}x = \mathrm{e}.$

Generalization. The authors remarked that the problem admits the following immediate generalization:

$$\lim_{n\to\infty} n \int_0^\infty x^k \mathrm{e}^{-x} \sqrt[n]{\mathrm{e}^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}}\,\mathrm{d}x = \mathrm{e}(k+1)!$$

for all $k \in \mathbb{N}$, since the inequalities corresponding to (2b) will replace $\Gamma\left(2 + \frac{1}{n}\right)$ with $\Gamma\left(k + 2 + \frac{1}{n}\right)$. Then the result is a consequence of $\Gamma(k+2) = (k+1)!$.

This problem had 16 *complete solutions in the contest. Based on the total number of points scored by the contestants, this problem ranked as the second most difficult.*

Problem 3. Let $A \in \mathcal{M}_n(\mathbb{C})$ such that $A^*A^2 = AA^*$. Prove that $A^2 = A$. (We denote by A^* the conjugate transpose of A, i.e., the matrix $\overline{A}^T$.)

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Author's solution. First, we will show that $\operatorname{Ker}(A^*) = \operatorname{Ker}\left(A^2\right)$. Let $\mathbf{x} \in \operatorname{Ker}(A^*)$. Then $AA^*\mathbf{x} = \mathbf{0}$, so $A^*A^2\mathbf{x} = \mathbf{0}$, wherefrom $\mathbf{x}^*(A^*)^2 A^2\mathbf{x} = 0$, and we obtain that $\left\|A^2\mathbf{x}\right\|^2 = 0$, from which it follows that $A^2\mathbf{x} = \mathbf{0}$, so $\mathbf{x} \in \operatorname{Ker}\left(A^2\right)$. Conversely, let $\mathbf{x} \in \operatorname{Ker}\left(A^2\right)$. Then $A^2\mathbf{x} = \mathbf{0}$, so $AA^*\mathbf{x} = \mathbf{0}$, wherefrom $\mathbf{x}^*AA^*\mathbf{x} = 0$, and we obtain that $\|A^*\mathbf{x}\|^2 = 0$, from which it follows that $A^*\mathbf{x} = \mathbf{0}$, so $\mathbf{x} \in \operatorname{Ker}(A^*)$.

Since $\operatorname{Ker}(A^*) = \operatorname{Ker}\left(A^2\right)$, we have that $\operatorname{Ker}\left(A^2\right) \perp \operatorname{Im} A$, and, since $\operatorname{Ker} A \subseteq \operatorname{Ker}\left(A^2\right)$, we obtain that $\operatorname{Ker} A \perp \operatorname{Im} A$. Taking into account that $\dim(\operatorname{Ker} A) + \dim(\operatorname{Im} A) = n$, it follows that we can construct an orthonormal basis for $\mathbb{C}^n$ with vectors from $\operatorname{Ker} A$ and $\operatorname{Im} A$. Since $\operatorname{Ker} A$ and $\operatorname{Im} A$ are invariant subspaces for A, there exists a unitary matrix, $U \in \mathcal{M}_n(\mathbb{C})$, such that

$$U^*AU = \begin{pmatrix} O_{n-r} & O_{n-r,r} \\ O_{r,n-r} & P \end{pmatrix},$$

where $P \in \mathcal{M}_r(\mathbb{C})$ is an invertible matrix.

Since $A^*A^2 = AA^*$, it follows that $P^*P^2 = PP^*$, which leads to $(P^*P^2)^* = (PP^*)^*$, wherefrom $(P^*)^2P = PP^*$, and we obtain that $P^*P^2 = (P^*)^2P$, so $P = P^*$, which implies that $P^3 = P^2$, so $P^2 = P$ and, finally, $A^2 = A$.

Second solution. *The following solution was given by Marian–Daniel Vasile, from West University of Timișoara, Romania (team leader).*

First, we will show that $\operatorname{Ker} A = \operatorname{Ker}\left(A^2\right) = \operatorname{Ker}(A^*)$. Let $\mathbf{x} \in \operatorname{Ker} A$. Then $A\mathbf{x} = \mathbf{0}$, so $A^2\mathbf{x} = \mathbf{0}$, wherefrom $AA^*\mathbf{x} = A^*A^2\mathbf{x} = \mathbf{0}$, and we obtain that $\mathbf{x}^*AA^*\mathbf{x} = 0$, from which it follows that $\|A^*\mathbf{x}\|^2 = \mathbf{0}$, which leads to $A^*\mathbf{x} = \mathbf{0}$ and, finally, $\mathbf{x} \in \operatorname{Ker}(A^*)$. We have thus obtained that $\operatorname{Ker} A \subseteq \operatorname{Ker} A^2 \subseteq \operatorname{Ker} A^*$. But $\dim(\operatorname{Ker} A) = \dim(\operatorname{Ker}(A^*))$, so $\operatorname{Ker} A = \operatorname{Ker}\left(A^2\right) = \operatorname{Ker}(A^*)$.

Taking the conjugates transpose of both sides of the equality from the statement we obtain that $(A^*)^2A = AA^* = A^*A^2$, so $A^*\left(A^*A - A^2\right) = O_n$. Since $\operatorname{Ker} A = \operatorname{Ker}(A^*)$, we obtain that $AA^*A = A^3$. We multiply the previous equality with A at right, and we obtain that $A^4 = AA^*A^2 = A^2A^*$, so $A^2(A^2 - A^*) = O_n$.

Now, we use the relation above and the fact that $\operatorname{Ker} A = \operatorname{Ker}\left(A^2\right) = \operatorname{Ker}(A^*)$ and we obtain that

$$A^*\left(A^2 - A^*\right) = O_n \tag{3a}$$

and

$$A\left(A^2 - A^*\right) = O_n. \tag{3b}$$

From (3a), we get that $A^*A^2 = (A^*)^2$ and, using equality from the statement, we obtain that $AA^* = (A^*)^2$. We consider the conjugate transpose of this last relation, and we find that

$$A^2 = AA^*. \tag{3c}$$

From (3b), we obtain that

$$A^3 = AA^*. \tag{3d}$$

From (3c) and (3d), we conclude that $A^3 = A^2$, so $A^2(A - I_n) = O_n$. Taking into account that $\operatorname{Ker} A = \operatorname{Ker}\left(A^2\right)$, it follows that $A^2 = A$.

Third solution. *The following solution was given by David–Mihai Rucărea–nu and Ștefan Solomon, from National University of Science and Technology Politehnica Bucharest, Romania (contestants).*

Recall that, for a matrix $X \in \mathcal{M}_n(\mathbb{C})$, its Frobenius norm is $\|X\|_F = \sqrt{\operatorname{Tr}(XX^*)}$.

We will show that $\left\|A^2 - A\right\|_F = 0$, which is equivalent to $A^2 - A = O_n$.

Taking the conjugates transpose of both sides of the equality from the statement we obtain that $(A^*)^2A = AA^*$, wherefrom $\mathrm{Tr}\,(AA^*) = \mathrm{Tr}\,\big((A^*)^2A\big) = \mathrm{Tr}\,\big(A(A^*)^2\big)$.

On the other hand, from trace properties and previous equality we have that $\mathrm{Tr}\,\big(A^2(A^*)^2\big) = \mathrm{Tr}\,\big((A^*)^2A^2\big) = \mathrm{Tr}\,(A^*AA^*) = \mathrm{Tr}\,\big((A^*)^2A\big) = \mathrm{Tr}\,(AA^*) = \mathrm{Tr}\,\big(A^*A^2\big) = \mathrm{Tr}\,\big(A^2A^*\big)$.

Therefore, we can write

$$\begin{aligned}\left\|A^2 - A\right\|_F^2 &= \mathrm{Tr}\,\big(A^2(A^*)^2 - A(A^*)^2 - A^2A^* + AA^*\big)\\ &= \mathrm{Tr}\,\big(A^2(A^*)^2\big) - \mathrm{Tr}\,\big(A^2A^*\big) + \mathrm{Tr}\,(AA^*) - \mathrm{Tr}\,\big(A(A^*)^2\big) = 0\end{aligned}$$

and the conclusion follows immediately.

Fourth solution. *The following solution was given by Marian Panțiruc, from Gheorghe Asachi Technical University of Iași, Romania (deputy leader).*

Let $\lambda \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n$, with $\mathbf{x} \neq \mathbf{0}$, such that $A\mathbf{x} = \lambda\mathbf{x}$. From the statement, we know that $A^*A^2\mathbf{x} = AA^*\mathbf{x}$, so $A\,(A^*\mathbf{x}) = \lambda^2\,(A^*\mathbf{x})$. It follows that either λ^2 is an eigenvalue of A or $A^*\mathbf{x} = \mathbf{0}$.

As in the second solution, it can be shown that $\mathrm{Ker}\,A = \mathrm{Ker}\,\big(A^2\big) = \mathrm{Ker}\,(A^*)$. Therefore, if $A^*\mathbf{x} = \mathbf{0}$, then $A\mathbf{x} = \mathbf{0}$, so $\lambda = 0$.

Otherwise, $\lambda \neq 0$ and $\lambda^2 \in \sigma\,(A)$, the spectrum of matrix A. It follows that $\lambda^{2^n} \in \sigma\,(A)$, for any non-negative integer n. Hence, $|\lambda| = 1$, otherwise $\sigma\,(A)$ would contain infinitely many distinct elements.

On the other hand, from $AA^*\mathbf{x} = \lambda^2A^*\mathbf{x}$, it follows that $\lambda^2\,\langle A^*\mathbf{x}, \mathbf{x}\rangle = \langle AA^*\mathbf{x}, \mathbf{x}\rangle$, so $\lambda^2\,\langle \mathbf{x}, A\mathbf{x}\rangle = \|A^*\mathbf{x}\|^2$, wherefrom $\lambda\,|\lambda\|^2\,\|\mathbf{x}\|^2 = \|A^*\mathbf{x}\|^2$, hence λ is a positive real number. Since $|\lambda| = 1$, it follows that $\lambda = 1$.

We have thus proved that $\sigma\,(A) \subset \{0, 1\}$.

Next, we will show that the algebraic and geometric multiplicities of the values 0 and 1, respectively, are equal.

If the algebraic multiplicity of 0 is greater than the geometric one, then there exists $\mathbf{y} \in \mathbb{C}^n$, with $\mathbf{y} \neq \mathbf{0}$, $A\mathbf{y} \neq \mathbf{0}$, but $A^2\mathbf{y} = \mathbf{0}$. From $A^*A^2\mathbf{y} = AA^*\mathbf{y}$, it follows that $AA^*\mathbf{y} = \mathbf{0}$, so $\mathbf{y}^*AA^*\mathbf{y} = \mathbf{0}$, wherefrom $\|A^*\mathbf{y}\|^2 = 0$, hence, $A^*\mathbf{y} = \mathbf{0}$, equivalent with $\mathbf{y} \in \mathrm{Ker}\,(A^*) = \mathrm{Ker}\,A$, so $A\mathbf{y} = \mathbf{0}$, contradiction.

If the algebraic multiplicity of 1 is greater than the geometric one, then there exists $\mathbf{z} \in \mathbb{C}^n$, with $\mathbf{z} \neq \mathbf{0}$, $(A - I_n)\,\mathbf{z} \neq \mathbf{0}$, but $(A - I_n)^2\,\mathbf{z} = \mathbf{0}$. Let $\mathbf{w} = A\mathbf{z} - \mathbf{z}$. Then $\|A^*\mathbf{w} - \mathbf{w}\|^2 = \langle A^*\mathbf{w} - \mathbf{w}, A^*\mathbf{w} - \mathbf{w}\rangle = \langle A^*\mathbf{w}, A^*\mathbf{w}\rangle + \|\mathbf{w}\|^2 - \langle A^*\mathbf{w}, \mathbf{w}\rangle - \langle \mathbf{w}, A^*\mathbf{w}\rangle = \langle AA^*\mathbf{w}, \mathbf{w}\rangle + \|\mathbf{w}\|^2 - \langle \mathbf{w}, A\mathbf{w}\rangle - \langle A\mathbf{w}, \mathbf{w}\rangle = \left\langle A^*A^2\mathbf{w}, \mathbf{w}\right\rangle - \|\mathbf{w}\|^2 = \langle A^*\mathbf{w}, \mathbf{w}\rangle - \|\mathbf{w}\|^2 = \langle \mathbf{w}, A\mathbf{w}\rangle - \|\mathbf{w}\|^2 = 0$, so $A^*\mathbf{w} = \mathbf{w}$. On the other hand, $0 < \|\mathbf{w}\|^2 = \|A\mathbf{z} - \mathbf{z}\|^2 = \langle A\mathbf{z} - \mathbf{z}, A\mathbf{z} - \mathbf{z}\rangle = \langle \mathbf{w}, A\mathbf{z} - \mathbf{z}\rangle = \langle \mathbf{w}, A\mathbf{z}\rangle - \langle \mathbf{w}, \mathbf{z}\rangle = \langle A^*\mathbf{w}, \mathbf{z}\rangle - \langle \mathbf{w}, \mathbf{z}\rangle = \langle \mathbf{w}, \mathbf{z}\rangle - \langle \mathbf{w}, \mathbf{z}\rangle = 0$, which is absurd.

Therefore, matrix A is diagonalizable, with 0 and/or 1 as eigenvalues, so $A^2 = A$.

Fifth solution. *The following solution was given by Petruț–Rareș Gheorghieș, from National University of Science and Technology Politehnica Bucharest, Romania (contestant).*

We multiply the equality from the statement with A at left, and we obtain that $(A^*)^2A^2 = A^*AA^*$. Taking the conjugates transpose of both sides of the previous equality we obtain that $(A^*)^2A^2 = AA^*A$, so $A^*AA^* = AA^*A$. On the other hand, the equality from the statement implies that $(AA^*)^2 = \left(A^*A^2\right)^2 = A^*A(AA^*A)A = A^*A(A^*AA^*)A$, so

$$(AA^*)^2 = (A^*A)^3\,. \tag{3e}$$

It is known that the matrices AA^* and A^*A have the same eigenvalues. So, if λ is an eigenvalue of AA^*, then $\lambda^{\frac{2}{3}}$ is also an eigenvalue of AA^*, and, by induction, $\lambda^{\frac{2k}{3}}$ is an eigenvalue of AA^*, for any positive integer k. Since the spectrum of the hermitian matrix AA^* has a finite number of elements, all of which are real, it follows that $\sigma\left(AA^*\right) \subset \{-1, 0, 1\}$. If -1 would be an eigenvalue of AA^*, then there would be another eigenvalue, μ, of AA^* such that $\mu^2 = (-1)^3$, impossible. Hence, $\sigma\left(AA^*\right) \subset \{0, 1\}$.

Since AA^* is a hermitian matrix, there exists a unitary matrix, say, $U \in \mathcal{M}_n(\mathbb{C})$, such that $AA^* = U\begin{pmatrix} I_r & O_{r,n-r} \\ O_{n-r,r} & O_{n-r}\end{pmatrix}U^*$. It follows that $\left(AA^*\right)^2 = AA^*$. From (3e), we get that $\left(AA^*\right)^2 = A^*\left(AA^*\right)^2A = A^*AA^*A = \left(A^*A\right)^2$, so

$$AA^* = \left(A^*A\right)^2. \tag{3f}$$

Since A^*A is also a hermitian matrix, there exists a unitary matrix, say, $V \in \mathcal{M}_n(\mathbb{C})$, such that $A^*A = V\begin{pmatrix} I_r & O_{r,n-r} \\ O_{n-r,r} & O_{n-r}\end{pmatrix}V^*$, so

$$\left(A^*A\right)^2 = A^*A. \tag{3g}$$

From (3f) and (3g), we obtain that $A^*A = AA^*$, so A is a normal matrix. Consequently, there exist a unitary matrix $P \in \mathcal{M}_n(\mathbb{C})$ and a diagonal matrix $D \in \mathcal{M}_n(\mathbb{C})$ with the eigenvalues of A on the main diagonal, such that $A = PDP^*$. Now, the equality from the statement implies that $\overline{D}D^2 = D\overline{D}$. Let δ be an eigenvalue of A. It follows that $\overline{\delta}\delta^2 = \delta\overline{\delta}$, so $\delta \in \{0, 1\}$. Hence, $A^2 = PD^2P^* = PDP^* = A$.

Sixth solution. *The following solution was given by Paul Burcă, from National University of Science and Technology Politehnica Bucharest, Romania (contestant).*

As in the previous solution, it can be shown that $A^*AA^* = AA^*A$. The equality from the statement implies that $AA^*A^2 = A^2A^*$, so $A^2A^* = (AA^*A)A = (A^*AA^*)A$, hence

$$\left(A^*A\right)^2 = A^2A^*. \tag{3h}$$

On the other hand, the equality from the statement also implies that $A^*A^2A^* = A(A^*)^2$. Taking the conjugates transpose of both sides of the previous equality, we obtain that $A(A^*)^2 A = A^2A^*$. From (3h), we get that $(A^*A)^2 = A(A^*)^2 A$. Taking the conjugates transpose of both sides of the previous equality, we obtain that $(A^*A)^2 = A^*A^2A^*$. Using the last two relations, it can be shown that the Frobenius norm of the matrix $A^*A - AA^*$ is zero, which means that A is a normal matrix. By repeating the final part of the previous proof, the conclusion follows.

Remarks. From the previous solutions, it follows that the matrices A verifying the relation $A^*A^2 = AA^*$ have the properties:

(i) $A^2 = A$, so they are projection matrices;

(ii) $A^*A = AA^*$, so they are normal matrices;

(iii) $A^* = A$, so they are self-adjoint matrices.

Hence, the linear transformation $A : \mathbb{C}^n \to \mathbb{C}^n$, with $A(\mathbf{x}) = A\mathbf{x}$, for any $\mathbf{x} \in \mathbb{C}^n$, is an orthogonal projection in the vector space $\mathbb{C}^n$. More precisely, each matrix A is uniquely determined by a subspace $V_1 \subset \mathbb{C}^n$, $V_1 = \operatorname{Fix} A = \{\mathbf{x} \in \mathbb{C}^n \mid A\mathbf{x} = \mathbf{x}\}$, and, if $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_1^{\perp}$, where $\mathbf{x}_1 \in V_1$ and $\mathbf{x}_1^{\perp} \in V_1^{\perp}$, then $A\mathbf{x} = \mathbf{x}_1$.

Moreover, there exists an orthogonal basis in $\mathbb{C}^n$ such that the matrix A has the form $J_A = \begin{pmatrix} I_k & O_{k,n-k} \\ O_{n-k,k} & O_{n-k} \end{pmatrix}$ in this basis.

Generalization. *Given by Vasile Pop, from Technical University of Cluj–Napoca, Romania.*

In the following, we will show that,
if k, p, n are positive integers and $A \in \mathcal{M}_n(\mathbb{C})$ such that $A^*A^k = A^pA^*$, then $A^k = A^p$.

The complex vector space $\mathcal{M}_n(\mathbb{C})$, equipped with Frobenius norm $\|X\|_F = \sqrt{\operatorname{Tr}(X^*X)}$, is a normed vector space in which we have $\|X\|_F = 0 \Leftrightarrow X = O_n$. Thus, it will be enough to show that $\operatorname{Tr} B = 0$, where $B = (A^k - A^p)^*\left(A^k - A^p\right) = (A^k)^*A^k - (A^k)^*A^p - (A^p)^*A^k + (A^p)^*A^p = (A^k)^*AA^{k-1} - (A^k)^*AA^{p-1} - (A^p)^*A^k + (A^p)^*A^p = B_1 - B_2 - B_3 + B_4$. From the equality in the statement we have that $(A^k)^*A = A(A^p)^*$ and taking into account the cyclic property of the trace, it follows that $\operatorname{Tr} B_1 = \operatorname{Tr}(((A^k)^*A)A^{k-1}) = \operatorname{Tr}((A(A^p)^*)A^{k-1}) = \operatorname{Tr}((A^p)^*A^{k-1}A) = \operatorname{Tr}((A^p)^*A^k) = \operatorname{Tr} B_3$ and $\operatorname{Tr} B_2 = \operatorname{Tr}((A^k)^*AA^{p-1}) = \operatorname{Tr}(A(A^p)^*A^{p-1}) = \operatorname{Tr}((A^p)^*A^{p-1}A) = \operatorname{Tr}((A^p)^*A^p) = \operatorname{Tr} B_4$. Hence, $\operatorname{Tr} B = (\operatorname{Tr} B_1 - \operatorname{Tr} B_3) - (\operatorname{Tr} B_2 - \operatorname{Tr} B_4) = 0$ and the conclusion follows immediately.

If $p = 0$ and k is a positive integer, the previous implication is not true anymore. For example, $A^*A^2 = A^* \not\Rightarrow A^2 = I_n$. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $A^* = A$ and $A^3 = A$, but $A^2 \neq I_3$.

This problem had 26 complete solutions in the contest. Also, based on the total number of points scored by the contestants, this problem ranked as the easiest problem in the contest.

Problem 4. Let (a_n) be a monotone decreasing sequence of real numbers that converges to 0. Prove that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent if and only if the sequence $(a_n \ln n)$ is bounded and $\sum_{n=1}^{\infty}(a_n - a_{n+1}) \ln n$ is convergent.

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Despite the jury's efforts to select only original problems, this one was later identified as a weaker version of Exercise 2.4 from Chapter IV, p. 37, of [3]. The stronger version (Exercise 2.4) asks that $(a_n \ln n)$ is convergent to 0 (written in the form of "*small o*" notation). Yet, [3] only provides some basic hints (see p. 181) on how to approach the problem and not a full solution.

Proposer's solution. *This solution was given by the* proposer *and accompanied the problem in* the short list.

Define the sequences (S_N) and (T_N) of partial sums:

$$S_N = \sum_{n=1}^{N} \frac{a_n}{n} \quad \text{and} \quad T_N = \sum_{n=1}^{N}(a_n - a_{n+1}) \ln n \qquad (N \geq 1).$$

Because (a_n) is decreasing to 0, $a_n \geq 0$ and $a_n - a_{n+1} \geq 0$ for all n, so both $(S_N)_{N\geq 1}$ and $(T_N)_{N\geq 1}$ are increasing. Let S and T be their respective limits (finite or ∞). In what follows, we make use of

$$\ln n - \ln(n-1) = \int_{n-1}^{n} \frac{\mathrm{d}x}{x} \in \left(\frac{1}{n}, \frac{1}{n-1}\right), \quad \text{for all } n \geq 2.$$

(⇒) Assume $S < \infty$. Let $N \geq 2$. Then

$$a_N \ln N = a_N \sum_{n=2}^{N}(\ln n - \ln(n-1)) \leq a_N \sum_{n=2}^{N} \frac{1}{n-1} \leq \sum_{n=2}^{N} \frac{a_{n-1}}{n-1} = S_{N-1} \leq S.$$

This implies that $(a_n \ln n)$ is bounded. Moreover,

$$\begin{aligned} T_N &= \sum_{n=1}^{N}(a_n - a_{n+1})\ln n = \sum_{n=1}^{N} a_n \ln n - \sum_{n=2}^{N+1} a_n \ln(n-1) \\ &= \sum_{n=2}^{N} a_n(\ln n - \ln(n-1)) - a_{N+1}\ln N \le \sum_{n=2}^{N}\frac{a_n}{n-1} \le \sum_{n=2}^{N}\frac{a_{n-1}}{n-1} = S_{N-1} \\ &\le S, \end{aligned}$$

hence $T \le S < \infty$.

($\Leftarrow$) Assume $T < \infty$ and $a_N \log N \le M$, for all $N \ge 2$, where M is some positive number. Then

$$\begin{aligned} S_N - a_1 &= \sum_{n=2}^{N}\frac{a_n}{n} \le \sum_{n=2}^{N} a_n(\ln n - \ln(n-1)) = \sum_{n=2}^{N} a_n \ln n - \sum_{n=1}^{N-1} a_{n+1}\ln n \\ &= \sum_{n=2}^{N-1}(a_n - a_{n+1})\ln n + a_N \ln N = T_{N-1} + a_N \log N \le T + M \end{aligned}$$

for all $N \ge 2$, hence $S \le a_1 + T + M < \infty$.

Alternative solution. *The following solution was given by Mircea Rus, from Technical University of Cluj-Napoca, Romania (team leader).*

This solution uses the *summation by parts* – the discrete analog of integration by parts that is used to evaluate or estimate sums, especially when one sequence is monotone; a variant of this method is the well known *Abel transformation*. Additionally, the second solution also proves, in the affirmative case when the series are convergent and $(a_n \ln n)$ is bounded, that $\lim_{n\to\infty} a_n \ln n = 0$.

For any sequence (x_n), define $\Delta x_n = x_{n+1} - x_n$. This operation (called *forward difference*) is the discrete analog of the derivative from calculus and satisfies $\Delta(x_n y_n) = x_n \cdot \Delta y_n + \Delta x_n \cdot y_{n+1}$, for any sequences $(x_n), (y_n)$. The verification is trivial. The *summation by parts* states that

$$\sum_{n=1}^{N} x_n \cdot \Delta y_n = (x_{N+1}y_{N+1} - x_1 y_1) - \sum_{n=1}^{N} \Delta x_n \cdot y_{n+1}$$

and is a direct consequence of the previous identity.

Because (a_n) and $(-\Delta a_n)$ have positive elements, $\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} = 1$ and $\lim_{n\to\infty}\frac{\Delta \ln n}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} = 1$, it follows by the *limit comparison test* that

$$\sum_{n=1}^{\infty}\frac{a_n}{n} < \infty \Leftrightarrow \sum_{n=1}^{\infty} a_n \cdot \Delta \ln n < \infty$$

$$\sum_{n=1}^{\infty}(-\Delta a_n)\cdot \ln n < \infty \Leftrightarrow \sum_{n=1}^{\infty}(-\Delta a_n)\cdot \ln(n+1) < \infty,$$

so the problem reduces to proving the equivalence

$$\sum_{n=1}^{\infty} a_n \cdot \Delta \ln n < \infty \Leftrightarrow (a_n \ln n) \text{ is bounded and } \sum_{n=1}^{\infty} (-\Delta a_n) \cdot \ln(n+1) < \infty. \quad (4a)$$

For series with positive elements, the convergence is equivalent to the (upper) boundedness. Because of this, (4a) follows straight from summation by parts, which gives

$$\sum_{n=1}^{N} a_n \cdot \Delta \ln n = a_{N+1} \ln(N+1) + \sum_{n=1}^{N} (-\Delta a_n) \cdot \ln(n+1), \quad \text{for } N \geq 1. \quad (4b)$$

Additional conclusions. In the affirmative case when the series are convergent and $(a_n \ln n)$ is bounded, it follows by (4b) that $(a_n \ln n)$ is convergent. Let $L = \lim_{n\to\infty} a_n \ln n$. Then $L = \lim_{n\to\infty} \frac{a_n}{n} \Big/ \frac{1}{n \ln n}$. Since $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent and $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ is divergent (this can be easily justified by the *Cauchy condensation test*), it follows by the *limit comparison test* that L must be 0. We also obtain $\sum_{n=1}^{\infty} a_n \cdot \ln\left(1 + \frac{1}{n}\right) = \sum_{n=1}^{\infty} (a_n - a_{n+1}) \cdot \ln(n+1)$ as consequence of (4b).

Although this was not an original problem, it turned out to be the most challenging of the contest, judging by the total number of points. Also, 16 *contestants achieved a full score on it.*

References

[1] A. Greenbaum, T.P. Chartier, *Numerical methods. Design, analysis, and computer implementation of algorithms*, Princeton University Press, 2012.

[2] R.A. Horn, C.R. Johnson, *Matrix analysis*, 2nd ed., Cambridge University Press, 2013.

[3] B.M. Makarov, M.G. Goluzina, A.A. Lodkin, A.N. Podkorytov, *Selected problems in real analysis*, translated from the Russian by H. H. McFaden, Translations of Mathematical Monographs, 107, American Mathematical Society, Providence, RI, 1992.

PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the authorsquare will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of November 2025**.

PROPOSED PROBLEMS

570. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative such that $f(0) = f(1) = 0$ and $\int\limits_0^1 x f(x)\,\mathrm{d}x = 0$. Prove that

$$\int_0^1 \left(f'(x)\right)^2 \,\mathrm{d}x \geq 192 \left(\int_0^1 f(x)\,\mathrm{d}x\right)^2.$$

(This problem was inspired by problem 563 form the 3-4/2024 issue of GMA. It states that if the condition $f(0) = f(1)$ from the hypothesis of problem 563 is strengthened to $f(0) = f(1) = 0$, then the constant 180 from problem 563 can be improved to 192.)

Proposed by Ulrich Abel and Vitaliy Kushnirevych, Technische Hochschule Mittelhessen, Germany.

571. Let $a_1, a_2, a_3, a_4 \geq 0$ be real numbers. Prove that

$$a_1 + a_2 + 3a_3 + 3a_4 + \sqrt{(a_1 - a_3)^2 + (a_2 - a_4)^2} \geq \sum_{k=1}^{4} \sqrt{a_k^2 + a_{k+1}^2},$$

where $a_5 = a_1$.

Proposed by Leonard Giugiuc, Middle School Greci, Mehedinți, Romania.

572. Prove that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the equation

$$n^2 \iint\limits_{[0,1]^2} x^n y^n e^{xyz} \,\mathrm{d}x\mathrm{d}y = e$$

has an unique real solution denoted by z_n and, moreover, $\lim\limits_{n\to\infty} n\,(z_n - 1) = 4$.

Proposed by Dumitru Popa, Department of Mathematics, Ovidius University of Constanța, Romania.

573. Let $\mathcal{B} = \bigcup\limits_{n\geq 1} \mathcal{B}_n$, where

$$\mathcal{B}_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \leq x_{i+2} \text{ for all } 1 \leq i \leq n-2\}.$$

On $\mathcal{B}$ we define the relations $\leq$ and $\prec$ as follows:
Let $x, y \in \mathcal{B}$, $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$.

We say that $x \le y$ if $m \ge n$ and for every $1 \le i \le n$ we have either $x_i \le y_i$ or $1 < i < m$ and $x_i + x_{i+1} \le y_{i-1} + y_i$. It is known that $(\mathcal{B}, \le)$ is partially ordered.[1]

We say that $x \prec y$ if $x \le y$, $m - n \le 2$, and for every $1 < i < m$ we have either $x_{i+1} \ge y_{i-1}$ or $i \le n$ and $x_i + x_{i+1} = y_{i-1} + y_i$.

If $x \prec y$ we denote $\delta(x, y) = \sum_{i=1}^{n} (-1)^i (x_i - y_i)_+$. (Here $x_+ := (x + |x|)/2$ is the positive part of x, with $x_+ = x$ if $x \ge 0$ and $x_+ = 0$ if $x \le 0$.)

Prove that if $x, y, z \in \mathcal{B}$ such that $x \le y \le z$ and $x \prec z$, then $x \prec y \prec z$ and $\delta(x, y) + \delta(y, z) = \delta(x, z)$.

Proposed by Constantin-Nicolae Beli, IMAR, București, Romania.

574. The matrices $A, B \in \mathcal{M}_n(\mathbb{C}), n \ge 2$, satisfy the relations:

$$(1)\ A^2 = B^2; \quad (2)\ A^3 + BAB = 2I_n.$$

(a) Prove that $A^3 = I_n$;
(b) Prove that if A and B have no common eigenvalues, then $A + B = O_n$.

Proposed by Vasile Pop and Constantin-Cosmin Todea, Technical University of Cluj-Napoca, Romania.

575. Let n be a positive integer with $n \ge 3$. Prove that $\dfrac{n}{4}$ is the least positive value of the constant k such that the inequality

$$a_1 + a_2 + \cdots + a_n + \frac{k(a_1 - a_n)^2}{a_1 + a_n} \ge \sqrt{n(a_1^2 + a_2^2 + \cdots + a_n^2)}$$

holds whenever $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$.

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania, and Leonard Giugiuc, Middle School Greci, Mehedinți, Romania.

576. We say that $x \in \mathbb{R}$ is a *nice number* if $\displaystyle\sum_{k=1}^{\infty} k^n x^k$ is an integer, for all $n \in \mathbb{N}$, $n \ge 1$.

(a) Find all the nice numbers.

(b) If x is a nice number, show that $\displaystyle\sum_{k=1}^{\infty} k^n x^k$ is an even natural number, for all $n \in \mathbb{N}$, $n \ge 1$.

Proposed by Mircea Rus, Technical University of Cluj-Napoca, Romania.

577. Let $a \in \mathbb{R} \setminus \{0\}$ and $f : \mathbb{R} \to \mathbb{R}$ a differentiable function with the property

$$|f'(x) + af(x)| \le 1, \quad \text{for all } x \in \mathbb{R}.$$

Prove that there exists a unique differentiable function $g : \mathbb{R} \to \mathbb{R}$ that satisfies

$$g'(x) + ag(x) = 0 \quad \text{and} \quad |f(x) - g(x)| \le \frac{1}{|a|}, \quad \text{for all } x \in \mathbb{R}.$$

Proposed by Dorian Popa, Technical University of Cluj-Napoca, Romania.

[1] See problem 331 from the 1-2/2011 issue of GMA, with solution in the 1-2/2012 issue.

SOLUTIONS

553. Let $ABCD$ be an isosceles tetrahedron with centroid G. Let M, N be two points such that $\overrightarrow{NG} = 3\overrightarrow{GM}$. Prove that

$$NA + NB + NC + ND \geq MA + MB + MC + MD.$$

Proposed by Leonard Giugiuc, Drobeta Turnu-Severin, Romania.

Solution by the author. We recall that in an isosceles tetrahedron the centroid G coincides with the circumcenter O. WLOG we may assume that the circumradius is 1.

We regard the 3D space where $ABCD$ lives in as a subspace of a 4D space identified as the quaternions $\mathbb{H}$, such that the zero element $0 \in \mathbb{H}$ coincides with $G = O$.

Recall that an element $w \in \mathbb{H}$ writes as $w = \alpha + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}$. Its conjugate is $\bar{w} = \alpha - \beta\mathbf{i} - \gamma\mathbf{j} - \delta\mathbf{k}$ and the length is given by $\|w\| = \sqrt{w\bar{w}} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}$.

Let $a, b, c, d \in \mathbb{H}$ be the coordinates of A, B, C, D and let z be the coordinate of M. Since $G = 0 \in \mathbb{H}$, we have $a + b + c + d = 0$ and $\overrightarrow{NG} = 3\overrightarrow{GM}$ writes as $N = -3M = -3z$. And since $O = 0 \in \mathbb{H}$, $OA = OB = OC = OD = 1$ write as $\|a\| = \|b\| = \|c\| = \|d\| = 1$.

We have

$$\begin{aligned}
NB + NC + ND &= \|b + 3z\| + \|c + 3z\| + \|d + 3z\| \\
&= \|b + 3z\| \cdot \|\bar{b}\| + \|c + 3z\| \cdot \|\bar{c}\| + \|d + 3z\| \cdot \|\bar{d}\| \\
&= \|b\bar{b} + 3z\bar{b}\| + \|c\bar{c} + 3z\bar{c}\| + \|d\bar{d} + 3z\bar{d}\| \\
&= \|1 + 3z\bar{b}\| + \|1 + 3z\bar{c}\| + \|1 + 3z\bar{d}\| \\
&\geq \|1 + 3z\bar{b} + 1 + 3z\bar{c} + 1 + 3z\bar{d}\| \\
&= 3\|1 + z(\bar{b} + \bar{c} + \bar{d}\| = 3\|1 - z\bar{a}\| \\
&= 3\|a\bar{a} - z\bar{a}\| = 3\|a - z\| \cdot \|\bar{a}\| = 3\|a - z\| = 3MA.
\end{aligned}$$

When we add the inequality above with the similar inequalities $NA + NC + ND \geq 3MB$, $NA + NB + ND \geq 3MC$, and $NA + NB + NC \geq 3MD$ and we divide by 3 we get $NA + NB + NC + ND \geq MA + MB + MC + MD$.

554. Let $n \in \mathbb{N}$, $n \geq 2$.

(a) Prove that $\det(A^2 - B^2)(C^2 - B^2) \geq 0$ for all $A, B, C \in \mathcal{M}_n(\mathbb{R})$ with $AB = BC$.

(b) Find all values $k \geq 1$ such that $\det(A^k - B^2)(C^k - B^2) \geq 0$ holds for all $A, B, C \in \mathcal{M}_n(\mathbb{R})$ with $AB = BC$.

Proposed by Mihai Opincariu, Brad, Romania, and Vasile Pop, Technical University of Cluj-Napoca, Romania.

Solution by the authors. **First solution for (a).** We consider the block matrix $M = \begin{bmatrix} A & B \\ -B & -C \end{bmatrix} \in \mathcal{M}_{2n}(\mathbb{R})$. Then

$$M^2 = \begin{bmatrix} A^2 - B^2 & AB - BC \\ -BA + CB & C^2 - B^2 \end{bmatrix} = \begin{bmatrix} A^2 - B^2 & O \\ CB - BA & C^2 - B^2 \end{bmatrix},$$

so $0 \le (\det M)^2 = \det M^2 = \det(A^2 - B^2)\det(C^2 - B^2)$.

Second solution for (a). If A and C are invertible, then $AB = BC$ leads to $B = A^{-1}BC$, hence $A^{-1}B = BC^{-1}$. Then

$$A^{-2}B^2 = A^{-1}(A^{-1}B)B = A^{-1}(BC^{-1})B = (A^{-1}B)C^{-1}B = BC^{-2}B.$$

From here,

$$\det(A^2 - B^2) = \det A^2 \det(I - A^{-2}B^2) = (\det A)^2 \det(I - BC^{-2}B). \tag{1}$$

Recall that if $P, Q \in \mathcal{M}_n(\mathbb{R})$, then PQ and QP have the same characteristic polynomial, so $\det(I - PQ) = \det(I - QP)$. Hence we have

$$\begin{aligned}\det(I - BC^{-2}B) &= \det(I - C^{-2}B^2) = \det(C^{-2})\det(C^2 - B^2)\\ &= \det(C^{-1})^2 \det(C^2 - B^2).\end{aligned} \tag{2}$$

Combining (1) and (2) leads to

$$\det(A^2 - B^2) = (\det A)^2 \det(C^{-1})^2 \det(C^2 - B^2),$$

so

$$\det(A^2 - B^2)\det(C^2 - B^2) = \big((\det A)\det(C^{-1})\det(C^2 - B^2)\big)^2 \ge 0.$$

Without the assumption that A and C are invertible, let $A_x = A - xI$ and $C_x = C - xI$, for any $x \in \mathbb{R}$. Then $A_xB = AB - xB = BC - xB = BC_x$. Since the set of values of x for which either A_x or C_x is non-invertible is finite (being the set of eigenvalues of A and C put together), it follows that there exists a sequence $(x_m)_{m\ge1}$ that is convergent to 0 such that A_{x_m} and C_{x_m} are invertible, for all $m \ge 1$. Now, we can apply the previously obtained result for the invertible matrices A_{x_m} and C_{x_m} to obtain $\det(A_{x_m}^2 - B^2)\det(C_{x_m}^2 - B^2) \ge 0$, i.e.,

$$\det\big((A - x_mI)^2 - B^2\big)\det\big((C - x_mI)^2 - B^2\big) \ge 0 \tag{3}$$

Taking the limit in (3) as $m \to \infty$ and using the continuity of the functions $\det\big((A - xI)^2 - B^2\big)$ and $\det\big((C - xI)^2 - B^2\big)$ (as functions of $x \in \mathbb{R}$), the conclusion follows.

(b) Let $A, B, C \in \mathcal{M}_n(\mathbb{R})$ such that $AB = BC$. It follows that

$$A^2B = A(AB) = A(BC) = (AB)C = (BC)C = BC^2,$$

and, by induction, $A^mB = BC^m$, for all $m \in \mathbb{N}^*$. Then (a) applied to the matrices A^m, B, C^m leads to $\det(A^{2m} - B^2)(C^{2m} - B^2) \ge 0$. This means that every even number k is a solution.

We can show by counter-examples that odd numbers k are not solutions.

If $n = 2$, then let $A = I_2$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. We can check that $AB = BC = B$ and $B^2 = O$. Therefore, if k is odd, then

$$\det\big(A^k - B^2\big)\det\big(C^k - B^2\big) = \det I_2\,(\det C)^k = (-1)^k = -1 < 0.$$

If $n \ge 3$, then we consider the block matrices

$$A = \begin{bmatrix} A_2 & O \\ O & I_{n-2} \end{bmatrix} = I_n,\ B = \begin{bmatrix} B_2 & O \\ O & O \end{bmatrix},\ \text{and } C = \begin{bmatrix} C_2 & O \\ O & I_{n-2} \end{bmatrix},$$

where $A_2 = I_2$, $B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $C_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ are the matrices from the case $n = 2$. Then, $AB = BC = B$, $B^2 = O$, and

$$\det\left(A^k - B^2\right) \cdot \det\left(C^k - B^2\right) = (-1)^k = -1 < 0$$

for all k odd, just like in the previous case.

We also received a solution from Marian-Daniel Vasile, Timișoara, Romania. The proof of (a) is essentially the same as the authors' first proof. For (b) the proof of $\det(A^k - B^2)(C^k - B^2) \geq 0$ for k even is the same as the authors', but for the proof of the necessity of k being even he uses a different example, namely, $A = I_n$, $B = O_n$, and $C = \operatorname{diag}(-1, 1, \ldots, 1)$. We have $AB = BC = O_n$ and the inequality $\det(A^k - B^2)(C^k - B^2) \geq 0$ writes as $(-1)^k \geq 0$, which happens only if k is even.

555. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative such that $f(1) = 0$ and $f'(1) = 1$. Prove that there exists $c \in (0,1)$ such that

$$f(c) = f'(c) \int_0^c f(x)dx.$$

`Proposed by Cezar Lupu, Beijing Institute of Mathematical Sciences and Applications (BIMSA) and Tsinghua University, Beijing, P.R. China.`

Solution by the author. Let us consider the differentiable function $\varphi : [0,1] \to \mathbb{R}$ defined by

$$\varphi(t) = te^{-f(t)} \int_0^t f(x)\,\mathrm{d}x.$$

A computation of the derivative shows that

$$\varphi'(t) = (e^{-f(t)} - te^{-f(t)} f'(t)) \int_0^t f(x)\,\mathrm{d}x + te^{-f(t)} f(t),$$

which is equivalent to

$$\varphi'(t) = e^{-f(t)} \left(\int_0^t f(x)\,\mathrm{d}x + t \left(f(t) - f'(t) \int_0^t f(x)\,\mathrm{d}x \right) \right), \forall t \in [0,1].$$

On the other hand, we have $\varphi'(0) = 0$ and

$$\begin{aligned} \varphi'(1) &= e^{-f(1)} \left(\int_0^1 f(x)\,\mathrm{d}x + \left(f(1) - f'(1) \int_0^1 f(x)\,\mathrm{d}x \right) \right) \\ &= \int_0^1 f(x)\,\mathrm{d}x - \int_0^1 f(x)\,\mathrm{d}x = 0. \end{aligned}$$

Now, by Flett's mean value theorem (see [1]) there exists $c \in (0,1)$ such that

$$\varphi'(c) = \frac{\varphi(c) - \varphi(0)}{c},$$

which is equivalent to

$$ce^{-f(c)} \left(\int_0^c f(x)\,\mathrm{d}x + c \left(f(c) - f'(c) \int_0^c f(x)\,\mathrm{d}x \right) \right) = ce^{-f(c)} \int_0^c f(x)\,\mathrm{d}x,$$

which is further equivalent to

$$f(c) = f'(c)\int_0^c f(x)\,\mathrm{d}x$$

and this is exactly what we wanted to prove.

REFERENCES

[1] T.M. Flett, A mean value problem, *The Mathematical Gazette*, **42** (1958), 38–39.

Solution by Marian-Daniel Vasile, Timișoara, Romania. Let $F : [0, 1] \to \mathbb{R}$, $F(x) = \int_0^x f(t)\,\mathrm{d}t$ and let $g : [0, 1] \to \mathbb{R}$, $g(x) = e^{-f(x)}F(x)$. We have $F(0) = 0$ and $F'(x) = f(x)$, so $g(0) = 0$ and $g'(x) = e^{-f(x)}(-f'(x)F(x) + f(x))$. Thus $g'(c) = 0$ is equivalent to $f(c) = f'(c)F(c) = f'(c)\int_0^c f(t)\,\mathrm{d}t$. Hence we must prove that there is $c \in (0, 1)$ with $g'(c) = 0$. Suppose the contrary.

By Darboux's theorem, g' has the intemediate value property. So our assumption that $g'(c) \neq 0$ for $c \in (0, 1)$ implies that $g'(x) > 0$ for all $x \in (0, 1)$ or $g'(x) < 0$ for all $x \in (0, 1)$.

If $g'(x) > 0$ for all $x \in (0, 1)$ then, again by the intermediate value property, $g'(1) \geq 0$. (If $g'(1) < 0$, since also $g'(1/2) > 0$, there is $c \in (1/2, 1) \subset (0, 1)$ with $g'(c) = 0$.) Also from $g'(x) > 0$ for all $x \in (0, 1)$ we deduce that g is stictly increasing on $[0, 1]$. In particular, $g(1) > g(0) = 0$.

Similarly, if $g'(x) < 0$ for all $x \in (0, 1)$, then $g'(1) \leq 0$ and $g(1) < 0$. In conclusion, we have either $g'(1) \geq 0$ and $g(1) > 0$ or $g'(1) \leq 0$ and $g(1) < 0$. In both cases, $g'(1)/g(1) \geq 0$. But $g'(x)/g(x) = -f'(x) + f(x)/F(x)$. Since $f(1) = 0$ and $f'(1) = 1$, we get $g'(1)/g(1) = -1 < 0$. Contradiction. This concludes the proof.

Note. From Marian-Daniel Vasile's proof we can see that the result remains true if in the hypothesis we replace the condition $f'(1) = 1$ by $f'(1) > 0$.

556. For given $n \geq 3$, prove that $k = 2n - 3$ is the smallest positive constant such that

$$\frac{1}{a_1 + k} + \frac{1}{a_2 + k} + \cdots + \frac{1}{a_n + k} \leq \frac{n}{1 + k}$$

holds for any nonnegative real numbers $a_1, \ldots, a_n$ such that at most one of them is > 1 and $\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$.

`Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania.`

Solution by the author. For $a_1 = \frac{2n-3}{n-2}$, $a_2 = \cdots = a_{n-1} = 1$, and $a_n = 0$, since $a_n = 0$, we have

$$\sum_{1 \leq i < j \leq n} a_i a_j = \sum_{1 \leq i < j \leq n-1} a_i a_j = a_1 \sum_{i=2}^{n-1} a_i + \sum_{2 \leq i < j \leq n-1} a_i a_j$$
$$= \frac{2n-3}{n-2} \cdot (n-2) + \frac{(n-2)(n-3)}{2} = \frac{n(n-1)}{2},$$

so the constraints are satisfied. And the inequality writes as

$$\frac{n-2}{2n-3+(n-2)k}+\frac{n-2}{1+k}+\frac{1}{k}\le\frac{n}{1+k},$$

i.e.,

$$\frac{n-2}{2n-3+(n-2)k}\le\frac{2}{1+k}-\frac{1}{k}=\frac{k-1}{k(1+k)},$$

which, after reductions, becomes $k\ge 2n-3$. To show that $2n-3$ is the smallest value of k, we need to show that $E\le\dfrac{n}{1+k}$ for $k=2n-3$, where

$$E=\frac{1}{a_1+k}+\frac{1}{a_2+k}+\cdots+\frac{1}{a_n+k}.$$

Assume that $a_1=\max\{a_1,a_2,\ldots,a_n\}$ and $a_n=\min\{a_1,a_2,\ldots,a_n\}$. If $a_1=1$ or $a_n=1$, then $a_1=a_2=\cdots=a_n=1$, and the inequality becomes an equality. Consider now $a_1>1$, $a_n<1$, and

$$a_2,\ldots,a_{n-1}\in[a_n,1].$$

If $S=\sum_{k=3}^{n}a_i$ and $Q=\sum_{3\le i<j\le n}a_ia_j$, then the constraint relation writes as

$$a_1a_2+(a_1+a_2)S+Q=\frac{n(n-1)}{2},$$

i.e., as $a_1=\dfrac{1}{a_2+S}\left(\dfrac{n(n-1)}{2}-a_2S-Q\right)$.

If $a_3,\ldots,a_n$ are fixed, then S and Q are constants and, by the constraint relation, a_1 is a function of a_2 and so is E. We claim that $E'(a_2)\ge 0$. If this claim is true, then $E(a_2)$ is increasing and it is maximum when a_2 is maximum, hence when $a_2=1$.

By differentiating the constraint relation, we get

$$(S+a_2)a_1'+S+a_1=0,\quad\text{i.e.,}\quad a_1'=-\frac{S+a_1}{S+a_2}.$$

Then

$$E'(a_2)=\frac{-a_1'}{(a_1+k)^2}-\frac{1}{(a_2+k)^2}=\frac{S+a_1}{(S+a_2)(a_1+k)^2}-\frac{1}{(a_2+k)^2}.$$

To prove that $E'(a_2)\ge 0$, we write it as follows:

$$\left(\frac{a_2+k}{a_1+k}\right)^2\ge\frac{S+a_2}{S+a_1},\quad\left(\frac{a_2+k}{a_1+k}\right)^2-1\ge\frac{S+a_2}{S+a_1}-1,$$

$$\frac{(a_2-a_1)(a_1+a_2+2k)}{(a_1+k)^2}\ge\frac{a_2-a_1}{S+a_1}.$$

Since $a_2-a_1<0$, this is equivalent to

$$\frac{a_1+a_2+2k}{(a_1+k)^2}\le\frac{1}{S+a_1},$$

i.e.,

$$k^2\ge a_1a_2+(a_1+a_2)S+2kS=\frac{n(n-1)}{2}-Q+2kS.$$

(Here we used the constraint relation $a_1a_2 + (a_1 + a_2)S + Q = \dfrac{n(n-1)}{2}$.)

Denoting

$$x_i = 1 - a_i \geq 0, \quad i = 3, \ldots, n,$$

we have

$$S = n - 2 - \sum_{i=3}^{n} x_i,$$

$$Q = \sum_{3\leq i<j\leq n} (1 - x_i)(1 - x_j) = \sum_{3\leq i<j\leq n} x_i x_j - (n-3)\sum_{i=3}^{n} x_i + \frac{(n-2)(n-3)}{2},$$

and the inequality becomes

$$k^2 \geq 2n - 3 + 2(n-2)k - \sum_{3\leq i<j\leq n} x_i x_j - (2k - n + 3)\sum_{i=3}^{n} x_i,$$

which, since $k = 2n - 3$, is equivalent to

$$\sum_{3\leq i<j\leq n} x_i x_j + 3(n-1)\sum_{i=3}^{n} x_i \geq 0.$$

Similarly, we can prove that for fixed $a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$, where $i \in \{3, \ldots, n-1\}$, the expression E is maximum when $a_i = 1$. So, we only need to prove the original inequality for $a_2 = a_3 = \cdots = a_{n-1} = 1$, i.e., to show that

$$\frac{1}{a_1 + 2n - 3} + \frac{1}{a_n + 2n - 3} \leq \frac{1}{n-1}$$

for $a_1a_n + (n-2)(a_1 + a_n) = 2n - 3$. It is easy to verify that this inequality is an identity.

For $k = 2n - 3$, $a_1 = \max\{a_1, a_2, \ldots, a_n\}$, and $a_n = \min\{a_1, a_2, \ldots, a_n\}$, the equality occurs when $a_2 = \cdots = a_{n-1} = 1$ and $a_1a_n + (n-2)(a_1 + a_n) = 2n - 3$, with $a_1 \geq 1 \geq a_n$.

557. Find the differentiable functions $f : (0, \infty) \to \mathbb{R}$ that satisfy the identity:

$$f'(x) = x \cdot f\left(\frac{1}{x}\right) \tag{1}$$

for all $x \in (0, \infty)$.

`Proposed by Dorian Popa, Technical University of Cluj-Napoca, Romania.`

Solutions by the author. **First solution.** Based on (1), f is twice differentiable and

$$f''(x) = \left(x \cdot f\left(\frac{1}{x}\right)\right)' = f\left(\frac{1}{x}\right) - \frac{1}{x}f'\left(\frac{1}{x}\right) \overset{(1)}{=} \frac{1}{x}f'(x) - \frac{1}{x}\cdot\frac{1}{x}f(x)$$
$$= \frac{1}{x^2}\left(x \cdot f'(x) - f(x)\right),$$

for all $x \in (0, \infty)$, which leads to

$$x^2 f''(x) - x \cdot f'(x) + f(x) = 0, \quad \text{for all } x \in (0, \infty). \tag{2}$$

Let $x = \mathrm{e}^t$ and $g(t) = f(x) = f(\mathrm{e}^t)$ in (2), for arbitrary $t \in \mathbb{R}$. Then

$$g'(t) = \mathrm{e}^t \cdot f'(\mathrm{e}^t) = x \cdot f'(x),$$
$$g''(t) = \mathrm{e}^t \cdot f'(\mathrm{e}^t) + \mathrm{e}^{2t} \cdot f''(\mathrm{e}^t) = x \cdot f'(x) + x^2 f''(x)$$

and (2) becomes

$$g''(t) - 2g'(t) + g(t) = 0, \quad \text{for all } t \in \mathbb{R}. \tag{3}$$

Equation (3) is a homogeneous linear ODE with constant coefficients with the general solution $g(t) = \mathrm{e}^t(a + bt), \quad a, b \in \mathbb{R}$ constants, that leads to

$$f(x) = g(\ln x) = x(a + b \cdot \ln x), \quad \text{for all } x \in (0, \infty).$$

Then the identity (1) becomes

$$a + b \cdot \ln x + b = x \cdot \frac{1}{x}(a - b \cdot \ln x), \quad \text{for all } x \in (0, \infty),$$

which is equivalent to $b = 0$.

Concluding, the solutions are the functions $f(x) = ax$, for all $x \in (0, \infty)$, with $a \in \mathbb{R}$ constant.

Alternative approach. It is possible to solve (2) without prior knowledge about linear ODEs, by letting $h(x) = \dfrac{f(x)}{x}$ for all $x \in (0, \infty)$, so $f(x) = x \cdot h(x)$. Then (2) becomes $xh''(x) + h'(x) = 0$, which means that $(x \cdot h'(x))' = 0$. This leads to $x \cdot h'(x) = a \in \mathbb{R}$ (constant), then $h'(x) = \dfrac{a}{x}$, with the solutions $h(x) = a \ln x + b$. Then $f(x) = x\,(a \ln x + b)$, and $b = 0$ follows as above.

Second solution. Writing $f(x) = x \cdot h(x)$, the initial identity (1) becomes

$$x \cdot h'(x) = h\left(\frac{1}{x}\right) - h(x), \quad \text{for all } x \in (0, \infty), \tag{4}$$

hence

$$x \cdot h'(x) = h\left(\frac{1}{x}\right) - h(x) = -\left(h(x) - h\left(\frac{1}{x}\right)\right) = -\frac{1}{x}h'\left(\frac{1}{x}\right), \quad \text{for all } x \in (0, \infty),$$

which leads to

$$h'(x) = -\frac{1}{x^2}h'\left(\frac{1}{x}\right) = \left(h\left(\frac{1}{x}\right)\right)', \quad \text{for all } x \in (0, \infty).$$

This means that $h(x) - h\left(\dfrac{1}{x}\right)$ is constant over $(0, \infty)$, hence, by (4), $x \cdot h'(x) = a \in \mathbb{R}$ for all $x \in (0, \infty)$. The rest follows as presented in the previous solution.

We received a solution from Marian-Daniel Vasile, Timișoara, Romania, which follows the same line as the author's **Alternative approach**.

We also received a solution from G.C. Greubel, Newport News, VA, USA, but this is not entirely correct. He considers the more general equation

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \alpha\, x\, f\left(\frac{1}{x}\right),$$

where α is a constant. Upon making the substitution $x\,u = 1$, which gives $\frac{\mathrm{d}u}{\mathrm{d}x} = -u^2$, so $\frac{\mathrm{d}}{\mathrm{d}x} = -u^2\,\frac{\mathrm{d}}{du}$, our equation becomes

$$u^3\,\frac{\mathrm{d}}{\mathrm{d}u}\,f\left(\frac{1}{u}\right) = -\alpha\,f(u).$$

Differentiating on both sides, we get

$$\left(u^3\,\frac{\mathrm{d}^2}{\mathrm{d}u^2} + 3\,u^2\,\frac{\mathrm{d}}{\mathrm{d}u}\right)\,f\left(\frac{1}{u}\right) = -\alpha\,\frac{\mathrm{d}f(u)}{\mathrm{d}u} = -\alpha^2\,u\,f\left(\frac{1}{u}\right),$$

which, after dividing by u, becomes

$$\left(u^2\,\frac{\mathrm{d}^2}{\mathrm{d}u^2} + 3\,u\,\frac{\mathrm{d}}{\mathrm{d}u} + \alpha^2\right)\,f\left(\frac{1}{u}\right) = 0.$$

So $u \mapsto f(1/u)$ is the solution of the differential equation $x^2y'' + 3xy' + \alpha^2 y = 0$. This is a Cauchy–Euler equation, which has solutions of the type $y(x) = x^p$, where p is a solution of the equation $p(p-1) + 3p + \alpha^2 = 0$, i.e., $p^2 + 2p + \alpha^2 = 0$. We have the solutions $p_{1,2} = -1 \pm \sqrt{1-\alpha^2}$, so

$$y(x) = A\,x^{-1+\sqrt{1-\alpha^2}} + B\,x^{-1-\sqrt{1-\alpha^2}},$$

from which we get

$$f(u) = A\,u^{1-\sqrt{1-\alpha^2}} + B\,u^{1+\sqrt{1-\alpha^2}}.$$

Using this form of $f(u)$ in the original equation it is determined that

$$f(u) = A\left(u^{1-\sqrt{1-\alpha^2}} + \frac{1-\sqrt{1-\alpha^2}}{\alpha}\,u^{1+\sqrt{1-\alpha^2}}\right), \tag{5}$$

where A is a constant. The constant α leads to some special cases:

$$\begin{cases} f(u) = 2A\,u & \alpha = 1, \\ f(u) = 0 & \alpha = -1, \\ f(u) = A & \alpha = 0. \end{cases}$$

Remarks. Here are some issues regarding G.C. Greubel's solution.

First note that, because of the denominator α, formula (5) doesn't apply when $\alpha = 0$. An alternative formula, which works for all α, is

$$f(u) = C\left(\alpha u^{1-\sqrt{1-\alpha^2}} + \left(1-\sqrt{1-\alpha^2}\right)\,u^{1+\sqrt{1-\alpha^2}}\right).$$

Next, recall that formula $y(x) = Ax^{p_1} + Bx^{p_2}$ doesn't apply if we have a double root, i.e., if $p_1 = p_2 =: p$. In this case the general solution of the Cauchy-Euler equation is $y(x) = x^p(A\ln x + B)$. In our case, $p^2 + 2p + \alpha^2 = 0$ has a double root iff $\alpha = \pm 1$ and this double root is -1, so $y(x) = x^{-1}(A\ln x + B)$. Then $f(u) = y(1/u) = u(-A\ln u + B)$. After plugging this formula in the original equation, we get $f(u) = Cu$ if $\alpha = 1$ and $f(u) = Cu(2\ln u + 1)$ if $\alpha = -1$.

Finally, if $p_{1,2}$ are complex conjugates, i.e., $p_{1,2} = a \pm bi$, with $b > 0$, then $x^{a\pm bi} = x^a(\cos b\ln x \pm i\sin b\ln x)$. So the solution of the Cauchy-Euler equation is spanned by $x^a\cos b\ln x$ and $x^a\sin b\ln x$. In our case, this happens when $|\alpha| > 1$, when we have $p_{1,2} = -1 \pm i\sqrt{\alpha^2-1}$. We get $y(x) = x^{-1}(A\cos\sqrt{\alpha^2-1}\ln x + B\sin\sqrt{\alpha^2-1}\ln x)$. Then $f(u) = y(1/u)$ writes as $f(u) = u(A\cos\sqrt{\alpha^2-1}\ln u -$

$B\sin\sqrt{\alpha^2-1}\ln u)$. We plug this formula in the original equation and we get $f(u) = Cu(\sqrt{\alpha+1}\cos\sqrt{\alpha^2-1}\ln u - \sqrt{\alpha-1}\sin\sqrt{\alpha^2-1}\ln u)$.

Hence we have a formula for $f(u)$ in each of the cases $\alpha = 1$, $\alpha = -1$, $|\alpha| < 1$ and $|\alpha| > 1$.

558. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 x^k f(x)\,\mathrm{d}x = 0 \quad \text{for} \quad 0 \le k \le n-1$$

and

$$\int_0^1 x^n f(x)\,\mathrm{d}x = 1.$$

Prove that

$$\int_0^1 f^2(x)\,\mathrm{d}x \ge (2n+1)\binom{2n}{n}^2.$$

Proposed by Cezar Lupu, Beijing Institute of Mathematical Sciences and Applications (BIMSA), Tsinghua University, Beijing, P.R. China.

Solution by the author. Let P_k be the k-th Legendre polynomial. By Rodrigues's formula,

$$P_k(x) = \frac{1}{2^k k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^2-1)^k.$$

Then P_k is of degree k, with the leading coefficient $\dfrac{1}{2^k}\dbinom{2k}{k}$, and we have the following property

$$\int_{-1}^1 P_j(x)P_k(x)\,\mathrm{d}x = \frac{2}{2k+1}\delta_{j,k}.$$

Next, we define the shifted Legendre polynomials $\widetilde{P}_k(x) = P_k(2x-1)$. Then, by a change of variables, we get

$$\int_0^1 \widetilde{P}_j(x)\widetilde{P}_k(x)\,\mathrm{d}x = \frac{1}{2k+1}\delta_{j,k}.$$

Also, the shifted Legendre polynomial $\widetilde{P}(x)$ has degree n and the leading coefficient equal to $\dbinom{2n}{n}$. Then the hypothesis implies that

$$\int_0^1 \widetilde{P}_n(x)f(x)\,\mathrm{d}x = \binom{2n}{n}$$

and

$$\int_0^1 \widetilde{P}_k(x)f(x)\,\mathrm{d}x = 0 \quad \text{for} \quad 0 \le k \le n-1.$$

If we denote by $\langle\cdot,\cdot\rangle$ the inner product and by $||\cdot||$ the norm from $L^2([0,1])$, then the above relations write as $\langle\widetilde{P}_j,\widetilde{P}_k\rangle = \dfrac{1}{2k+1}\delta_{j,k}$, $\langle\widetilde{P}_n, f\rangle = \dbinom{2n}{n}$, and $\langle\widetilde{P}_k, f\rangle = 0$ for $0 \le k \le n-1$.

If $e_k(x) = \sqrt{2k+1}\,\widetilde{P}_k(x)$, then $e_1, e_2, \ldots$ is a sequence of orthonormal functions in $L^2([0,1])$ and we have $\langle f, e_n\rangle = \sqrt{2n+1}\binom{2n}{n}$ and $\langle f, e_k\rangle = 0$ for $0 \le k \le n-1$. Then, by Bessel's inequality, we have

$$||f|| \ge \sum_{k=1}^{n} |\langle f, e_k\rangle|^2,$$

which translates to

$$\int_0^1 f^2(x)\,\mathrm{d}x \ge \left(\sqrt{2n+1}\cdot\binom{2n}{n}\right)^2 = (2n+1)\binom{2n}{n}^2$$

and this concludes the proof.

The equality holds iff

$$f = \sum_{k=1}^{n}\langle f, e_k\rangle e_k = \sqrt{2n+1}\binom{2n}{n} e_n = (2n+1)\binom{2n}{n}\widetilde{P}(x).$$

Note. As pointed out by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany, this result is a consequence of Satz 2.1 in [1], which states that if $a < b$ and $\int_a^b x^k f(x)\,\mathrm{d}x = 0$ for $k = 0, \ldots, n-1$, then

$$\left|\int_a^b x^n f(x)\,\mathrm{d}x\right| \le C_n \left(\int_a^b |f(x)|^2\,\mathrm{d}x\right)^{1/2},$$

where

$$C_n = \sqrt{\frac{(n!)^4(b-a)^{2n+1}}{(2n)!(2n+1)!}} = \frac{(b-a)^{n+1/2}}{\sqrt{2n+1}}\binom{2n}{n}^{-1},$$

with equality iff f is of the form $f(x) = c\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}[(x-a)(x-b)]^n$.

If $\int_a^b x^n f(x)\,\mathrm{d}x = 1$, this implies $\int_a^b |f(x)|^2\,\mathrm{d}x \ge C_n^{-2} = \dfrac{2n+1}{(b-a)^{2n+1}}\binom{2n}{n}^2$. In our case $a = 0$ and $b = 1$, so we get the claimed equality.

References

[1] U. Abel, Integralungleichungen aus der Hilbertraum-Theorie, *Elemente der Mathematik*, **38** (1983), 144–152.

559. Let $f : [0,1] \to [-1,1]$ be a continuous function, with finite derivative in 0 and $f(0) = 1$. Find $\lim_{n\to\infty}\int_0^1 f^n(x^n)\,\mathrm{d}x$.

`Proposed by Mircea Rus, Technical University of Cluj-Napoca, Romania.`

Solution by the author. Since $|f^n(x^n)| \le 1$, for all $n \ge 1$ and all $x \in [0,1]$, we may find the limit by using *the bounded convergence theorem*, so we need to investigate de pointwise limit $\lim_{n\to\infty} f^n(x^n)$.

Since f has finite derivative at 0, there exists $a \in (0,1)$ such that

$$\left|\frac{f(t)-f(0)}{t} - f'(0)\right| \leq 1,$$

for all $t \in (0,a)$, which leads to

$$|f(t)-1| \leq bt, \quad \text{for all } t \in [0,a),$$

where $b = 1 + |f'(0)|$. Also $|f(t)| \leq 1$, so

$$|f^n(t)-1| = \left|f^{n-1}(t) + \cdots + 1\right| \cdot |f(t)-1| \leq n\,|f(t)-1|$$

for all $t \in [0,1]$ and $n \geq 1$. Hence

$$|f^n(t)-1| \leq bnt, \text{ for all } t \in [0,a) \text{ and } n \geq 1. \tag{1}$$

Now, let $x \in [0,1)$. Since $\lim\limits_{n\to\infty} x^n = 0$, there exists $k \in \mathbb{N}$ such that $x^n \in [0,a)$, for all $n \geq k$. Letting $t = x^n$ in (1), we obtain

$$|f^n(x^n)-1| \leq bnx^n, \text{ for all } n \geq k. \tag{2}$$

Since $\lim\limits_{n\to\infty} nx^n = 0$, it follows from (2) that

$$\lim_{n\to\infty} f^n(x^n) = 1, \text{ for all } x \in [0,1).$$

We can apply now the *bounded convergence theorem* and obtain

$$\lim_{n\to\infty} \int_0^1 f^n(x^n)\,\mathrm{d}x = \int_0^1 \mathrm{d}x = 1.$$

560. Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_1 \in (0,1)$ and $x_{n+1} = x_n - \frac{x_n^2}{2^n}$ for all $n \geq 1$. Prove that the sequence $(x_n)_{n\geq 1}$ is convergent to a limit $C > 0$ and moreover,

$$\lim_{n\to\infty} 8^{n-1}\left(x_n - C - \frac{C^2}{2^{n-1}} - \frac{C^3}{3\cdot 4^{n-2}}\right) = \frac{12C^4 + 32C^3}{21}.$$

Proposed by Dumitru Popa, Ovidius University of Constanța, Romania.

Solution by the author. The sequence $(x_n)_{n\geq 1}$ is obviously decreasing and, by induction, we deduce that $x_n \in (0,1)$ for all $n \geq 1$. From $x_n - x_{n+1} = \frac{x_n^2}{2^n}$ for all $n \geq 1$ we deduce that $x_1 - x_{n+1} = \sum\limits_{k=1}^{n} \frac{x_n^2}{2^n} \leq \sum\limits_{k=1}^{n} \frac{x_1^2}{2^n} = x_1^2\left(1 - \frac{1}{2^n}\right) < x_1^2$ (the sequence is decreasing positive). Thus $x_n \geq x_1 - x_1^2$. Since $(x_n)_{n\geq 1}$ is decreasing and bounded bellow by $x_1 - x_1^2$, it is convergent and $\lim\limits_{n\to\infty} x_n = C$, with $C \in [x_1 - x_1^2, x_1] \subset (0,1)$.

Let $y_n = x_n - C$. From the recurrence relation we deduce that

$$\lim_{n\to\infty} \frac{y_{n+1} - y_n}{\frac{1}{2^n} - \frac{1}{2^{n-1}}} = \lim_{n\to\infty} \frac{x_{n+1} - x_n}{-\frac{1}{2^n}} = \lim_{n\to\infty} x_n^2 = C^2$$

By the $\frac{0}{0}$ case of Cesàro's lemma, we get $\lim\limits_{n\to\infty} \frac{y_n}{\frac{1}{2^{n-1}}} = C^2$, i.e., $\lim\limits_{n\to\infty} \frac{x_n - C}{\frac{1}{2^{n-1}}} = C^2$.

Now let $z_n = y_n - \frac{C^2}{2^{n-1}} = x_n - C - \frac{C^2}{2^{n-1}}$. We have

$$z_{n+1} - z_n = x_{n+1} - x_n + \frac{C^2}{2^n} = -\frac{x_n^2}{2^n} + \frac{C^2}{2^n} = -\frac{1}{2^n}(x_n - C)(x_n + C).$$

Hence

$$\lim_{n\to\infty} \frac{z_{n+1}-z_n}{\frac{1}{4^{n-1}}-\frac{1}{4^{n-2}}} = \lim_{n\to\infty} \frac{-\frac{1}{2^n}(x_n-C)(x_n+C)}{-\frac{3}{4^{n-1}}} = \lim_{n\to\infty} \frac{x_n-C}{-\frac{3}{2^{n-2}}}\cdot(x_n+C)$$
$$= \frac{C^2}{6}\cdot 2C = \frac{C^3}{3}.$$

Again by the case $\frac{0}{0}$ of Cesàro's lemma, $\lim_{n\to\infty} \frac{z_n}{\frac{1}{4^{n-2}}} = \frac{C^3}{3}$, that is $\lim_{n\to\infty} 4^{n-2} z_n = \frac{C^3}{3}$. Since $z_n = y_n - \frac{C^2}{2^{n-1}} = x_n - C - \frac{C^2}{2^{n-1}}$ we get

$$\lim_{n\to\infty} 4^{n-2}\left(x_n - C - \frac{C^2}{2^{n-1}}\right) = \frac{C^3}{3}. \tag{1}$$

Finally, let $w_n = z_n - \frac{C^3}{3\cdot 4^{n-2}} = x_n - C - \frac{C^2}{2^{n-1}} - \frac{C^3}{3\cdot 4^{n-2}}$. Note that

$$w_{n+1}-w_n = z_{n+1}-z_n - \frac{C^3}{4^{n-1}} = -\frac{x_n^2}{2^n} + \frac{C^2}{2^n} + \frac{C^3}{4^{n-1}} = -\frac{1}{2^n}\left(x_n^2 - C^2 - \frac{C^3}{2^{n-2}}\right).$$

From (1) we get $x_n = C + \frac{2C^2}{2^n} + \frac{16C^3}{3\cdot 4^n} + o\left(\frac{1}{4^n}\right)$, which implies that

$$x_n^2 = \left(C + \frac{2C^2}{2^n} + \frac{16C^3}{3\cdot 4^n}\right)^2 + o\left(\frac{1}{4^n}\right) = C^2 + \frac{C^3}{2^{n-2}} + \frac{4C^4}{4^n} + \frac{32C^3}{3\cdot 4^n} + o\left(\frac{1}{4^n}\right)$$
$$= C^2 + \frac{C^3}{2^{n-2}} + \frac{12C^4+32C^3}{3\cdot 4^n} + o\left(\frac{1}{4^n}\right).$$

It follows that

$$w_{n+1}-w_n = -\frac{1}{2^n}\left(x_n^2 - C^2 - \frac{C^3}{2^{n-1}}\right) = -\frac{1}{2^n}\left(\frac{12C^4+32C^3}{3\cdot 4^n} + o\left(\frac{1}{4^n}\right)\right)$$
$$= -\frac{12C^4+32C^3}{3\cdot 8^n} + o\left(\frac{1}{8^n}\right),$$

or, equivalently, $\lim_{n\to\infty} 8^n(w_{n+1}-w_n) = -\frac{12C^4+32C^3}{3}$. Then

$$\lim_{n\to\infty} \frac{w_{n+1}-w_n}{\frac{1}{8^n}-\frac{1}{8^{n-1}}} = \lim_{n\to\infty} -\frac{8^n(w_{n+1}-w_n)}{7} = \frac{12C^4+32C^3}{21}.$$

Then, by the case $\frac{0}{0}$ of Cesàro's lemma, $\lim_{n\to\infty} \frac{w_n}{\frac{1}{8^{n-1}}} = \frac{12C^4+32C^3}{3}$, i.e. $\lim_{n\to\infty} 8^{n-1}w_n = \frac{12C^4+32C^3}{21}$ and the proof is finished.

561. Calculate

$$\sum_{n=1}^{\infty}\left[n^2\left(\frac{1}{n^3} - \frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} - \cdots\right) - \frac{1}{2n}\right].$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania.

Solution by the authors. The series equals $\frac{\pi^2}{12}$.

Let $a_n = \frac{1}{n^3} - \frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} - \cdots$. We observe that $a_n + a_{n+1} = \frac{1}{n^3}$, $\forall n \geq 1$.

Using the relation $\frac{1}{k^3} = \frac{1}{2}\int_0^1 x^{k-1}\ln^2 x\,\mathrm{d}x$ for $k \geq 1$, we get

$$a_n = \frac{1}{n^3} - \frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} - \cdots = \frac{1}{2}\int_0^1 \left(x^{n-1} - x^n + x^{n+1} + \cdots\right)\ln^2 x\,\mathrm{d}x$$
$$= \frac{1}{2}\int_0^1 \frac{x^{n-1}}{1+x}\ln^2 x\,\mathrm{d}x < \frac{1}{2}\int_0^1 x^{n-1}\ln^2 x\,\mathrm{d}x = \frac{1}{n^3}.$$

It follows that $0 < n^2 a_n < \frac{1}{n}$, and so $\lim_{n\to\infty} n^2 a_n = 0$.

We calculate

$$\sum_{n=1}^{\infty}\left[n^2\left(\frac{1}{n^3} - \frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} - \cdots\right) - \frac{1}{2n}\right] = \sum_{n=1}^{\infty}\left(n^2 a_n - \frac{1}{2n}\right)$$
$$= \sum_{n=1}^{\infty} n^2\left(a_n - \frac{1}{2n^3}\right) = \sum_{n=1}^{\infty} n^2\left(a_n - \frac{a_n + a_{n+1}}{2}\right) = \frac{1}{2}\sum_{n=1}^{\infty}(n^2 a_n - n^2 a_{n+1})$$
$$= \frac{1}{2}\sum_{n=1}^{\infty}(n^2 a_n - (n+1)^2 a_{n+1}) + \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)a_{n+1}$$
$$= \frac{1}{2}\left(a_1 - \lim_{n\to\infty}(n+1)^2 a_{n+1}\right) + \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)a_{n+1}$$
$$= \frac{1}{2}a_1 + \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)a_{n+1} = \frac{1}{2}\sum_{n=1}^{\infty}(2n-1)a_n.$$

We calculate the preceding series. We have

$$\sum_{n=1}^{\infty}(2n-1)a_n = \sum_{n=1}^{\infty}(2n-1)\frac{1}{2}\int_0^1 \frac{x^{n-1}}{1+x}\ln^2 x\,\mathrm{d}x$$
$$= \frac{1}{2}\int_0^1 \frac{\ln^2 x}{1+x}\left(\sum_{n=1}^{\infty}(2n-1)x^{n-1}\right)\mathrm{d}x$$
$$= \frac{1}{2}\int_0^1 \frac{\ln^2 x}{1+x}\cdot\frac{1+x}{(1-x)^2}\mathrm{d}x = \frac{1}{2}\int_0^1 \frac{\ln^2 x}{(1-x)^2}\mathrm{d}x$$
$$= \frac{1}{2}\int_0^1 \ln^2 x\left(\sum_{n=1}^{\infty} n x^{n-1}\right)\mathrm{d}x = \frac{1}{2}\sum_{n=1}^{\infty} n\int_0^1 x^{n-1}\ln^2 x\mathrm{d}x$$
$$= \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6},$$

and it follows that

$$\sum_{n=1}^{\infty}\left[n^2\left(\frac{1}{n^3} - \frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} - \cdots\right) - \frac{1}{2n}\right] = \frac{\pi^2}{12}.$$

The problem is solved.

562. For any matrix M, let $M^* = \overline{M}^t$ denote the transpose conjugate of M. The matrix M is called *anti-Hermitian* if $M^* = -M$.

Prove that if $A \in \mathcal{M}_n(\mathbb{C})$ is invertible and anti-Hermitian, then the function

$$f : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C}), \quad f(X) = AX - XA^2, \quad X \in \mathcal{M}_n(\mathbb{C}),$$

is bijective.

Proposed by Mihai Opincariu, Brad, Romania, and Vasile Pop, Technical University of Cluj-Napoca, Romania.

Solution by the authors. We first prove that all eigenvalues of A are non-zero purely imaginary numbers.

Indeed, let $\lambda \in \mathbb{C}$ be an eigenvalue of A, and U a corresponding eigenvector, i.e., $AU = \lambda U$. Then

$$U^*AU = (U^*A)U = -(U^*A^*)U = -(AU)^*U = -(\lambda U)^*U = -\overline{\lambda}(U^*U)$$

and, at the same time,

$$U^*AU = U^*(AU) = U^*\lambda U = \lambda(U^*U),$$

therefore $(\lambda + \overline{\lambda})(U^*U) = 0$. With $U = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix}^t \neq O$, it follows that $U^*U = \sum_{k=1}^n |u_k|^2 \neq 0$, so $\lambda + \overline{\lambda} = 0$, proving that $\Re\lambda = 0$, i.e., λ is purely imaginary. Moreover, $\lambda \neq 0$, since A is invertible.

Now, f is a linear function between two vector spaces over $\mathbb{C}$ of the same dimension n^2. Therefore, in order to prove its bijectivity, it suffices to prove it is injective, i.e., that $\ker f = 0$. This means that the only solution of $f(X) = AX - XA^2 = O$ is $X = O$. Henceforth, it remains to show that if X satisfies

$$AX = XA^2, \tag{1}$$

then $X = O$.

Let $X \in \mathcal{M}_n(\mathbb{C})$ verifying (1). Then $A^2X = A(AX) = A(XA^2) = (AX)A^2 = (XA^2)A^2 = XA^4$ and by induction,

$$A^kX = XA^{2k}, \quad \text{for all } k \in \mathbb{N}. \tag{2}$$

Let $P_A(z) = c_0 + c_1z + \cdots + c_nz^n$ be the characteristic polynomial of A. Then, by (2), we have $\sum_{k=0}^n c_kA^kX = \sum_{k=0}^n c_kXA^{2k}$, i.e., $P_A(A)X = XP_A(A^2)$, which leads to

$$XP_A(A^2) = O, \tag{3}$$

since $P_A(A) = O$. The eigenvalues of A are $ia_1, \dots, ia_n$, for some $a_1, \dots, a_n \in \mathbb{R}^*$. Then $P_A(A^2) = \big(A^2 - (ia_1)I_n\big) \cdots \big(A^2 - (ia_n)I_n\big)$. But the eigenvalues of A^2 are $(ia_1)^2, \dots, (ia_n)^2$, that is, $-a_1^2, \dots, -a_n^2$. Then, for every k we have $ia_k \notin \{-a_1^2, \dots, -a_n^2\}$, so ia_k is not an eigenvalue for A^2 and so $\det\big(A^2 - (ia_k)I_n\big) \neq 0$. It follows that $\det P_A(A^2) \neq 0$, so $P_A(A^2)$ is invertible. Based on (3), we obtain $X = O$, which concludes the proof.

Solution by Robert Rogoszan, Baia Mare, Romania. Recall Sylvester's equation $AX - XB = C$. It is known that for $A, B, C \in \mathcal{M}_n(\mathbb{C})$ this equation has a unique solution iff $\operatorname{Spec}(A) \cap \operatorname{Spec}(B) = \emptyset$. (Here $\operatorname{Spec}(X)$ denotes the spectrum of X, i.e., the set of all eigenvalues of X.)

We will now prove that if $A = -A^*$, then $\operatorname{Spec}(A) \cap \operatorname{Spec}(A^2) = \emptyset$, which, by the above property of Sylvester's equation, is equivalent to f being bijective.

It can be easily proved that if A is anti-Hermitian, then all its eigenvalues are purely imaginary numbers, i.e., $\mathrm{Spec}(A) \subset i\mathbb{R}$. As A is invertible, we have $0 \notin \mathrm{Spec}(A)$, so $\mathrm{Spec}(A) \subset i\mathbb{R}^*$.

Now every eigenvalue of A^2 is of the form λ^2, for some $\lambda \in \mathrm{Spec}(A)$. Then $\lambda = ai$ for some $a \in \mathbb{R}^*$, so $\lambda^2 = -a^2 \in \mathbb{R}^*$. Thus $\mathrm{Spec}(A^2) \subset \mathbb{R}^*$. Since $\mathrm{Spec}(A) \subset i\mathbb{R}^*$, we have $\mathrm{Spec}(A) \cap \mathrm{Spec}(A^2) = \emptyset$ and so f is bijective.