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On the limit of a sequence of certain integrals

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Abstract. The solution to a problem, which was published in the problem section of *The College Mathematics Journal*, is not correct. While the outcome is the correct formula in a special case, the approach cannot be considered as a proof. We give a correct solution and extend the problem. To this end we demonstrate the application of a general method to derive asymptotic expansions for integrals. Furthermore, we deal with a sequence of very similar integrals, which appeared as a problem of the 14th South Eastern European Mathematical Olympiad for University Students (SEEMOUS 2020) and was presented in Gazeta Matematică, Seria A.

Keywords: One-variable calculus, asymptotic expansions.

MSC: Primary 26A06; Secondary 41A60.

1. Introduction

Ovidiu Furdui and Alina Sîntămărian [1] posed the Problem 1181 to calculate, for each real number $k > 0$, the limit

$$L = \lim_{n\to\infty} I_n$$

for the sequence of integrals

$$I_n = \int_0^1 \left(\frac{\sqrt[n]{x} + k - 1}{k} \right)^n \mathrm{d}x. \tag{1}$$

Moreover, they asked for the value of the limit $\lim_{n\to\infty} n\,(I_n - L)$. The published solution [6] to Problem 1181 is not mathematically rigorous, and the case $k \in (0,1)$ is not discussed in the given solution. The main purpose

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of this note is to present a correct and complete solution. Furthermore we extend the solution by deriving an asymptotic expansion.

A similarly looking problem, viz., determine the limit of the sequence of integrals

$$J_n = \int_0^1 \left(\frac{k}{\sqrt[n]{x} + k - 1} \right)^n \mathrm{d}x, \tag{2}$$

appeared as a problem of the 14th South Eastern European Mathematical Olympiad for University Students (SEEMOUS 2020) [5, Problem 2]. This problem was proposed, for $k > 1$, in [2]. Six solutions by different solvers were presented in [3]. A further solution was presented in [4, No 35/38, The Asymptotic Behaviour of Integrals, pages 486–487]. We derive an asymptotic expansion also for the sequence of integrals (2).

2. The integrals I_n

In [6] it was claimed that, for $k > 0$, it holds $L = k/(k+1)$ as well as $\lim_{n\to\infty} n(I_n - L) = k(k-1)/(k+1)^3$. Both values are correct for $k \geq 1$, but the limit $\lim_{n\to\infty} I_n$ does not exist for small positive values of the parameter k, such that the second limit cannot be accepted. More precisely, for $0 < k \leq 3 - \sqrt{8} \approx 0.171573$, the sequence $(I_n)_{n\geq 1}$ is oscillating and $(-1)^n I_n$ tends to infinity. Furthermore, the asymptotic relation

$$\left(\frac{\sqrt[n]{x} + k - 1}{k} \right)^n = x^{1/k} + \frac{(k-1)\, x^{1/k} \log^2(x)}{2k^2 n} + O\left(n^{-2}\right) \tag{3}$$

as $n \to \infty$, was used to conclude that

$$I_n = \int_0^1 x^{1/k} \mathrm{d}x + \frac{k-1}{2k^2 n} \int_0^1 x^{1/k} \log^2(x)\, \mathrm{d}x + O\left(n^{-2}\right) \tag{4}$$

as $n \to \infty$. While the outcome is partially correct, this approach cannot be considered as a proof, because it needs a precise study in which kind the Landau term $O\left(n^{-2}\right)$ in Eq. (3) depends on the variable x. A well-known counter-example is $f_n(x) = 2nx \exp\left(-nx^2\right) = o(1)$ as $n \to \infty$, for each $x \in [0,1]$, while $\lim_{n\to\infty} \int_0^1 f_n(x)\, \mathrm{d}x = 1 \neq 0 = \int_0^1 \lim_{n\to\infty} f_n(x)\, \mathrm{d}x$.

Evaluation of the integrals in (4) leads to the asymptotic relation

$$I_n = \frac{k}{k+1} + \frac{k(k-1)}{(k+1)^3 n} + O\left(n^{-2}\right) \tag{5}$$

as $n \to \infty$, which is the desired result.

In this note we give a closer inspection of the sequence $(I_n)_{n\geq 1}$ in dependence of the real parameter $k > 0$. Moreover, we use a method to derive a complete asymptotic expansion

$$I_n \sim \sum_{j=0}^{\infty} \frac{c_j(k)}{n^j} \qquad (n \to \infty)$$

and show how to determine the coefficients $c_j(k)$ in explicit form. The latter relation means, that for each $q \in \mathbb{N}_0$,

$$I_n \sim \sum_{j=0}^{q} \frac{c_j(k)}{n^j} + o\left(n^{-q}\right) \qquad (n \to \infty).$$

Formula (5) provides the coefficients

$$c_0(k) = k/(k+1) \quad \text{and} \quad c_1(k) = k(k-1)/(k+1)^3.$$

By the change of variable $z = 1 - \sqrt[n]{x}$, i.e., $x = (1-z)^n$, we obtain the representation

$$I_n = n \int_0^1 \left(\frac{k-z}{k}\right)^n (1-z)^{n-1} \,\mathrm{d}z.$$

Let us first consider the case $k > 1$. Then $(k-z)(1-z) > 0$, for $z \in [0,1)$, and we make a further change of variable:

$$\begin{aligned} e^{-t} &= \frac{1}{k}(k-z)(1-z), \\ -e^{-t}\mathrm{d}t &= \frac{1}{k}(2z-k-1)\,\mathrm{d}z. \end{aligned}$$

Since $0 \le z < 1$, the quadratic equation $ke^{-t} = z^2 - (k+1)z + k$ yields

$$z = \frac{k+1}{2} - \sqrt{\left(\frac{k+1}{2}\right)^2 - k\left(1-e^{-t}\right)} = \frac{1}{2}\left(k+1-\sqrt{(k-1)^2 + 4ke^{-t}}\right).$$

An easy calculation leads to the representation as a Laplace integral

$$I_n = \frac{n}{2}\int_0^\infty \left(1 + \frac{k-1}{\sqrt{(k-1)^2 + 4ke^{-t}}}\right) e^{-nt}\mathrm{d}t.$$

Thus, we can apply Watson's lemma (see, e.g., [7, p. 22]) which states that

$$\int_0^\delta e^{-st} F(t)\,\mathrm{d}t \sim \sum_{j=0}^{\infty} a_j \frac{\Gamma(\lambda_j)}{s^{\lambda_j}} \qquad (s \to \infty),$$

provided that the integral exists for a certain $s \ge 0$ and $F(t) \sim \sum_{j=0}^{\infty} a_j t^{\lambda_j - 1}$ as $t \to 0+$, where $0 < \lambda_0 < \lambda_1 < \cdots$, and Γ denotes the gamma function. Taylor expansion of

$$F(t) := 1 + \frac{k-1}{\sqrt{(k-1)^2 + 4ke^{-t}}}$$

yields, for sufficiently small t,

$$\begin{aligned} F(t) &= \frac{2k}{k+1} + \frac{2k(k-1)}{(k+1)^3} t - \frac{k(k-1)\left(k^2-4k+1\right)}{(k+1)^5} t^2 \\ &\quad + \frac{k(k-1)\left(k^4-14k^3+30k^2-14k+1\right)}{3(k+1)^7} t^3 + \cdots, \end{aligned}$$

such that

$$\begin{aligned} I_n &\sim \frac{k}{k+1} + \frac{k(k-1)}{(k+1)^3 n} - \frac{k(k-1)\left(k^2-4k+1\right)}{(k+1)^5 n^2} \\ &\quad + \frac{k(k-1)\left(k^4-14k^3+30k^2-14k+1\right)}{(k+1)^7 n^3} + \cdots \qquad (6) \end{aligned}$$

as $n \to \infty$, for $k > 1$. In particular,

$$L := \lim_{n\to\infty} I_n = \frac{k}{k+1}$$

and

$$\lim_{n\to\infty} n(I_n - L) = \frac{k(k-1)}{(k+1)^3}.$$

In the trivial case $k = 1$ direct evaluation yields $I_n = 1/2$.

Now we turn to the case $0 < k < 1$. To this end we split the integral I_n into two parts

$$I_n = n\int_0^k \left(\frac{k-z}{k}\right)^n (1-z)^{n-1}\,\mathrm{d}z + n\int_k^1 \left(\frac{k-z}{k}\right)^n (1-z)^{n-1}\,\mathrm{d}z = I_n' + I_n'',$$

say. I_n' has the same asymptotic behavior like I_n. We rewrite the second integral in the form

$$I_n'' = (-1)^n \frac{n}{k^n} \int_k^1 (z-k)(g(z))^{n-1}\,\mathrm{d}z,$$

where

$$g(z) = (z-k)(1-z).$$

The quadratic polynomial g is nonnegative on the interval $[k, 1]$ and attains its maximum $g(z_k) = \left(\frac{1-k}{2}\right)^2$ in the middle of the interval at the point $z_k := (k+1)/2$, where $0 < k < z_k < 1$. With $m_k := (1-k)/2$ we have

$$g(z) = m_k^2 - (z - z_k)^2.$$

Noting that $k - z_k = -m_k$ and $1 - z_k = m_k$, we obtain

$$I_n'' = (-1)^n \frac{n}{k^n} \int_{-m_k}^{m_k} (z + m_k)\left(m_k^2 - z^2\right)^{n-1}\,\mathrm{d}z.$$

Observing that $z\left(m_k^2-z^2\right)^{n-1}$ is an odd function, the representation simplifies to

$$I_n''=(-1)^n\frac{2n}{k^n}m_k\int_0^{m_k}\left(m_k^2-z^2\right)^{n-1}\mathrm{d}z.$$

The change of variable $z=m_k\sqrt{t}$ implies that

$$I_n''=(-1)^n\frac{n}{k^n}m_k^{2n}\int_0^1(1-t)^{n-1}t^{-1/2}\mathrm{d}t=(-1)^n\frac{n}{k^n}m_k^{2n}B\left(n,\frac{1}{2}\right),$$

where $B(\cdot,\cdot)$ denotes the Euler beta function. Taking advantage of the relations $B\left(n,\frac{1}{2}\right)=\sqrt{\pi}\Gamma(n)/\Gamma(n+1/2)$ $(n>0)$ and $\Gamma(n+1/2)/\Gamma(n)\sim\sqrt{n}\to\infty$ as $n\to\infty$, we arrive at

$$I_n''\sim(-1)^n\frac{\sqrt{\pi n}}{k^n}m_k^{2n}\qquad(n\to\infty).$$

Now,

$$(-1)^n I_n''\sim\frac{\sqrt{\pi n}}{k^n}m_k^{2n}=\frac{\sqrt{\pi n}}{k^n}\left(\frac{1-k}{2}\right)^{2n}=\sqrt{\pi n}\left(\frac{1-2k+k^2}{4k}\right)^n$$

decays to zero if and only if $1-2k+k^2<4k$. For $0<k<1$, this is equivalent to $k>3-\sqrt{8}\approx 0.171573$. Note that $\sqrt{\pi n}k^{-n}m_k^{2n}=\sqrt{\pi n}$ for $k=3-\sqrt{8}$.

Summarizing, we have the following result:

For $k>3-\sqrt{8}\approx 0.171573$, the asymptotic expansion (6) holds true. In the case $0<k\le 3-\sqrt{8}$ the sequence $(I_n)_{n\ge1}$ has an oscillating behaviour such that $\lim_{n\to\infty}(-1)^n I_n=\infty$.

3. The integrals J_n

In this section we derive an asymptotic expansion for the sequence of integrals (2). It turns out that this task is easier than the corresponding result for the integrals (1). We follow the approach [4, No 35/38, The Asymptotic Behaviour of Integrals, pages 486–487] by making the change of variable

$$x=\left(\frac{(k-1)s}{k-s}\right)^n$$

and obtain

$$J_n=kn\int_0^1\frac{1}{k-s}s^{n-1}\mathrm{d}s.$$

A further change of variable $s=e^{-t}$ leads to the Laplace integral

$$J_n=kn\int_0^\infty\frac{1}{k-e^{-t}}e^{-nt}\mathrm{d}t.$$

As in the previous section we can apply Watson's lemma. Taylor expansion of $F(t) := 1/\left(k - e^{-t}\right)$ yields, for sufficiently small t,

$$\begin{aligned} F(t) &= \frac{1}{k-1} - \frac{t}{(k-1)^2} + \frac{k+1}{2(k-1)^3}t^2 - \frac{k^2+4k+1}{6(k-1)^4}t^3 \\ &\quad + \frac{k^3+11k^2+11k+1}{24(k-1)^5}t^4 + \cdots, \end{aligned}$$

such that

$$\begin{aligned} J_n &\sim \frac{k}{k-1} - \frac{k}{(k-1)^2 n} + \frac{k(k+1)}{(k-1)^3 n^2} - \frac{k^3+4k^2+k}{(k-1)^4 n^3} \\ &\quad + \frac{k^4+11k^3+11k^2+k}{(k-1)^5 n^4} + \cdots \end{aligned}$$

as $n \to \infty$, for $k > 1$. In particular,

$$L_J := \lim_{n\to\infty} J_n = \frac{k}{k-1}$$

and

$$\lim_{n\to\infty} n\left(L_J - J_n\right) = \frac{k}{(k-1)^2}.$$

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Asymptotic expansions of the sum $\sum_{k=1}^{n} f\left(\sqrt[kn]{kn}\right)$

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Abstract. Let $\eta > 0$ and $f : [1, 1+\eta) \to \mathbb{R}$ be a function. We prove:
a) if f is differentiable at 1, then

$$\lim_{n\to\infty} \frac{n}{\ln^2 n} \sum_{k=1}^{n} \left[f\left(\sqrt[kn]{kn}\right) - f(1) \right] = \frac{3f'(1)}{2}$$

b) if f is twice differentiable at 1, then

$$\sum_{k=1}^{n} \left[f\left(\sqrt[kn]{kn}\right) - f(1) \right] = \frac{3f'(1)}{2} \cdot \frac{\ln^2 n}{n} + \gamma f'(1) \cdot \frac{\ln n}{n} + \gamma_1 f'(1) \cdot \frac{1}{n} + \frac{\pi^2 \left[f'(1) + f''(1)\right]}{12} \cdot \frac{\ln^2 n}{n^2} + o\left(\frac{\ln^2 n}{n^2}\right),$$

where γ is the Euler constant and γ_1 the first Stieltjes constant. As an application we deduce that

$$\prod_{k=1}^{n} \left(1 - \sqrt[kn]{kn} + \sqrt[kn]{k^2 n^2}\right) = 1 + \frac{3\ln^2 n}{2n} + \frac{\gamma \ln n}{n} + \frac{\gamma_1}{n} + \frac{9\gamma \ln^4 n}{8n^2} + \frac{3\gamma \ln^3 n}{2n^2} + \frac{\left(\pi^2 + 16\gamma_1 + 6\gamma^2\right) \ln^2 n}{12n^2} + o\left(\frac{\ln^2 n}{n^2}\right).$$

Keywords: Convergence and divergence of series and sequences, Euler-Maclaurin summation formula, orders of infinity, asymptotic expansion of a function, asymptotic expansion of a sequence, Stolz–Cesàro lemma, Euler constant, Stieltjes constant.
MSC: 40A05, 26A12, 40A25.

1. Introduction

In [4] the author proposed the following

Problem 1. *Find the value of the limit* $\lim_{n\to\infty} \frac{n}{\ln^2 n} \sum_{k=1}^{n} \left(\sqrt[kn]{kn} - 1\right)$.

The main purpose of this paper is to refine this evaluation. Moreover, this example suggested finding the asymptotic expansion of the sum $\sum_{k=1}^{n} f\left(\sqrt[kn]{kn}\right)$ in the case when f is differentiable, and when f is twice differentiable. As application we find the asymptotic expansion of the product $\prod_{k=1}^{n} \varphi\left(\sqrt[kn]{kn}\right)$ for φ twice differentiable at 1.

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In this paper all notations and notions used are standard. We recall just that the notation $a_n = o(b_n)$ means that $\forall\varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$ we have $|a_n| \leq \varepsilon |b_n|$. If $b_n \neq 0$ $\forall n \geq n_0$, the condition $a_n = o(b_n)$ is equivalent to $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$, while the notation $a_n \backsimeq b_n$ means $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$.

Our solution for Problem 1 is the following. Let $\varepsilon > 0$. Since $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$, there exists $\delta_\varepsilon > 0$ such that for all $0 < x < \delta_\varepsilon$ we have $\left|\frac{e^x-1}{x} - 1\right| < \varepsilon$, whence

$$|e^x - 1 - x| \leq \varepsilon x \text{ for all } 0 \leq x < \delta_\varepsilon. \tag{1}$$

Since $\lim\limits_{n\to\infty} \frac{2\ln n}{n} = 0$, for $\delta_\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$ we have $\frac{2\ln n}{n} < \delta_\varepsilon$. Let us take $n \geq n_\varepsilon$. For all $k = 1, \ldots, n$ we have $0 \leq \frac{\ln(kn)}{kn} \leq \frac{\ln(n^2)}{kn} \leq \frac{\ln(n^2)}{n} = \frac{2\ln n}{n} < \delta_\varepsilon$ and from (1), (for $x = \frac{\ln(kn)}{kn}$), it follows that $\left|e^{\frac{\ln(kn)}{kn}} - 1 - \frac{\ln(kn)}{kn}\right| \leq \frac{\varepsilon\ln(kn)}{kn}$, or $\left|\sqrt[kn]{kn} - 1 - \frac{\ln(kn)}{kn}\right| \leq \frac{\varepsilon\ln(kn)}{kn}$. We deduce that

$$\begin{aligned} \left|\sum_{k=1}^{n}\left(\sqrt[kn]{kn} - 1\right) - \sum_{k=1}^{n}\frac{\ln(kn)}{kn}\right| &\leq \sum_{k=1}^{n}\left|\left(\sqrt[kn]{kn} - 1\right) - \frac{\ln(kn)}{kn}\right| \\ &\leq \varepsilon\sum_{k=1}^{n}\frac{\ln(kn)}{kn} \end{aligned}$$

or equivalently

$$\left|\frac{\sum\limits_{k=1}^{n}\left(\sqrt[kn]{kn} - 1\right)}{\sum\limits_{k=1}^{n}\frac{\ln(kn)}{kn}} - 1\right| \leq \varepsilon,$$

hence

$$\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n}\left(\sqrt[kn]{kn} - 1\right)}{\sum\limits_{k=1}^{n}\frac{\ln(kn)}{kn}} = 1. \tag{2}$$

Now let us note that $\sum\limits_{k=1}^{n}\frac{\ln(kn)}{kn} = \frac{1}{n}S_n + \frac{\ln n}{n}H_n$, where $S_n = \sum\limits_{k=1}^{n}\frac{\ln k}{k}$, $H_n = \sum\limits_{k=1}^{n}\frac{1}{k}$. Then, since from the Stolz–Cesàro lemma, $\lim\limits_{n\to\infty}\frac{S_n}{\ln^2 n} = \frac{1}{2}$, $\lim\limits_{n\to\infty}\frac{H_n}{\ln n} = 1$, we deduce that

$$\lim_{n\to\infty}\frac{n}{\ln^2 n}\sum_{k=1}^{n}\frac{\ln(kn)}{kn} = \frac{3}{2}. \tag{3}$$

From the relations (2) and (3) we deduce that the desired limit is $\frac{3}{2}$.

Remark 1. A different solution for Problem 1 was indicated to us by E. Păltănea.

2. A first refinement

We begin with a refinement of the evaluation from Problem 1. Let us note here the appearance of the Euler constant γ and the first Stieltjes constant γ_1, that is, $\gamma_1 = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\ln k}{k} - \frac{\ln^2 n}{2}\right)$, see [2].

Theorem 2. *The following estimate holds*

$$\frac{n}{\ln^2 n}\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right) = \frac{3}{2} + \frac{\gamma}{\ln n} + \frac{\gamma_1}{\ln^2 n} + \frac{\pi^2}{12}\cdot\frac{1}{n} + o\left(\frac{1}{n}\right).$$

Proof. Let $\varepsilon > 0$. Since $\lim_{x\to 0}\frac{e^x-1-x}{x^2} = \frac{1}{2}$, there exists $\delta_\varepsilon > 0$ such that $\forall 0 < x < \delta_\varepsilon$ we have $\left|\frac{e^x-1-x}{x^2} - \frac{1}{2}\right| < \varepsilon$, or equivalently

$$\left|e^x - 1 - x - \frac{x^2}{2}\right| \le \varepsilon x^2 \text{ for all } 0 \le x < \delta_\varepsilon. \tag{4}$$

Since $\lim_{n\to\infty}\frac{2\ln n}{n} = 0$, for $\delta_\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\forall n \ge n_\varepsilon$ we have $\frac{2\ln n}{n} < \delta_\varepsilon$. Let $n \ge n_\varepsilon$. For all $k = 1, \dots, n$ we have $0 \le \frac{\ln(kn)}{kn} \le \frac{2\ln n}{n} < \delta_\varepsilon$ and by the relation (3) we deduce that $\left|e^{\frac{\ln(kn)}{kn}} - 1 - \frac{\ln(kn)}{kn} - \frac{\ln^2(kn)}{2k^2n^2}\right| \le \frac{\varepsilon\ln^2(kn)}{k^2n^2}$, that is, $\left|\sqrt[kn]{kn} - 1 - \frac{\ln(kn)}{kn} - \frac{\ln^2(kn)}{2k^2n^2}\right| \le \frac{\varepsilon\ln^2(kn)}{k^2n^2}$. By an obvious summation we get

$$\left|\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right) - \sum_{k=1}^{n}\frac{\ln(kn)}{kn} - \frac{1}{2}\sum_{k=1}^{n}\frac{\ln^2(kn)}{k^2n^2}\right| \le \varepsilon\sum_{k=1}^{n}\frac{\ln^2(kn)}{k^2n^2},$$

or

$$\left|\frac{\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right) - \sum_{k=1}^{n}\frac{\ln(kn)}{kn}}{\sum_{k=1}^{n}\frac{\ln^2(kn)}{k^2n^2}} - \frac{1}{2}\right| \le \varepsilon,$$

hence

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right) - \sum_{k=1}^{n}\frac{\ln(kn)}{kn}}{\sum_{k=1}^{n}\frac{\ln^2(kn)}{k^2n^2}} = \frac{1}{2}.$$

From the equality

$$\sum_{k=1}^{n}\frac{\ln^2(kn)}{k^2n^2} = \frac{1}{n^2}\sum_{k=1}^{n}\frac{\ln^2 k}{k^2} + \frac{2\ln n}{n^2}\sum_{k=1}^{n}\frac{\ln k}{k^2} + \frac{\ln^2 n}{n^2}\sum_{k=1}^{n}\frac{1}{k^2}$$

it follows that

$$\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n}\frac{\ln^2(kn)}{k^2n^2}}{\frac{\ln^2 n}{n^2}}=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k^2}+\frac{2}{\ln n}\sum_{k=1}^{n}\frac{\ln k}{k^2}+\frac{1}{\ln^2 n}\sum_{k=1}^{n}\frac{\ln^2 k}{k^2}\right)=\frac{\pi^2}{6}.$$

Hence $\lim\limits_{n\to\infty}\frac{\sum\limits_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right)-\sum\limits_{k=1}^{n}\frac{\ln(kn)}{kn}}{\frac{\ln^2 n}{n^2}}=\frac{\pi^2}{12}$, or equivalently

$$\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right)=\sum_{k=1}^{n}\frac{\ln(kn)}{kn}+\frac{\pi^2}{12}\frac{\ln^2 n}{n^2}+o\left(\frac{\ln^2 n}{n^2}\right). \tag{5}$$

From the well-known evaluation $\sum\limits_{k=1}^{n}\frac{1}{k}=\ln n+\gamma+\frac{1}{2n}+o\left(\frac{1}{n}\right)$, see [1, 2], it follows that

$$\frac{\ln n}{n}\sum_{k=1}^{n}\frac{1}{k}=\frac{\ln^2 n}{n}+\frac{\gamma\ln n}{n}+\frac{\ln n}{2n^2}+o\left(\frac{\ln n}{n^2}\right)=\frac{\ln^2 n}{n}+\frac{\gamma\ln n}{n}+o\left(\frac{\ln^2 n}{n^2}\right).$$

Similarly, from $\sum\limits_{k=1}^{n}\frac{\ln k}{k}=\frac{\ln^2 n}{2}+\gamma_1+\frac{\ln n}{2n}+o\left(\frac{\ln n}{n}\right)$, see [2], we get

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\ln k}{k}=\frac{\ln^2 n}{2n}+\frac{\gamma_1}{n}+\frac{\ln n}{2n^2}+o\left(\frac{\ln n}{n^2}\right)=\frac{\ln^2 n}{2n}+\frac{\gamma_1}{n}+o\left(\frac{\ln^2 n}{n^2}\right).$$

We deduce that

$$\sum_{k=1}^{n}\frac{\ln(kn)}{kn}=\frac{1}{n}\sum_{k=1}^{n}\frac{\ln k}{k}+\frac{\ln n}{n}\sum_{k=1}^{n}\frac{1}{k}=\frac{3\ln^2 n}{2n}+\frac{\gamma\ln n}{n}+\frac{\gamma_1}{n}+o\left(\frac{\ln^2 n}{n^2}\right). \tag{6}$$

Replacing (6) in (5) we get

$$\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right)=\frac{3\ln^2 n}{2n}+\frac{\gamma\ln n}{n}+\frac{\gamma_1}{n}+\frac{\pi^2}{12}\frac{\ln^2 n}{n^2}+o\left(\frac{\ln^2 n}{n^2}\right),$$

which is equivalent to the evaluation stated in Theorem 2. □

3. The case of differentiable functions

Theorem 3. *Let $\eta>0$ and $f:[1,1+\eta)\to\mathbb{R}$ be differentiable at 1. Then*

$$\lim_{n\to\infty}\frac{n}{\ln^2 n}\sum_{k=1}^{n}\left[f\left(\sqrt[kn]{kn}\right)-f(1)\right]=\frac{3f'(1)}{2}.$$

Proof. Let $\varepsilon > 0$. Since f is differentiable at 1, $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = f'(1)$, thus there exists $\delta_\varepsilon > 0$ such that $\forall x \in [1, 1+\eta)$ with the property that $|x-1| < \delta_\varepsilon$, $x \neq 1$, it follows that $\left|\frac{f(x)-f(1)}{x-1} - f'(1)\right| < \varepsilon$, or

$$\left|f(x) - f(1) - f'(1)(x-1)\right| \leq \varepsilon (x-1), \tag{7}$$

for all $x \in [1, 1+\eta)$ with $0 \leq x-1 < \delta_\varepsilon$.

Let us define $\nu_\varepsilon = \min(\eta, \delta_\varepsilon) > 0$ and note that from

$$\lim_{n\to\infty} \left(e^{\frac{2\ln n}{n}} - 1\right) = 0,$$

for $\nu_\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$ we have $0 < e^{\frac{2\ln n}{n}} - 1 < \nu_\varepsilon$. Let $n \geq n_\varepsilon$. For all $k = 1, \ldots, n$ we have $0 \leq \sqrt[kn]{kn} - 1 = e^{\frac{\ln(kn)}{kn}} - 1 \leq e^{\frac{2\ln n}{n}} - 1 < \nu_\varepsilon$ and, by (4),

$$\left|f\left(\sqrt[kn]{kn}\right) - f(1) - f'(1)\left(\sqrt[kn]{kn} - 1\right)\right| \leq \varepsilon \left(\sqrt[kn]{kn} - 1\right).$$

Then we have

$$\left|\sum_{k=1}^{n}\left[f\left(\sqrt[kn]{kn}\right) - f(1)\right] - f'(1)\sum_{k=1}^{n}\left(\sqrt[kn]{kn} - 1\right)\right|$$

$$\leq \sum_{k=1}^{n}\left|\left[f\left(\sqrt[kn]{kn}\right) - f(1)\right] - f'(1)\left(\sqrt[kn]{kn} - 1\right)\right| \leq \varepsilon \sum_{k=1}^{n}\left(\sqrt[kn]{kn} - 1\right),$$

or

$$\left|\frac{\sum_{k=1}^{n}\left[f\left(\sqrt[kn]{kn}\right) - f(1)\right]}{\sum_{k=1}^{n}\left(\sqrt[kn]{kn} - 1\right)} - f'(1)\right| \leq \varepsilon,$$

hence

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n}\left[f\left(\sqrt[kn]{kn}\right) - f(1)\right]}{\sum_{k=1}^{n}\left(\sqrt[kn]{kn} - 1\right)} = f'(1)$$

and by Problem 1 we obtain the limit from the statement. □

4. The case of twice differentiable functions

Proposition 4. *The following estimate holds*

$$\lim_{n\to\infty} \frac{n^2}{\ln^2 n} \sum_{k=1}^{n}\left(\sqrt[kn]{kn} - 1\right)^2 = \frac{\pi^2}{6}.$$

Proof. Let $\varepsilon > 0$. Since $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^2 = 1$, there exists $\delta_\varepsilon > 0$ such that $\forall 0 < x < \delta_\varepsilon$ we have $\left|\left(\frac{e^x-1}{x}\right)^2 - 1\right| < \varepsilon$, or equivalently

$$\left|(e^x-1)^2 - x^2\right| \leq \varepsilon x^2,\ \forall 0 \leq x < \delta_\varepsilon. \tag{8}$$

We continue as in the proof of Theorem 2. There exists $n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$ we have $\frac{2\ln n}{n} < \delta_\varepsilon$. For all $n \geq n_\varepsilon$ and all $k = 1, \ldots, n$ we have $0 \leq \frac{\ln(kn)}{kn} < \delta_\varepsilon$ and, by (5), $\left|\left(e^{\frac{\ln(kn)}{kn}} - 1\right)^2 - \frac{\ln^2(kn)}{k^2n^2}\right| \leq \frac{\varepsilon \ln^2(kn)}{k^2n^2}$, or $\left|\left(\sqrt[kn]{kn} - 1\right)^2 - \frac{\ln^2(kn)}{k^2n^2}\right| \leq \frac{\varepsilon \ln^2(kn)}{k^2n^2}$. We deduce

$$\begin{aligned}\left|\sum_{k=1}^n \left(\sqrt[kn]{kn}-1\right)^2 - \sum_{k=1}^n \frac{\ln^2(kn)}{k^2n^2}\right| &\leq \sum_{k=1}^n \left|\left(\sqrt[kn]{kn}-1\right)^2 - \frac{\ln^2(kn)}{k^2n^2}\right| \\ &\leq \varepsilon \sum_{k=1}^n \frac{\ln^2(kn)}{k^2n^2},\end{aligned}$$

or

$$\left|\frac{\sum\limits_{k=1}^n \left(\sqrt[kn]{kn}-1\right)^2}{\sum\limits_{k=1}^n \frac{\ln^2(kn)}{k^2n^2}} - 1\right| \leq \varepsilon,$$

hence

$$\lim_{n\to\infty} \frac{\sum\limits_{k=1}^n \left(\sqrt[kn]{kn}-1\right)^2}{\sum\limits_{k=1}^n \frac{\ln^2(kn)}{k^2n^2}} = 1.$$

From

$$\begin{aligned}\sum_{k=1}^n \frac{\ln^2(kn)}{k^2n^2} &= \frac{1}{n^2}\left(\sum_{k=1}^n \frac{\ln^2 k}{k^2} + 2(\ln n)\sum_{k=1}^n \frac{\ln k}{k^2} + (\ln n)^2 \sum_{k=1}^n \frac{1}{k^2}\right) \\ &\backsimeq \frac{(\ln n)^2}{n^2}\cdot\frac{\pi^2}{6}\end{aligned}$$

we deduce that $\lim_{n\to\infty} \frac{\sum_{k=1}^n \left(\sqrt[kn]{kn}-1\right)^2}{\frac{(\ln n)^2}{n^2}} = \frac{\pi^2}{6}$. □

We prove now the evaluation for the case of twice differentiable functions.

Theorem 5. *Let* $\eta > 0$ *and* $f : [1, 1+\eta) \to \mathbb{R}$ *be twice differentiable at* 1. *Then the following estimate holds*

$$\sum_{k=1}^{n}\left[f\left(\sqrt[kn]{kn}\right)-f(1)\right] = \frac{3f'(1)}{2}\cdot\frac{\ln^2 n}{n}+\gamma f'(1)\cdot\frac{\ln n}{n}+\gamma_1 f'(1)\cdot\frac{1}{n} +\frac{\pi^2\left[f'(1)+f''(1)\right]}{12}\cdot\frac{\ln^2 n}{n^2}+o\left(\frac{\ln^2 n}{n^2}\right).$$

Proof. Let $\varepsilon > 0$. Since f is twice differentiable at 1,

$$\lim_{x\to 1}\frac{f(x)-f(1)-f'(1)(x-1)}{(x-1)^2}=\frac{f''(1)}{2},$$

thus there exists $\delta_\varepsilon > 0$ such that $\forall x \in [1, 1+\eta)$ with $|x-1| < \delta_\varepsilon$, $x \neq 1$, we have $\left|\frac{f(x)-f(1)-f'(1)(x-1)}{(x-1)^2}-\frac{f''(1)}{2}\right| < \varepsilon$, or $\forall x \in [1, 1+\eta)$, $|x-1| < \delta_\varepsilon$ the following relation holds

$$\left|f(x)-f(1)-\alpha(x-1)-\beta(x-1)^2\right| \le \varepsilon(x-1)^2, \tag{9}$$

where $\alpha = f'(1)$, $\beta = \frac{f''(1)}{2}$. Let us define $\nu_\varepsilon = \min(\eta, \delta_\varepsilon) > 0$ and note that, from $\lim_{n\to\infty}\left(e^{\frac{2\ln n}{n}}-1\right)=0$, for $\nu_\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\forall n \ge n_\varepsilon$ we have $0 < e^{\frac{2\ln n}{n}}-1 < \nu_\varepsilon$. Let $n \ge n_\varepsilon$. For all $k = 1, \ldots, n$ we have $0 \le \sqrt[kn]{kn}-1 = e^{\frac{\ln(kn)}{kn}}-1 \le e^{\frac{2\ln n}{n}}-1 < \nu_\varepsilon$ and, by (6),

$$\left|f\left(\sqrt[kn]{kn}\right)-f(1)-\alpha\left(\sqrt[kn]{kn}-1\right)-\beta\left(\sqrt[kn]{kn}-1\right)^2\right| \le \varepsilon\left(\sqrt[kn]{kn}-1\right)^2.$$

We deduce that

$$\left|\sum_{k=1}^{n} f\left(\sqrt[kn]{kn}\right)-f(1)n-\alpha\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right)-\beta\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right)^2\right| \le \varepsilon\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right)^2,$$

or

$$\left|\frac{\sum_{k=1}^{n} f\left(\sqrt[n]{k}\right)-f(1)n-\alpha\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right)}{\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right)^2}-\beta\right| \le \varepsilon.$$

Thus

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\left[f\left(\sqrt[kn]{kn}\right)-f(1)\right]-\alpha\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right)}{\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right)^2}=\beta$$

and, by Proposition 4, we deduce that

$$\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n}\left[f\left(\sqrt[kn]{kn}\right)-f(1)\right]-\alpha\sum\limits_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right)}{\frac{\ln^2 n}{n^2}}=\beta\cdot\frac{\pi^2}{6}.$$

This is equivalent to

$$\begin{aligned}\sum_{k=1}^{n}\left[f\left(\sqrt[kn]{kn}\right)-f(1)\right] &= f'(1)\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right)\\ &\quad+\frac{\pi^2 f''(1)}{12}\cdot\frac{\ln^2 n}{n^2}+o\left(\frac{\ln^2 n}{n^2}\right),\end{aligned}$$

which, by Theorem 2, gives us

$$\begin{aligned}\sum_{k=1}^{n}\left[f\left(\sqrt[kn]{kn}\right)-f(1)\right] &= f'(1)\left(\frac{3\ln^2 n}{2n}+\frac{\gamma\ln n}{n}+\frac{\gamma_1}{n}+\frac{\pi^2}{12}\frac{\ln^2 n}{n^2}\right)\\ &\quad+\frac{\pi^2 f''(1)}{12}\frac{\ln^2 n}{n^2}+o\left(\frac{\ln^2 n}{n^2}\right)\\ &= \frac{3f'(1)}{2}\cdot\frac{\ln^2 n}{n}+\gamma f'(1)\cdot\frac{\ln n}{n}+f'(1)\,\gamma_1\cdot\frac{1}{n}\\ &\quad+\frac{\pi^2\left[f'(1)+f''(1)\right]}{12}\cdot\frac{\ln^2 n}{n^2}+o\left(\frac{\ln^2 n}{n^2}\right).\end{aligned}$$

□

5. Some examples

The first application of Theorem 5 represents an extension of Proposition 4.

Proposition 6. *Let $\eta>0$ and $\varphi:[1,1+\eta)\to\mathbb{R}$ be a function of the class C^1. Then the following estimate holds*

$$\sum_{k=1}^{n}\left(\sqrt[kn]{kn}-1\right)^2\varphi\left(\sqrt[kn]{kn}\right)=\frac{\pi^2\varphi(1)}{6}\cdot\frac{\ln^2 n}{n^2}+o\left(\frac{\ln^2 n}{n^2}\right).$$

Proof. Let us define $f:[1,1+\eta)\to\mathbb{R}$ by $f(x)=(x-1)^2\varphi(x)$. Then $f'(x)=2(x-1)\varphi(x)+(x-1)^2\varphi'(x)$ and since φ is of the class C^1, f is twice differentiable at 1 with $f''(1)=2\varphi(1)$. We apply Theorem 5. □

Proposition 7. *(i) If $\lim\limits_{n\to\infty}a_n=0$, $\lim\limits_{n\to\infty}\frac{a_n}{b_n}\in\mathbb{R}$, then*

$$e^{a_n}=1+a_n+\frac{a_n^2}{2}+\frac{a_n^3}{6}+o\left(b_n^3\right).$$

(ii) If $\lim_{n\to\infty} x_n = 0$, $\lim_{n\to\infty} y_n = 0$, $\lim_{n\to\infty} \frac{x_n}{b_n} \in \mathbb{R}$, $\lim_{n\to\infty} \frac{y_n}{b_n} = 0$, *then*

$$e^{x_n+y_n} = 1 + x_n + y_n + \frac{x_n^2}{2} + x_n y_n + \frac{y_n^2}{2} + \frac{x_n^3}{6} + o\left(b_n^3\right).$$

(iii) For all real numbers a, b, c, d we have

$$\begin{aligned} & e^{\frac{a\ln^2 n}{n}+\frac{b\ln n}{n}+\frac{c}{n}+\frac{d\ln^2 n}{n^2}+o\left(\frac{\ln^2 n}{n^2}\right)} \\ = \; & 1 + \frac{a\ln^2 n}{n} + \frac{b\ln n}{n} + \frac{c}{n} + \frac{a^2\ln^4 n}{2n^2} + \frac{ab\ln^3 n}{n^2} + \frac{\left(d+ac+\frac{b^2}{2}\right)\ln^2 n}{n^2} \\ & + o\left(\frac{\ln^2 n}{n^2}\right). \end{aligned}$$

Proof. (i) From $\lim_{n\to\infty} \frac{a_n}{b_n} = l$ it follows that there exists $n_0 \in \mathbb{N}$ such that $\left|\frac{a_n}{b_n}\right| < |l| + 1 := M$, $\forall n \geq n_0$. From $\lim_{t\to 0} \frac{e^t-1-t-\frac{t^2}{2}}{t^3} = \frac{1}{6}$ we deduce that $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that $\left|e^t - 1 - t - \frac{t^2}{2} - \frac{t^3}{6}\right| \leq \frac{\varepsilon}{M^3}|t|^3$, $\forall |t| < \delta_\varepsilon$. From $\lim_{n\to\infty} a_n = 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $|a_n| < \delta_\varepsilon$, $\forall n \geq n_\varepsilon$. We deduce that $\left|e^{a_n} - 1 - a_n - \frac{a_n^2}{2} - \frac{a_n^3}{6}\right| \leq \frac{\varepsilon}{M^3}|a_n|^3$, $\forall n \geq n_\varepsilon$, and hence

$$\left|e^{a_n} - 1 - a_n - \frac{a_n^2}{2} - \frac{a_n^3}{6}\right| \leq \varepsilon |b_n|^3, \forall n \geq \max(n_0, n_\varepsilon).$$

(ii) From *(i)* applied for $a_n = x_n + y_n$ we get

$$\begin{aligned} e^{x_n+y_n} &= 1 + x_n + y_n + \frac{(x_n+y_n)^2}{2} + \frac{(x_n+y_n)^3}{6} + o\left(b_n^3\right) \\ &= 1 + x_n + y_n + \frac{x_n^2 + 2x_n y_n + y_n^2}{2} + \frac{x_n^3 + 3x_n^2 y_n + 3x_n y_n^2 + y_n^3}{6} \\ & \quad + o\left(b_n^3\right). \end{aligned}$$

Now from the hypotheses $\lim_{n\to\infty} \frac{x_n^2 y_n}{b_n^3} = \lim_{n\to\infty} \left(\frac{x_n^2}{b_n^2} \cdot \frac{y_n}{b_n}\right) = 0$, that is $x_n^2 y_n = o\left(b_n^3\right)$; $\lim_{n\to\infty} \frac{x_n y_n^2}{b_n^3} = \lim_{n\to\infty} \left(\frac{x_n}{b_n} \cdot \frac{y_n^2}{b_n^2}\right) = 0$, that is $x_n y_n^2 = o\left(b_n^3\right)$ and $y_n^3 = o\left(b_n^3\right)$. The stated evaluation follows.
(iii) For $x_n = \frac{a\ln^2 n}{n}$, $y_n = \frac{b\ln n}{n} + \frac{c}{n} + \frac{d\ln^2 n}{n^2} + o\left(\frac{\ln^2 n}{n^2}\right)$, and $b_n = \frac{\ln^2 n}{n}$, the result in *(ii)* gives us that

$$
\begin{aligned}
& e^{\frac{a\ln^2 n}{n}+\frac{b\ln n}{n}+\frac{c}{n}+\frac{d\ln^2 n}{n^2}+o\left(\frac{\ln^2 n}{n^2}\right)} \\
= \; & 1+\frac{a\ln^2 n}{n}+\frac{b\ln n}{n}+\frac{c}{n}+\frac{d\ln^2 n}{n^2}+o\left(\frac{\ln^2 n}{n^2}\right) \\
& +\frac{a^2\ln^4 n}{2n^2}+\frac{a\ln^2 n}{n}\left(\frac{b\ln n}{n}+\frac{c}{n}+\frac{d\ln^2 n}{n^2}+o\left(\frac{\ln^2 n}{n^2}\right)\right) \\
& +\frac{1}{2}\left(\frac{b\ln n}{n}+\frac{c}{n}+\frac{d\ln^2 n}{n^2}+o\left(\frac{\ln^2 n}{n^2}\right)\right)^2+\frac{a^3\ln^6 n}{6n^3}+o\left(\frac{\ln^6 n}{n^3}\right) \\
= \; & 1+\frac{a\ln^2 n}{n}+\frac{b\ln n}{n}+\frac{c}{n}+\frac{d\ln^2 n}{n^2}+\frac{a^2\ln^4 n}{2n^2}+\frac{ab\ln^3 n}{n^2}+\frac{ac\ln^2 n}{n^2} \\
& +\frac{b^2\ln^2 n}{2n^2}+o\left(\frac{\ln^2 n}{n^2}\right) \\
= \; & 1+\frac{a\ln^2 n}{n}+\frac{b\ln n}{n}+\frac{c}{n}+\frac{a^2\ln^4 n}{2n^2}+\frac{ab\ln^3 n}{n^2}+\frac{\left(d+ac+\frac{b^2}{2}\right)\ln^2 n}{n^2} \\
& +o\left(\frac{\ln^2 n}{n^2}\right).
\end{aligned}
$$

□

We use now Theorem 5 to find an estimate of a product.

Proposition 8. *Let $\eta > 0$ and $\varphi : [1, 1+\eta) \to (0,\infty)$ be twice differentiable at 1. Then the following estimate holds*

$$
\begin{aligned}
\frac{\prod_{k=1}^{n}\varphi\left(\sqrt[kn]{kn}\right)}{[\varphi(1)]^n} = \; & 1+\frac{3\varphi'(1)}{2\varphi(1)}\cdot\frac{\ln^2 n}{n}+\frac{\gamma\varphi'(1)}{\varphi(1)}\cdot\frac{\ln n}{n}+\frac{\gamma_1\varphi'(1)}{\varphi(1)}\cdot\frac{1}{n} \\
& +\frac{9}{8}\left[\frac{\varphi'(1)}{\varphi(1)}\right]^2\cdot\frac{\ln^4 n}{n^2}+\frac{3\gamma}{2}\left[\frac{\varphi'(1)}{\varphi(1)}\right]^2\cdot\frac{\ln^3 n}{n^2}+F\cdot\frac{\ln^2 n}{n^2} \\
& +o\left(\frac{\ln^2 n}{n^2}\right),
\end{aligned}
$$

where $F=\frac{\pi^2}{12}\left(\frac{\varphi'(1)}{\varphi(1)}+\frac{\varphi''(1)}{\varphi(1)}-\left[\frac{\varphi'(1)}{\varphi(1)}\right]^2\right)+\frac{3\gamma_1}{2}\left[\frac{\varphi'(1)}{\varphi(1)}\right]^2+\frac{\gamma^2}{2}\left[\frac{\varphi'(1)}{\varphi(1)}\right]^2.$

Proof. Let us define $f : [1, 1+\eta) \to \mathbb{R}$, $f(x) = \ln\varphi(x)$. Then $f'(x) = \frac{\varphi'(x)}{\varphi(x)}$, $f''(x) = \frac{\varphi''(x)\varphi(x)-[\varphi'(x)]^2}{\varphi^2(x)} = \frac{\varphi''(x)}{\varphi(x)} - \left[\frac{\varphi'(x)}{\varphi(x)}\right]^2$. From Theorem 5 we have

$$\sum_{k=1}^{n}\left[\ln\varphi\left(\sqrt[kn]{kn}\right)-\ln\varphi(1)\right] = \frac{3\varphi'(1)}{2\varphi(1)}\cdot\frac{\ln^2 n}{n}+\frac{\gamma\varphi'(1)}{\varphi(1)}\cdot\frac{\ln n}{n}+\frac{\gamma_1\varphi'(1)}{\varphi(1)}\cdot\frac{1}{n}$$
$$+\frac{\pi^2\left(\frac{\varphi'(1)}{\varphi(1)}+\frac{\varphi''(1)}{\varphi(1)}-\left[\frac{\varphi'(1)}{\varphi(1)}\right]^2\right)}{12}\cdot\frac{\ln^2 n}{n^2}$$
$$+o\left(\frac{\ln^2 n}{n^2}\right),$$

that is,

$$\frac{\prod_{k=1}^{n}\varphi\left(\sqrt[kn]{kn}\right)}{[\varphi(1)]^n}$$
$$= e^{\frac{3\varphi'(1)}{2\varphi(1)}\cdot\frac{\ln^2 n}{n}+\frac{\gamma\varphi'(1)}{\varphi(1)}\cdot\frac{\ln n}{n}+\frac{\gamma_1\varphi'(1)}{\varphi(1)}\cdot\frac{1}{n}+\frac{\pi^2\left(\frac{\varphi'(1)}{\varphi(1)}+\frac{\varphi''(1)\varphi(1)-[\varphi'(1)]^2}{\varphi^2(1)}\right)}{12}\cdot\frac{\ln^2 n}{n^2}+o\left(\frac{\ln^2 n}{n^2}\right)}.$$

From Proposition 7, after some calculations, we get the stated evaluation. □

As a concrete example of Proposition 8 we give

Corollary 9. *The following estimate holds*

$$\prod_{k=1}^{n}\left(1-\sqrt[kn]{kn}+\sqrt[kn]{k^2n^2}\right)=1+\frac{3\ln^2 n}{2n}+\frac{\gamma\ln n}{n}+\frac{\gamma_1}{n}$$
$$+\frac{9\ln^4 n}{8n^2}+\frac{3\gamma\ln^3 n}{2n^2}+\frac{\left(\pi^2+16\gamma_1+6\gamma^2\right)\ln^2 n}{12n^2}+o\left(\frac{\ln^2 n}{n^2}\right).$$

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Traian Lalescu national mathematics contest for university students, 2024 edition

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Abstract. We present the problems from the Sections A and B of the 2024 edition of Traian Lalescu National Mathematics Contest for University Students, hosted by the Ferdinand I Military Technical Academy of Bucharest between May 9 and May 11. We also include alternative solutions provided by members of the jury or by contestants.

Keywords: Series, Stolz–Cesàro lemma, Taylor formula, rank, kernel, Jordan canonical form, cyclic group.

MSC: 40A05, 26A24, 15A03, 26A42, 20K01.

The 2024 edition of the Traian Lalescu National Mathematics Contest for University Students and of the National Session of Scientific Communications for University Students was organized between May 9 and May 11 by the Ferdinand I Military Technical Academy of Bucharest, with the support of the Traian Lalescu Foundation and the Ministry of Education.

The participating students represented 14 universities from Brașov, Bucharest, Cluj-Napoca, Constanța, Craiova, Iași, Timișoara:

- Section A — first- and second-year students from faculties of Mathematics: 21 students;
- Section B — first-year students that follow some specialization in Electrical Engineering or in Computer Science: 20 students;
- Section C — first-year students from technical faculties with a specialization outside the field of Electrical Engineering: 28 students;
- Section D — second-year students that follow some specialization in Electrical Engineering: 17 students;
- Section E — second-year students from technical faculties with a specialization outside the field of Electrical Engineering: 12 students;
- Scientific Communications, Mathematics Section: 9 students;
- Scientific Communications, Computer Science Section: 8 students.

The interested reader may find additional details at the competition's website: `https://lalescu.mta.ro/`.

We present the statements and solutions of the problems given at Sections A and B of the contest, together with alternative solutions provided by members of the jury or by the contestants.

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1. Section A

Problem 1. Consider $n \geq 2$ and $A \in \mathcal{M}_n(\mathbb{C})$. Prove that there exists $k \in \mathbb{N}, 1 \leq k \leq n$, and $B \in \mathcal{M}_n(\mathbb{C})$ such that

$$B^2 = B, \quad \operatorname{Ker} B = \operatorname{Ker} A^k, \quad \text{and} \quad \operatorname{Im} B = \operatorname{Im} A^k.$$

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Author's solution. Let $J_A = \begin{bmatrix} J_1 & O \\ O & J_2 \end{bmatrix} = P^{-1}AP$ be the Jordan canonical form associated to matrix A, where $J_1 \in \mathcal{M}_m(\mathbb{C})$ contains all the Jordan cells corresponding to nonzero eigenvalues (J_1 is invertible), and the Jordan block $J_2 \in \mathcal{M}_{n-m}(\mathbb{C})$ contains all the Jordan cells corresponding to the eigenvalue $\lambda = 0$ (it is possible that J_2 is empty).

If k is the maximal dimension of a Jordan cell from J_2 block, then we have

$$J_A^k = \begin{bmatrix} J_1^k & O \\ O & O \end{bmatrix} = P^{-1}A^kP$$

(if $m = n$, then J_2 is empty and k can be taken equal to 1). We define $J_B = \begin{bmatrix} I_m & O \\ O & O \end{bmatrix}$ and $B = PJ_BP^{-1}$, for which we verify that the conclusion of the problem is satisfied. More precisely,

$$B^2 = PJ_B^2P^{-1} = PJ_BP^{-1} = B.$$

For any $X \in \mathbb{C}^n$, we denote by $X_1 \in \mathbb{C}^n$ and $X_0 \in \mathbb{C}^{n-m}$ the blocks from the representation $X = \begin{bmatrix} X_1 \\ X_0 \end{bmatrix}$ (with X_0 possibly empty).

For $X \in \mathbb{C}^n$, let $Y = P^{-1}X$. Then the following equivalences hold:

$$\begin{aligned} &X \in \operatorname{Ker} B \iff BX = O \iff PJ_BP^{-1}X = O \\ &\iff P \begin{bmatrix} I_m & O \\ O & O \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_0 \end{bmatrix} = O \iff Y_1 = O. \end{aligned}$$

On the other hand,

$$\begin{aligned} &X \in \operatorname{Ker} A^k \iff A^kX = O \iff PJ_A^kP^{-1}X = O \\ &\iff P \begin{bmatrix} J_1^k & O \\ O & O \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_0 \end{bmatrix} = O \iff J_1^kY_1 = O \iff Y_1 = O. \end{aligned}$$

Therefore, $\operatorname{Ker} B = \operatorname{Ker} A^k$.

If $Z \in \mathbb{C}^n$, then

$$Z \in \operatorname{Im} B \iff \text{there exists } X \in \mathbb{C}^n : Z = BX = PJ_BP^{-1}X$$
$$\iff \text{there exists } Y \in \mathbb{C}^n : Z = PJ_BY$$
$$\iff \text{there exists } Y \in \mathbb{C}^n : Z = P\begin{bmatrix} I_m & O \\ O & O \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_0 \end{bmatrix}$$
$$\iff \text{there exists } Y_1 \in \mathbb{C}^m : Z = P\begin{bmatrix} Y_1 \\ O \end{bmatrix}$$

(we used the equivalence $Y = P^{-1}X \iff X = PY$). Similarly,

$$Z \in \operatorname{Im} A^k \iff \text{there exists } U \in \mathbb{C}^n : Z = A^kU = PJ_A^kP^{-1}U$$
$$\iff \text{there exists } V \in \mathbb{C}^n : Z = PJ_A^kV$$
$$\iff \text{there exists } V \in \mathbb{C}^n : Z = P\begin{bmatrix} J_1^k & O \\ O & O \end{bmatrix}\begin{bmatrix} V_1 \\ V_0 \end{bmatrix}$$
$$\iff \text{there exists } V_1 \in \mathbb{C}^m : Z = P\begin{bmatrix} J_1^kV_1 \\ O \end{bmatrix}$$

(we used the equivalence $V = P^{-1}U \iff U = PV$). Since

$$Y_1 = J_1^kV_1 \iff V_1 = \left(J_1^{-1}\right)^k Y_1,$$

we conclude that $\operatorname{Im} B = \operatorname{Im} A^k$.

Alternative solution. *The following solution was given by Ana-Maria Negoiţă, from University of Bucharest (contestant).*

First, we observe that $\operatorname{Ker} A^k \subset \operatorname{Ker} A^{k+1}$, which holds for every positive integer k. If $\dim \operatorname{Ker} A = 0$ then we have $\operatorname{Ker} A^k = \{0\}$, for every positive integer k.

Assume now that $\dim \operatorname{Ker} A \geq 1$. Then we have the following inequalities $1 \leq \dim \operatorname{Ker} A \leq \cdots \leq \dim \operatorname{Ker} A^{n+1} \leq n$. This implies the existence of $k \in \mathbb{N}^*$ such that $\dim \operatorname{Ker} A^k = \dim \operatorname{Ker} A^{k+1}$, which means that $\operatorname{Ker} A^k = \operatorname{Ker} A^{k+1}$. By induction, for every positive integer s we get that $\operatorname{Ker} A^k = \operatorname{Ker} A^{k+s}$.

So there exists a positive integer $k \leq n$ such that $\operatorname{Ker} A^k = \operatorname{Ker} A^{k+s}$ for every positive integer s. Let $v \in \operatorname{Ker} A^k \cap \operatorname{Im} A^k$. Then we have $A^kv = 0$, and there exists u such that $v = A^ku$. This implies that $A^{2k}u = 0$, and using the fact that $\operatorname{Ker} A^k = \operatorname{Ker} A^{2k}$ we get that $A^ku = 0$, which gives $v = 0$. Therefore, by using the Rank-Nullity Theorem, it holds $\operatorname{Ker} A^k \oplus \operatorname{Im} A^k = \mathbb{C}^n$.

From this, we can write each $x \in \mathbb{C}^n$ uniquely as $u+v$, where $u \in \operatorname{Ker} A^k$ and $v \in \operatorname{Im} A^k$. We define $T : \mathbb{C}^n \to \mathbb{C}^n$ such that $T(x) = v$, where v is as above. Take $U = \operatorname{Ker} A^k$, $V = \operatorname{Im} A^k$, and observe that $T = \operatorname{pr}_V^U$.

Let B be the matrix associated to T with respect to the canonical basis. This means that $\operatorname{Ker} B = \operatorname{Ker} T = U$, and $\operatorname{Im} B = \operatorname{Im} T = V$.

We see that $T(T(x)) = T(v) = v = T(x)$, so $T \circ T = T$, which implies $B^2 = B$.

So B and k verify the conditions of the problem.

Remark. In a similar way, contestants Mario Drăguţ and Cezar Tulceanu from University of Bucharest determined $k \geq 1$ such that $\operatorname{Ker} A^k \oplus \operatorname{Im} A^k = \mathbb{C}^n$. We take $\{e_1, \ldots, e_n\}$ basis for $\mathbb{C}^n$ such that $\{e_1, \ldots, e_t\}$ is a basis for $\operatorname{Ker} A^k$ and $\{e_{t+1}, \ldots, e_n\}$ is a basis for $\operatorname{Im} A^k$. We take B the matrix with respect to this basis, which has the following form:

$$B = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

Problem 2. Let $n \geq 3$ be a positive integer. Decide if the multiplicative group $U(\mathbb{Z}/2^n\mathbb{Z})$ is cyclic.

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Author's solution. We prove that the group is not cyclic. Suppose, by contradiction, that the multiplicative group $U(\mathbb{Z}/2^n\mathbb{Z})$ is cyclic. Using the Euler totient function, we have

$$\operatorname{card} U(\mathbb{Z}/2^n\mathbb{Z}) = \varphi\left(2^n\right) = 2^n - 2^{n-1} = 2^{n-1}.$$

Hence, there exists a nonzero integer k such that

$$U(\mathbb{Z}/2^n\mathbb{Z}) = \left\{\hat{1}, \hat{k}, \hat{k}^2, \hat{k}^3, \ldots, \hat{k}^{2^{n-1}-1}\right\}$$

and all the 2^{n-1} classes are mutually disjoint.

Since $\hat{k}$ is invertible in $U(\mathbb{Z}/2^n\mathbb{Z})$, k is necessarily odd. Furthermore, it can be proven by induction that, for any natural number n greater than 2,

$$k^{2^{n-2}} \equiv 1 \left(\operatorname{mod}\ 2^n\right),$$

that is, $\hat{k}^{2^{n-2}} = \hat{1}$. Contradiction.

Remark. We can also see that the multiplicative group $U(\mathbb{Z}/2^n\mathbb{Z})$ is not cyclic for $n \geq 3$ using the *Lifting the Exponent Lemma* (LTE). For each $k \in U(\mathbb{Z}/2^n\mathbb{Z})$, by using LTE, and also the fact that $8 \mid k^2 - 1$, we get that

$$v_2\left(k^{2^{n-2}} - 1\right) = v_2(k^2 - 1) + v_2(2^{n-3}) \geq 3 + n - 3 = n,$$

because $v_2(k^2 - 1) \geq 3$.

Problem 3. Let $A, B, C \in \mathcal{M}_n(\mathbb{C})$ with $A \neq O$ such that $A + BAC = BA + AC$. Prove that 1 is an eigenvalue either for matrix B or for matrix C, with algebraic multiplicity at least $\dfrac{\operatorname{rank} A}{2}$.

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Author's solution. We look for information concerning the kernels of the matrices $B - I_n$ and $C - I_n$. The given relation can be rewritten as

$$(B - I_n) A (C - I_n) = O.$$

Using the Frobenius rank inequality for the matrices $B - I_n$, A, and $C - I_n$, we deduce that

$$\begin{aligned} 0 &= \operatorname{rank}((B - I_n) A (C - I_n)) \\ &\geq \operatorname{rank}((B - I_n) A) + \operatorname{rank}(A (C - I_n)) - \operatorname{rank} A. \end{aligned}$$

By applying the Sylvester rank inequality for $(B - I_n) A$ and $A (C - I_n)$, we obtain

$$0 \geq \operatorname{rank}(B - I_n) + \operatorname{rank}(C - I_n) + \operatorname{rank} A - 2n.$$

The conclusion follows if we observe that, from the last inequality, we have

$$n - \frac{\operatorname{rank} A}{2} \geq \min\{\operatorname{rank}(B - I_n), \operatorname{rank}(C - I_n)\}.$$

Problem 4. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n - na_{n+1}) = \ell \in \mathbb{R}.$$

Prove that the series $\sum\limits_{n=1}^{\infty} a_n$ converges and determine its sum.

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Author's solution. For any positive integer n, denote $S_n = a_1 + a_2 + \cdots + a_n$ and $T_n = S_n - na_{n+1}$. We prove that $(S_n)_{n\geq 1}$ converges to ℓ.

Observe that, for any n, we have

$$S_{n+1} - S_n = a_{n+1} = \frac{S_n}{n} - \frac{T_n}{n} \Longrightarrow \frac{T_n}{n(n+1)} = \frac{S_n}{n} - \frac{S_{n+1}}{n+1}. \tag{1}$$

It follows that, for any $p \geq 1$,

$$\sum_{n=1}^{p} \frac{T_n}{n(n+1)} = \sum_{n=1}^{p} \left(\frac{S_n}{n} - \frac{S_{n+1}}{n+1} \right) = S_1 - \frac{S_{p+1}}{p+1} = S_1 - \frac{T_{p+1}}{p+1} - a_{p+2}.$$

Since $a_n \to 0$, we obtain that $(a_n)_{n\geq 1}$ is bounded, hence from relation above it follows that the series $\sum\limits_{n=1}^{\infty} \dfrac{T_n}{n(n+1)}$ is absolutely convergent.

Moreover, we deduce that the remainder of the series is

$$\sum_{n=p+1}^{\infty} \frac{T_n}{n\,(n+1)} = \frac{S_{p+1}}{p+1}.$$

Since $T_n \to \ell$, we have that

$$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall n \geq n_\varepsilon : \ell - \varepsilon < T_n < \ell + \varepsilon.$$

By combining this relation with the expression of the remainder, we obtain that, for any $p \geq n_\varepsilon$,

$$\frac{S_{p+1}}{p+1} = \sum_{n=p+1}^{\infty} \frac{T_n}{n\,(n+1)} \leq \sum_{n=p+1}^{\infty} (\ell+\varepsilon) \frac{1}{n\,(n+1)} = (\ell+\varepsilon) \frac{1}{p+1},$$

hence $S_{p+1} \leq \ell + \varepsilon$ for all $p \geq n_\varepsilon$. Similarly, $S_{p+1} \geq \ell - \varepsilon$ for all $p \geq n_\varepsilon$. This shows that $\lim\limits_{n\to\infty} S_n = \ell \in \mathbb{R}$.

Alternative solution. Some contestants and members of the jury observed that relation (1) implies

$$T_n = \frac{\frac{S_{n+1}}{n+1} - \frac{S_n}{n}}{\frac{1}{n+1} - \frac{1}{n}}, \forall n \geq 1,$$

and since

$$\ell = \lim_{n\to\infty} T_n = \lim_{n\to\infty} \frac{\frac{S_{n+1}}{n+1} - \frac{S_n}{n}}{\frac{1}{n+1} - \frac{1}{n}},$$

by applying the Stolz–Cesàro lemma, we obtain that

$$\lim_{n\to\infty} \frac{\frac{S_n}{n}}{\frac{1}{n}} = \lim_{n\to\infty} S_n = \ell \in \mathbb{R}.$$

2. Section B

Problem 1. Let $P, Q \in \mathbb{R}[X]$ be nonconstant monic polynomials such that $k = \deg P = \deg Q + 1$.

a) Prove that

$$\lim_{x\to\infty} \int_x^{2x} \frac{t^{k-1}}{P\,(t)}\,\mathrm{d}t = \ln 2.$$

b) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim\limits_{x\to\infty} \dfrac{f\,(x)}{x} = 1$, compute the limit

$$\lim_{x\to\infty} \int_x^{2x} \frac{f\,(Q(t))}{P(f\,(t))}\,\mathrm{d}t.$$

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Author's solution. a) Observe that there exist $\delta > 0$ and constants $m, M \in \mathbb{R}$ such that

$$t^k + mt^{k-1} \le P(t) \le t^k + Mt^{k-1}, \quad \forall t > \delta. \tag{2}$$

This easily follows if we suppose that $P(t) = t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0$ and we remark that

$$\lim_{t\to\infty} \frac{P(t) - t^k}{t^{k-1}} = a_{k-1} \in \mathbb{R}.$$

Relation (2) implies that, for any t sufficiently large (for which also $P(t) > 0$),

$$\frac{1}{t+M} \le \frac{t^{k-1}}{P(t)} \le \frac{1}{t+m}.$$

We deduce that, for any x sufficiently large,

$$\begin{aligned} \ln\left(\frac{2x+M}{x+M}\right) &= \int_x^{2x} \frac{1}{t+M}\,\mathrm{d}t \le \int_x^{2x} \frac{t^{k-1}}{P(t)}\,\mathrm{d}t \\ &\le \int_x^{2x} \frac{1}{t+m}\,\mathrm{d}t = \ln\left(\frac{2x+m}{x+m}\right), \end{aligned}$$

so passing to the limit for $x \to \infty$ we get the conclusion of assertion a).

b) Observe first that the fraction $\dfrac{f(Q(t))}{P(f(t))}$ is well defined for t sufficiently large.

We easily deduce that $\lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty} Q(t) = \lim\limits_{t\to\infty} f(t) = \infty$, and

$$\lim_{t\to\infty} \frac{f(Q(t))}{t^{k-1}} = \lim_{t\to\infty} \frac{f(Q(t))}{Q(t)} \cdot \frac{Q(t)}{t^{k-1}} = 1,$$

$$\lim_{t\to\infty} \frac{P(f(t))}{t^k} = \lim_{t\to\infty} \frac{P(f(t))}{(f(t))^k} \cdot \frac{(f(t))^k}{t^k} = 1.$$

Let $\varepsilon \in (0,1)$. From the previous relations, we get the existence of $\delta > 0$ such that, for any $t > \delta$, the above fractions are well defined and

$$\left|\frac{f(Q(t))}{t^{k-1}} - 1\right| < \varepsilon \quad \text{and} \quad \left|\frac{P(f(t))}{t^k} - 1\right| < \varepsilon,$$

i.e.,

$$(1-\varepsilon)\,t^{k-1} < f(Q(t)) < (1+\varepsilon)\,t^{k-1},$$
$$(1-\varepsilon)\,t^k < P(f(t)) < (1+\varepsilon)\,t^k.$$

From here,

$$\frac{1-\varepsilon}{1+\varepsilon} \cdot \frac{1}{t} < \frac{f(Q(t))}{P(f(t))} < \frac{1+\varepsilon}{1-\varepsilon} \cdot \frac{1}{t}, \quad \forall t > \delta,$$

and then

$$\frac{1-\varepsilon}{1+\varepsilon} \cdot \ln 2 = \frac{1-\varepsilon}{1+\varepsilon} \cdot \int_x^{2x} \frac{1}{t}\,\mathrm{d}t \le \int_x^{2x} \frac{f(Q(t))}{P(f(t))}\,\mathrm{d}t \le \frac{1+\varepsilon}{1-\varepsilon} \cdot \ln 2, \quad \forall x > \delta.$$

By denoting, for every $x > \delta$,

$$G(x) := \int_x^{2x} \frac{f(Q(t))}{P(f(t))}\,\mathrm{d}t,$$

we obtain that

$$\forall \varepsilon \in (0,1), \exists \delta > 0, \forall x > \delta : -\frac{2\varepsilon}{1+\varepsilon} \cdot \ln 2 \le G(x) - \ln 2 \le \frac{2\varepsilon}{1-\varepsilon} \cdot \ln 2,$$

from where we deduce that the desired value of the limit is $\ln 2$.

Problem 2. Consider $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $AB + BA = A + B$. Prove that the following assertions are equivalent:

a) $\operatorname{rank} AB + n = \operatorname{rank} A + \operatorname{rank} B$,

b) $\operatorname{Ker} B \subset \operatorname{Im} A$.

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Authors' solution. We use the equality case in the Sylvester rank inequality

$$\operatorname{rank} AB + n \ge \operatorname{rank} A + \operatorname{rank} B,$$

with equality if and only if $\operatorname{Ker} A \subset \operatorname{Im} B$.

Using this characterization, assertion b) is equivalent to

$$\operatorname{rank} BA + n = \operatorname{rank} A + \operatorname{rank} B,$$

and we are driven to the idea that $\operatorname{rank} AB = \operatorname{rank} BA$, which we prove in the following.

From $AB + BA = A + B$, it follows that

$$\begin{aligned} AB &= A + B - BA = A + B(I_n - A) = I_n - (I_n - A) + B(I_n - A) \\ &= I_n - (I_n - B)(I_n - A) = I_n - DC, \end{aligned}$$

with $D = I_n - B$, $C = I_n - A$. Similarly, we obtain $BA = I_n - CD$, hence

$$\operatorname{rank} AB = \operatorname{rank} BA \iff \operatorname{rank}(I_n - CD) = \operatorname{rank}(I_n - DC). \tag{3}$$

We give two different methods to prove relation (3).

Method 1. For any matrices $C, D \in \mathcal{M}_n(\mathbb{C})$, we prove the inclusions:

$$\operatorname{Ker}(I_n - CD) \subset \operatorname{Im} C, \tag{4}$$

$$\operatorname{Ker} C \subset \operatorname{Im}(I_n - DC). \tag{5}$$

We have

$$X \in \operatorname{Ker}(I_n - CD) \iff X = C(DX) \implies X \in \operatorname{Im} C,$$

$$X \in \operatorname{Ker} C \implies X = X - D(CX) = (I_n - DC)X \implies X \in \operatorname{Im}(I_n - DC).$$

According to the equality case in Sylvester rank inequality, we have

$$(4) \iff \operatorname{rank}((I_n - CD)C) + n = \operatorname{rank}(I_n - CD) + \operatorname{rank} C,$$
$$(5) \iff \operatorname{rank}(C(I_n - DC)) + n = \operatorname{rank}(I_n - DC) + \operatorname{rank} C,$$

and since $(I_n - CD)C = C(I_n - DC)$, it follows that $\operatorname{rank}(I_n - CD) = \operatorname{rank}(I_n - DC)$.

<u>*Method 2.*</u> We have

$$\operatorname{rank}(I_n - CD) + n = \operatorname{rank}\left[\begin{array}{c|c} I_n - CD & O \\ \hline O & I_n \end{array}\right] \overset{L_1}{=} \operatorname{rank}\left[\begin{array}{c|c} I_n - CD & C \\ \hline O & I_n \end{array}\right]$$
$$\overset{C_1}{=} \operatorname{rank}\left[\begin{array}{c|c} I_n & C \\ \hline D & I_n \end{array}\right] \overset{C_2}{=} \operatorname{rank}\left[\begin{array}{c|c} I_n & O \\ \hline D & I_n - DC \end{array}\right] \overset{L_2}{=} \operatorname{rank}\left[\begin{array}{c|c} I_n & O \\ \hline O & I_n - DC \end{array}\right]$$
$$= n + \operatorname{rank}(I_n - DC).$$

The matrices of elementary transformations on lines and columns are:

$$L_1 = \left[\begin{array}{c|c} I_n & C \\ \hline O & I_n \end{array}\right], C_1 = \left[\begin{array}{c|c} I_n & O \\ \hline D & I_n \end{array}\right], C_2 = \left[\begin{array}{c|c} I_n & -C \\ \hline O & I_n \end{array}\right], L_2 = \left[\begin{array}{c|c} I_n & O \\ \hline -D & I_n \end{array}\right].$$

In conclusion,

$$a) \iff \operatorname{rank} AB + n = \operatorname{rank} A + \operatorname{rank} B \iff$$
$$\operatorname{rank} BA + n = \operatorname{rank} A + \operatorname{rank} B \iff b).$$

Problem 3. Let V be a finite dimensional vector space and $T : V \to V$ be an endomorphism.

a) Prove that there exist a positive integer k and an endomorphism $P : V \to V$ such that

$$P \circ P = P, \quad \operatorname{Ker} P = \operatorname{Ker} T^k, \quad \text{and} \quad \operatorname{Im} P = \operatorname{Im} T^k.$$

b) The assertion of a) remains true if V is infinite dimensional ?

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Author's solution. a) Consider the sequence of subspaces of V: $\operatorname{Ker} T \subseteq \operatorname{Ker} T^2 \subseteq \cdots \subseteq \operatorname{Ker} T^n \subseteq \cdots$. Since V has finite dimension, it follows that there exists $k \geq 1$ such that $\operatorname{Ker} T^k = \operatorname{Ker} T^{k+1}$. We prove by induction that $\operatorname{Ker} T^{k+p} = \operatorname{Ker} T^k$, for any $p \geq 1$.

For $p = 1$ we have $\operatorname{Ker} T^k = \operatorname{Ker} T^{k+1}$. At the inductive step, suppose that $\operatorname{Ker} T^{k+p} = \operatorname{Ker} T^k$. The inclusion $\operatorname{Ker} T^{k+p+1} \subseteq \operatorname{Ker} T^k$ follows because

$$\begin{aligned} x \in \operatorname{Ker} T^{k+p+1} &\iff T^{k+p+1}(x) = 0 \iff T^{k+p}(T(x)) = 0 \\ &\iff T(x) \in \operatorname{Ker} T^{k+p} = \operatorname{Ker} T^k \\ &\iff T^{k+1}(x) = 0 \iff x \in \operatorname{Ker} T^{k+1} = \operatorname{Ker} T^k \\ &\implies x \in \operatorname{Ker} T^k, \end{aligned}$$

which closes the induction argument.

We prove now that $V = \operatorname{Ker} T^k \oplus \operatorname{Im} T^k$. Since

$$\dim \operatorname{Ker} T^k + \dim \operatorname{Im} T^k = \dim V,$$

it is sufficient to prove that

$$\operatorname{Ker} T^k \cap \operatorname{Im} T^k = \{0\}.$$

Indeed, if $x \in \operatorname{Ker} T^k \cap \operatorname{Im} T^k$, then $T^k(x) = 0$ and there exists $x_1 \in V$ such that $x = T^k(x_1)$. Hence, $T^{2k}(x_1) = T^k(x) = 0$, so $x_1 \in \operatorname{Ker} T^{2k} = \operatorname{Ker} T^k$, therefore $x = T^k(x_1) = 0$.

Denote $V_1 = \operatorname{Im} T^k$ and $V_2 = \operatorname{Ker} T^k$, such that $V = V_1 \oplus V_2$. We can define $P : V \to V$ by $P(x_1 + x_2) = x_1$, for any $x_1 \in V_1$, $x_2 \in V_2$ (P is the projection on V_1 parallel to V_2). Obviously, P satisfies the conclusion of the problem.

b) The assertion a) is no longer true if V is infinite dimensional.

For instance, take $V = \mathbb{R}[X]$ and $T : \mathbb{R}[X] \to \mathbb{R}[X]$, $T(f) = f'$ (the differentiation operator). Then, for any positive k we have $\operatorname{Ker} T^k = \mathbb{R}_{k-1}[X]$ (the vector subspace consisting of all polynomials of degree less than k) and $\operatorname{Im} T^k = \mathbb{R}[X]$. If $P : \mathbb{R}[X] \to \mathbb{R}[X]$ satisfies the properties from a) for a certain $k \geq 1$, then $1 \in \mathbb{R}_{k-1}[X] = \operatorname{Ker} T^k = \operatorname{Ker} P$, so $P(1) = 0$. Also, $1 \in \mathbb{R}[X] = \operatorname{Im} T^k = \operatorname{Im} P$, so there exists $f \in \mathbb{R}[X]$ such that $1 = P(f)$. Since $P^2 = P$, we obtain the following contradiction: $1 = P(f) = P^2(f) = P(1) = 0$.

Remark. We observe that the item a) of Problem 3 above is nothing else than the reformulation in terms of operators of Problem 1 from Section A.

Problem 4. We consider an even function $f : [-1, 1] \to \mathbb{R}$ of class C^2 for which $f(0) = 0$ and $f''(0) > 0$. Define the sequence $(a_n)_{n\geq 1}$ by

$$a_n = \sum_{k=1}^{n} f\left(\frac{k}{n\sqrt{n}}\right), \quad \forall n \geq 1.$$

a) Prove that the sequence $(a_n)_{n\geq 1}$ converges and compute $\ell = \lim\limits_{n\to\infty} a_n$.

b) Suppose, additionally, that f is of class C^4, with $f^{(4)}(0) > 0$. Study the convergence of the series $\sum\limits_{n=1}^{\infty} (a_n - \ell)$ and $\sum\limits_{n=1}^{\infty} f(a_n - \ell)$.

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Author's solution. Observe first that f' is an odd function, hence $f'(0) = 0$. Since f is of class C^2 and $f''(0) > 0$, we get that $f''(x) > 0$ in a neighborhood of 0, hence f' is increasing in that neighborhood. Furthermore, since $f'(0) = 0$, we have that there exists $\varepsilon > 0$ such that $f'(x) \geq 0$ on $[0, \varepsilon)$, so f is nondecreasing, hence it has also nonnegative values on $[0, \varepsilon)$.

a) Using the monotonicity of f, we will have successively, for any $n \geq 1$, that

$$f\left(\frac{k}{n\sqrt{n}}\right) \leq \int_k^{k+1} f\left(\frac{x}{n\sqrt{n}}\right) \mathrm{d}x \leq f\left(\frac{k+1}{n\sqrt{n}}\right), \quad \forall k = \overline{0, n-1},$$

$$\begin{aligned} a_n + f(0) - f\left(\frac{1}{\sqrt{n}}\right) &= \sum_{k=0}^{n-1} f\left(\frac{k}{n\sqrt{n}}\right) \leq \int_0^n f\left(\frac{x}{n\sqrt{n}}\right) \mathrm{d}x \\ &\leq \sum_{k=1}^{n} f\left(\frac{k}{n\sqrt{n}}\right) = a_n, \end{aligned}$$

and, finally, that

$$\int_0^n f\left(\frac{x}{n\sqrt{n}}\right) \mathrm{d}x \leq a_n \leq \int_0^n f\left(\frac{x}{n\sqrt{n}}\right) \mathrm{d}x + f\left(\frac{1}{\sqrt{n}}\right) - f(0). \quad (6)$$

Since f is continuous, it admits antiderivatives, so denoting by F an antiderivative which vanishes at 0, we have

$$F(0) = F'(0) = F''(0) = 0.$$

Using the Taylor formula for the function F, we deduce

$$F\left(\frac{1}{\sqrt{n}}\right) = \frac{F'''(0)}{6n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right),$$

hence

$$\lim_{n\to\infty} \int_0^n f\left(\frac{x}{n\sqrt{n}}\right) \mathrm{d}x = \lim_{n\to\infty} n\sqrt{n} F\left(\frac{1}{\sqrt{n}}\right) = \frac{f''(0)}{6}. \quad (7)$$

By relation (6), using also the fact that $\lim_{n\to\infty} f\left(\frac{1}{\sqrt{n}}\right) = f(0)$ from the continuity of f, it follows that $\lim_{n\to\infty} a_n = \dfrac{f''(0)}{6} = \ell \in \mathbb{R}$.

b) As above, from f' odd it follows that f'' is an even function, then f''' is odd, so $f'''(0) = 0$. Since f is of class C^4, we can use the Taylor formula of order 4 for F, hence

$$F\left(\frac{1}{\sqrt{n}}\right) = \frac{f''(0)}{6n\sqrt{n}} + \frac{f^{(4)}(0)}{120n^2\sqrt{n}} + o\left(\frac{1}{n^2\sqrt{n}}\right), \quad (8)$$

so

$$\begin{aligned} n(a_n - \ell) &= n\left(a_n - \frac{f''(0)}{6}\right) \geq n\left(\int_0^n f\left(\frac{x}{n\sqrt{n}}\right) \mathrm{d}x - \frac{f''(0)}{6}\right) \\ &= n\left(n\sqrt{n} F\left(\frac{1}{\sqrt{n}}\right) - \frac{f''(0)}{6}\right) \to \frac{f^{(4)}(0)}{120} > 0. \quad (9) \end{aligned}$$

We deduce, by the comparison test, that the series $\sum_{n=1}^{\infty} (a_n - \ell)$ diverges.

From (6) and (8), we have

$$n\left(a_n-\ell\right)\leq n\left(\int_0^n f\left(\frac{x}{n\sqrt{n}}\right)\mathrm{d}x-\frac{f''(0)}{6}\right)+n\left(f\left(\frac{1}{\sqrt{n}}\right)-f(0)\right).$$

But, using again the Taylor formula,

$$\lim_{n\to\infty} n\left(f\left(\frac{1}{\sqrt{n}}\right)-f(0)\right)=\lim_{n\to\infty} n\left(\frac{f''(0)}{2n}+o\left(\frac{1}{n}\right)\right)=\frac{f''(0)}{2}.$$

Hence, we have that

$$\frac{f^{(4)}(0)}{120}\leq\lim_{n\to\infty} n\left(a_n-\ell\right)\leq\frac{f^{(4)}(0)}{120}+\frac{f''(0)}{2}. \tag{10}$$

Moreover, for any x in the neighborhood of 0, we will have

$$f(x)=f(0)+f'(0)\cdot x+\frac{f''(0)}{2}x^2+o\left(x^2\right)=\frac{f''(0)}{2}x^2+o\left(x^2\right).$$

Therefore

$$\lim_{n\to\infty}\frac{f\left(a_n-\ell\right)}{\left(a_n-\ell\right)^2}=\frac{f''(0)}{2}\in(0,\infty),$$

so, by the comparison limit test, we get

$$\sum_{n=1}^{\infty} f\left(a_n-\ell\right)\sim\sum_{n=1}^{\infty}\left(a_n-\ell\right)^2.$$

By relation (10), we obtain that $\sum_{n=1}^{\infty}\left(a_n-\ell\right)^2\sim\sum_{n=1}^{\infty}\frac{1}{n^2}$, so it converges.

MATHEMATICAL NOTES

All in: the ultimate strategy

MIRCEA MARTIN[1)]

Abstract. Motivated by a problem from V. I. Arnol'd, A Mathematical Trivium, the article elaborates on the proof of a general result concerned with limits of difference quotients involving two functions and their inverses, under a minimal set of requirements. Properly implemented, the transition from a particular problem to a natural generalization provides a reliable problem solving strategy, illustrated as part of the article. The requisites include differential calculus techniques related to higher order chain rules.

Keywords: Limits, derivatives, l'Hôpital rule, higher order chain rules, inverse function differentiation rules.

MSC: 00A99, 97I40

1. INTRODUCTION

The following excerpt from V. I. Arnol'd [1], an essay addressed to mathematics teachers, emphasizes the importance of a teaching method and recommends its use in unambiguous terms:

The only way to actually determine what our students have been taught is to list some problems that they should be able to solve as a result of their instruction.

In full agreement with this statement, my students enrolled in the Problem Solving Seminar at Baker University were tested by offering them a set of five problems at the beginning of the semester. On one occasion, that list included the second problem from Mathematical Trivium,

Problem 1. *Find the limit*

$$\lim_{x\to 0} \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)} . \tag{1}$$

Though the problem looks like a standard exercise from a Differential Calculus textbook, a caveat consisting of comments and four specific tasks was issued. Basically, the students have been asked to

(1) outline a feasible approach,
(2) identify special features,
(3) describe techniques that might help in solving the problem,
(4) state a generalization and prove it.

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As expected, all students suggested a repeated use of L'Hôpital Rule, and noticed that along the way they will need to rely on Chain Rules for higher order derivatives of composite functions. Such rules make it possible to derive formulas for higher order derivatives of inverse functions, which for obvious reasons would help, too. The last task, to generalize the setting and eventually prove the generalization, turned out to be hard and my students were not able to carry it out on their own. However, after a little bit of coaching focussed on examples, heuristic geometric arguments, numerical and graphical explorations of the limit in the original problem using a graphing calculator, and proofs of the subsequent plausible result when $n = 2$ or $n = 3$, the students accepted the next generalization as true.

Theorem 1. *Suppose f and g are smooth functions defined on open intervals of real numbers that include $0 \in \mathbb{R}$, with $f(0) = g(0) = 0$. Let $n \geq 2$ be an integer such that*

(i) $D^i f(0) = D^i g(0), \;\; 1 \leq i \leq n-1,$

(ii) $D^n f(0) \neq D^n g(0).$

In addition, assume that f and g have inverse functions denoted by f^{-1} and g^{-1}, respectively. Then,

$$\lim_{x \to 0} \frac{f(x) - g(x)}{g^{-1}(x) - f^{-1}(x)} = [Df(0) \cdot Dg(0)]^{(n+1)/2}. \tag{2}$$

For convenience, we recall that a function $f : I \to \mathbb{R}$ with $I \subseteq \mathbb{R}$ an open set is smooth provided f has continuous derivatives on the entire domain, denoted by $D^i f$, for all integers $i \geq 0$. By convention, $D^0 f = f$.

Corollary 2. *The answer to the second problem from* Mathematical Trivium *is*

$$\lim_{x \to 0} \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)} = 1\,.$$

The All In Problem Solving Strategy, based on transitions from intricate particular problems to natural generalizations, proves useful in many instances. To make the point, generalizations single out the relevant setting and assumptions, by discarding immaterial features and streamlining a course of action.

2. Requisites and Proof

Proofs rely on requisites, which sometimes are extensions of results the students are familiar with. A brief discussion about chain rules for higher order derivatives of composite functions would be appropriate with regard to our goal. Specifically, let f and u be two smooth functions with the range of

u included in the domain of f. The derivative of $f \circ u$ is given by the well known rule

$$D(f \circ u) = Df \circ u \cdot Du. \tag{3}$$

Successive applications of rule (3) in conjunction with the product rule yield the second and third order derivatives of $f \circ u$,

$$D^2(f \circ u) = D^2 f \circ u \cdot (Du)^2 + Df \circ u \cdot D^2 u, \tag{4}$$

$$D^3(f \circ u) = D^3 f \circ u \cdot (Du)^3 + 3\, D^2 f \circ u \cdot Du \cdot D^2 u + Df \circ u \cdot D^3 u. \tag{5}$$

Equations (3), (4), (5) are specific forms, when $i = 1, 2, 3$, of the following explicit general formula discovered by C. F. Faà di Bruno [2],

$$D^i(f \circ u) = \sum_{K \in \mathcal{K}_{i,k}} \frac{i!}{k_1!\, k_2! \,\cdots\, k_i!}\, D^k f \circ u \cdot \prod_{j=1}^{i} \Big[\frac{D^j u}{j\,!} \Big]^{k_j}, \tag{6}$$

where the sum is over all i-tuples $K = (k_1,\, k_2,\, \ldots\,,\, k_i) \in \mathcal{K}_{i,k}$ of non-negative integers such that $k_1 + 2\, k_2 + \cdots + i\, k_i = i$ and $k_1 + k_2 + \cdots + k_i = k$. For details on this discovery my students were required to read the articles by W. P. Johnson [3] and S. Roman [4].

It should be noted that the only term in (6) that includes $D^i f$ corresponds to the i-tuple $(i,\, 0,\, \ldots,\, 0)$, hence that initial term is $D^i f \circ u \cdot (Du)^i$. At the same time, there is a unique term involving $D^i u$, corresponding to the i-tuple $(0,\, 0,\, \ldots,\, 1)$, hence the last term in (6) equals $Df \circ u \cdot D^i u$. For a later use, we denote by $S_i(f, u)$ the sum of all the other terms, if any, in which the derivatives of f and u have orders less than i, and record formula (6) as

$$D^i(f \circ u) = D^i f \circ u \cdot (Du)^i + S_i(f, u) + Df \circ u \cdot D^i u, \;\; i \geq 2. \tag{7}$$

When $i = 1$, (7) reduces to (3), and for $i = 2$, $i = 3$ we get (4), (5).

We are now in a position to proceed with the proof of the theorem. We start with a few preliminary steps.

Suppose first that f is a smooth function defined on an open interval $I \subseteq \mathbb{R}$, and assume that f has an inverse function, f^{-1}. Using equation (7) when $u = f^{-1}$, and based on a proof by induction, we conclude that f^{-1} is smooth, too. Moreover, we observe that each derivative $D^i f^{-1}$, $i \geq 1$, only depends on derivatives of f up to order i, and derivatives of f^{-1} of orders less than i. The next technical result is a straightforward consequence of the previous remarks, combined with a second elementary proof by induction on $i \geq 1$.

Lemma 3. *Suppose functions f and g satisfy the assumptions in our* Theorem 1. *Then $f^{-1}(0) = g^{-1}(0) = 0$, and*

(iii) $D^i f^{-1}(0) = D^i g^{-1}(0), \; 1 \leq i \leq n - 1,$

(iv) $D^n f^{-1}(0) \neq D^n g^{-1}(0).$

Consequently, as my students anticipated, a repeated use of L'Hôpital Rule is now fully justified, with the following consequence.

Lemma 4. *Under the previous assumptions, we get that*

$$\lim_{x\to 0} \frac{f(x)-g(x)}{g^{-1}(x)-f^{-1}(x)} = \frac{D^n f(0) - D^n g(0)}{D^n g^{-1}(0) - D^n f^{-1}(0)}, \quad n \geq 2. \tag{8}$$

The proof of the Theorem 1 is not yet complete. We next rely on equation (7) at $x = 0$ when $u = f^{-1}$, note that $D^n(f \circ f^{-1}) = 0$ because $n \geq 2$, and get

$$0 = D^n f(0) \cdot (Df^{-1}(0))^n + S_n(f, f^{-1})(0) + Df(0) \cdot D^n f^{-1}(0).$$

Since $Df(0) = 1/Df^{-1}(0)$, the above equation implies

$$D^n f^{-1}(0) = -D^n f(0) \cdot (Df^{-1}(0))^{n+1} - S_n(f, f^{-1})(0) \cdot Df^{-1}(0). \tag{9}$$

For g and g^{-1}, the similar equation at $x = 0$ is

$$D^n g^{-1}(0) = -D^n g(0) \cdot (Dg^{-1}(0))^{n+1} - S_n(g, g^{-1})(0) \cdot Dg^{-1}(0). \tag{10}$$

From statement (iii) in Lemma 3 we have $Df^{-1}(0) = Dg^{-1}(0)$ and, at the same time, $S_n(f, f^{-1})(0) = S_n(g, g^{-1})(0)$. For convenience, set $Df(0) = Dg(0) = \delta$, whence $Df^{-1}(0) = Dg^{-1}(0) = \delta^{-1}$, subtract (9) from (10), cancel the equal parts, and record the resulting equation,

$$D^n g^{-1}(0) - D^n f^{-1}(0) = [D^n f(0) - D^n g(0)] \cdot \delta^{-(n+1)}. \tag{11}$$

Finally, we substitute (11) into (8) and get

$$\lim_{x\to 0} \frac{f(x)-g(x)}{g^{-1}(x)-f^{-1}(x)} = \delta^{n+1} = [Df(0) \cdot Dg(0)]^{(n+1)/2}.$$

The proof of the Theorem 1 is complete. □

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Two-fold extension of a set function

George Stoica[1)]

Abstract. We give a general condition under which a finitely additive set function defined on the finite subsets of a countable set X has a countably additive extension defined on all subsets of X.

Keywords: Finitely/countably additive set function, finite/countable subsets of a countable set.

MSC: 97I20, 97K50

We are given a countable set X and a set function P defined on the finite subsets of X, with values in $[0,1]$ for simplicity, and which is finitely additive, i.e.,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2)$$

for any $A_1, A_2 \subset X$ finite, such that $A_1 \cap A_2 = \emptyset$. We are looking for a simultaneous two-fold extension of P, namely to a countably additive set function defined on all subsets of X. As the latter can be regarded as the hereditary σ-ring generated by the class of finite subsets of X (cf. [2], p. 24), we shall see below that the right candidate for such an extension is the inner measure induced by P (cf. [2], p. 58). Specifically, we have the following result.

Theorem. *Let X be a countable set and P a finitely additive set function defined on the finite subsets of X, taking values in $[0,1]$. In addition, we assume the following condition:*

For any $\varepsilon \in (0,1)$, there is a finite subset B_ε of X such that $P(B_\varepsilon) > 1-\varepsilon$. $(*)$

Then the formula

$$Q(A) := \sup\{P(A \cap F) \mid F \subset X, F \text{ finite}\}$$

defines an extension of P to all subsets $A \subseteq X$, which takes values in $[0,1]$, and is not only finitely additive, but also countably additive, i.e., for every disjoint sequence of sets $(A_n)_{n\geq 1}$ in X whose union is also in X, we have

$$Q\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} Q(A_n).$$

Examples. (i) Typical set functions defined on the subsets of $\mathbb{N}$ that satisfy the above theorem are as follows: to each $n \in \mathbb{N}$ associate a real number

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$\varepsilon_n > 0$ such that $\sum_{n\in\mathbb{N}} \varepsilon_n < \infty$ and, for $A \subset \mathbb{N}$, define

$$P(A) = \sum_{n\in A} \varepsilon_n \Big/ \sum_{n\in\mathbb{N}} \varepsilon_n \ .$$

To show that condition $(*)$ is satisfied in this case, let $\varepsilon \in (0,1)$. We need to exhibit a set $B_\varepsilon \subset \mathbb{N}$ such that

$$\sum_{n\in B_\varepsilon} \varepsilon_n > (1-\varepsilon) \sum_{n\in\mathbb{N}} \varepsilon_n.$$

As the sequence of partial sums converges increasingly to the sum of a positive series, it suffices to take $B_\varepsilon = \{1, 2, \ldots, n_\varepsilon\}$ with n_ε sufficiently large such that the corresponding partial sum exceeds a fraction (i.e., $1-\varepsilon$) of the whole sum.

(ii) Classes of set functions with no atoms[1] introduced in [1] and [4] satisfy condition $(*)$ when $X = \mathbb{N}$ or $\mathbb{Z}$. They include set functions with the intermediate value (or Darboux) property[2] (cf. [5]).

Remarks. (i) The countably additive extension in the above theorem is the most one can expect in our context, in the sense that any collection $\{A_i\}_{i\in I}$ of mutually disjoint subsets of X, with $P(A_i) > 0$ for any $i \in I$, is at most countable. Indeed, for any natural $n \geq 1$, let $I_n := \{i \in I \mid P(A_i) > 1/n\}$. As the events $\{A_i\}_{i\in I}$ are mutually disjoint, each I_n has at most n elements, so the set $I = \bigcup_{n\geq 1} I_n$ is at most countable.
(ii) Under condition $(*)$, Q is the unique extension of P with the required properties (see [3] p. 148, ex. 10.57 and 10.58(a)). Also note that condition $(*)$ is equivalent to $\sup\{P(F) \mid F \subset X, F \text{ finite}\} = 1$ (in general, one only has ≤ 1 in the above statement).
(iii) If hypothesis $(*)$ is excluded, the above theorem is false, as proved in [7] when $X = \mathbb{N}$, and in [6] in more general situations. For instance, an atomic finitely additive set function that does not satisfy the theorem (and indeed does not have a countably additive extension) is defined on the subsets of $\mathbb{N}$ by: $P(A) = 0$ if $A \subseteq \mathbb{N}$ is finite, and $P(A) = 1$ if $\mathbb{N} \setminus A$ is finite.

Proof of the Theorem. We easily obtain from the definitions that both P, Q are monotonically increasing with respect to inclusion: if $A \subseteq B$ are finite subsets of X, then

$$P(B) = P\big(A \cup (B \setminus A)\big) = P(A) + P(B \setminus A) \geq P(A).$$

In particular, if $A \subseteq B$ are any subsets of X and F is a finite subset of X, then $P(A \cap F) \leq P(B \cap F)$. By definition, $P(B \cap F) \leq Q(B)$, hence

[1] $A \subset X$ is called an *atom* if $P(A) > 0$ and there is no $B \subset A$ such that $0 < P(B) < P(A)$.
[2] If $P(A) > 0$ and $0 < b < P(A)$ are given, then there is $B \subset A$ such that $P(B) = b$.

$P(A \cap F) \leq Q(B)$. Finally, we pass to sup along all finite subsets F of X, to obtain that $Q(A) \leq Q(B)$.

To prove that Q is finitely additive on the subsets of X, we shall prove that Q is, at the same time, super- and sub-additive. Indeed, let $A_1, A_2 \subseteq X$ with $A_1 \cap A_2 = \emptyset$. On the one hand. we have

$$\begin{aligned}
Q(A_1 \cup A_2) &= \sup\{P\big((A_1 \cup A_2) \cap F\big) \mid F \subset X, F \text{ finite}\} \\
&= \sup\{P\big((A_1 \cap F) \cup (A_2 \cap F)\big) \mid F \subset X, F \text{ finite}\} \\
&= \sup\{P(A_1 \cap F) + P(A_2 \cap F) \mid F \subset X, F \text{ finite}\} \\
&\leq \sup\{P(A_1 \cap F) \mid F \subset X, F \text{ finite}\} \\
&\quad + \sup\{P(A_2 \cap F) \mid F \subset X, F \text{ finite}\} \\
&= Q(A_1) + Q(A_2).
\end{aligned}$$

On the other hand, if $F, G \subset X$ are finite, we have

$$P(A_1 \cap F) + P(A_2 \cap G) \leq P(A_1) + P(A_2) = P(A_1 \cup A_2) \leq Q(A_1 \cup A_2).$$

We keep G fixed and pass to sup along all finite subsets F of X, to obtain that $Q(A_1) + P(A_2 \cap G) \leq Q(B)$; and finally we pass to sup along all finite subsets G of X, to obtain that $Q(A_1) + Q(A_2) \leq Q(A_1 \cup A_2)$.

Let us show that, in fact, Q is countably additive on all subsets of X. Let $(A_n)_{n \geq 1}$ be a collection of mutually disjoint subsets of X and $A = \bigcup\limits_{n=1}^{\infty} A_n$.

We have that $\bigcup\limits_{n=1}^{k} A_n \subseteq A$ for any $k \geq 1$, so

$$Q(A) \geq Q\Big(\bigcup_{n=1}^{k} A_n\Big) = \sum_{n=1}^{k} Q(A_n) \text{ for any } k \geq 1.$$

Let $\varepsilon > 0$. We need to find a rank $N = N_\varepsilon \geq 1$ such that

$$Q(A) - \sum_{n=1}^{k} Q(A_n) < \varepsilon \text{ for all } k \geq N.$$

According to condition $(*)$, there exists $B_\varepsilon \subset X$, B_ε finite, such that $P(B_\varepsilon) > 1 - \varepsilon$. Only a finite number of A_n intersects B_ε, so there exists $N \geq 1$ such that $A_n \cap B_\varepsilon = \emptyset$ for any $n > N$. Thus, for $k \geq N$ we have

$$\begin{aligned}
\sum_{n=1}^{k} Q(A_n) &\geq \sum_{n=1}^{k} Q(A_n \cap B_\varepsilon) = Q\Big(\bigcup_{n=1}^{N} (A_n \cap B_\varepsilon)\Big) \\
&= Q\Big(\bigcup_{n=1}^{\infty} (A_n \cap B_\varepsilon)\Big) = Q(A \cap B_\varepsilon).
\end{aligned}$$

Finally, we deduce that

$$\begin{aligned} Q(A) - \sum_{n=1}^{k} Q(A_n) &\leq Q(A) - Q(A \cap B_\varepsilon) = Q(A \setminus B_\varepsilon) \\ &\leq Q(X \setminus B_\varepsilon) \leq 1 - Q(B_\varepsilon) = 1 - P(B_\varepsilon) < \varepsilon, \end{aligned}$$

exactly what we wanted. □

Open problem. The set functions listed in Examples (i) and those with the intermediate value (or Darboux) property in Examples (ii) satisfy, besides condition $(*)$, the much nicer condition

$$P(\{x\}) \neq 0 \text{ for any } x \in X.$$

It would be interesting to see if the extension result presented here is valid under this new condition.

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PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of May 2025**.

PROPOSED PROBLEMS

563. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative such that $f(0) = f(1)$ and $\displaystyle\int_0^1 x f(x)\,\mathrm{d}x = 0$. Prove that

$$\int_0^1 (f'(x))^2\,\mathrm{d}x \geq 180 \left(\int_0^1 f(x)\,\mathrm{d}x\right)^2.$$

Proposed by Cezar Lupu, Beijing Institute of Mathematical Sciences and Applications (BIMSA), Tsinghua University, Beijing, P. R. China, and Tudorel Lupu, Decebal High school, Constanţa, Romania.

564. Let $a_1, \ldots, a_7$ be nonnegative real numbers such that $a_1a_2 + a_2a_3 + \cdots + a_7a_1 = 7$. Prove that

(a) $$\frac{1}{5a_1+3} + \frac{1}{5a_2+3} + \cdots + \frac{1}{5a_7+3} \geq \frac{7}{8}.$$

(b)$*$ $$\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_7+1} \geq \frac{7}{2}.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania.

565. Let $f \in C^{2n+1}([0,1], \mathbb{R})$ for some $n \geq 0$ such that $f(1/2) = f'(1/2) = f''(1/2) = \cdots = f^{(2n)}(1/2) = 0$. Prove that

$$\int_0^1 \left(f^{(2n+1)}(x)\right)^2\,\mathrm{d}x - \left(\int_0^1 f^{(2n+1)}(x)\,\mathrm{d}x\right)^2$$

$$\geq 2^{4n+2}(4n+3)((2n+1)!)^2 \left(\int_0^1 f(x)\,\mathrm{d}x\right)^2.$$

Proposed by Florin Stănescu, Șerban Cioculescu School, Găești, Romania.

566. Prove that for all natural numbers $n \geq 4$ the equation $\sum\limits_{k=1}^{n} \frac{1}{k^2+2k+x} = \frac{3}{4}$ has a unique solution in the interval $(-1, 0)$, denoted by x_n, and find the value of the limit $\lim\limits_{n\to\infty} nx_n$.

Proposed by Dumitru Popa, Department of Mathematics, Ovidius University of Constanța, Romania.

567. Let $n \geq 4$ and let $a_1, \ldots, a_n \geq 0$ such that

$$a_1^2 + \cdots + a_n^2 + (n^2 - 3n + 1)\, a_1 \cdots a_n \geq (n-1)^2.$$

Prove that

$$a_1 + \cdots + a_n + \frac{1}{n-1}\sqrt{\frac{1}{n-1} \sum_{1 \leq i \leq j \leq n} (a_i - a_j)^2} \geq n.$$

Proposed by Leonard Giugiuc, Greci School, Mehedinți, Romania.

568. Let $f : [0, 1] \to \mathbb{R}$ be a twice differentiable function, with f'' continuous, such that $f(0) = f(1)$. Prove that there exists $\xi \in (0, 1)$ such that

$$\xi^2 f(\xi) = 2 \int_0^{\xi} x f(x)\, \mathrm{d}x.$$

Proposed by Cezar Lupu, Beijing Institute of Mathematical Sciences and Applications (BIMSA) and Tsinghua University, Beijing, P.R. China.

569. For an odd positive integer x put $x!! = x(x-2)\cdots 3 \cdot 1$. Let a and b be positive real numbers. Calculate

$$\lim_{n\to\infty} \left(\sqrt[n]{a \cdot (2n+1)!!} - \sqrt[n-1]{b \cdot (2n-1)!!} \right).$$

2) Let $f, g \in \mathbb{R}[x]$ be polynomials of the same degree with the coefficient of the leading term positive. Calculate

$$\lim_{n\to\infty} \left(\sqrt[n]{(2n+1)!! f(n)} - \sqrt[n-1]{(2n-1)!! g(n-1)} \right).$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania.

SOLUTIONS

544. Let $f:[0,1]\to\mathbb{R}$ be a differentiable function with continuous derivative on $[0,1]$ such that $f(0)=f(1/2)=f(1)=0$. Show that

$$\int_0^1 (f'(x))^2\,\mathrm{d}x \geq 48\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2.$$

Proposed by Robert Dragomirescu, Stanford University, USA, and Cezar Lupu, Yanqi Lake Beijing Institute of Mathematical Sciences and Applications (BIMSA) and Tsinghua University, P. R. China

Solution by the authors. By applying Cauchy-Schwarz's inequality, we have

$$\int_0^{\frac{1}{2}} (f'(x))^2\,\mathrm{d}x \int_0^{\frac{1}{2}}\left(x-\frac{1}{4}\right)^2\mathrm{d}x \geq \left(\int_0^{\frac{1}{2}}\left(x-\frac{1}{4}\right)f'(x)\,\mathrm{d}x\right)^2,$$

$$\int_{\frac{1}{2}}^1 (f'(x))^2\,\mathrm{d}x \int_{\frac{1}{2}}^1\left(x-\frac{3}{4}\right)^2\mathrm{d}x \geq \left(\int_{\frac{1}{2}}^1\left(x-\frac{3}{4}\right)f'(x)\,\mathrm{d}x\right)^2.$$

As

$$\int_0^{\frac{1}{2}}\left(x-\frac{1}{4}\right)^2\mathrm{d}x = \frac{1}{3}\left(x-\frac{1}{4}\right)^3\Big]_0^{\frac{1}{2}} = \frac{1}{96}$$

and

$$\int_{\frac{1}{2}}^1\left(x-\frac{3}{4}\right)^2\mathrm{d}x = \frac{1}{3}\left(x-\frac{3}{4}\right)^3\Big]_{\frac{1}{2}}^1 = \frac{1}{96},$$

by using integration by parts and the condition $f(0)=f(1/2)=f(1)=0$, we get

$$\int_0^{\frac{1}{2}}\left(x-\frac{1}{4}\right)f'(x)\,\mathrm{d}x = \left(x-\frac{1}{4}\right)f(x)\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x = -\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x,$$

$$\int_{\frac{1}{2}}^1\left(x-\frac{3}{4}\right)f'(x)\,\mathrm{d}x = \left(x-\frac{3}{4}\right)f(x)\Big]_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 f(x)\,\mathrm{d}x = -\int_{\frac{1}{2}}^1 f(x)\,\mathrm{d}x.$$

Then the two inequalities above write as

$$\int_0^{\frac{1}{2}} (f'(x))^2\,\mathrm{d}x \geq 96\left(\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x\right)^2$$

and

$$\int_{\frac{1}{2}}^1 (f'(x))^2\,\mathrm{d}x \geq 96\left(\int_{\frac{1}{2}}^1 f(x)\,\mathrm{d}x\right)^2.$$

By adding these inequalities and using the elementary inequality $\alpha^2 + \beta^2 \geq \dfrac{(\alpha+\beta)^2}{2}$, we obtain

$$\int_0^1 (f'(x))^2\,\mathrm{d}x \geq 96\frac{\left(\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x + \int_{\frac{1}{2}}^1 f(x)\,\mathrm{d}x\right)^2}{2} = 48\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2.$$

Solution by Moubinool Omarjee, Paris, France. For $a, b \in \mathbb{R}$ arbitrary we integrate by parts and we get

$$\begin{aligned}\int_0^{1/2} f(x)\,\mathrm{d}x &= [(x+a)f(x)]_0^{1/2} - \int_0^{1/2}(x+a)f'(x)\,\mathrm{d}x\\ &= -\int_0^{1/2}(x+a)f'(x)\,\mathrm{d}x,\end{aligned}$$

$$\int_{1/2}^1 f(x)\,\mathrm{d}x = [(x+b)f(x)]_{1/2}^1 - \int_{1/2}^1 (x+b)f'(x)\,\mathrm{d}x = -\int_{1/2}^1 (x+b)f'(x)\,\mathrm{d}x.$$

It follows that

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 k_{a,b}(x)f'(x)\;dx\,,\ \text{where } k_{a,b}(x) = \begin{cases} -x-a & 0 \leq x \leq 1/2.\\ -x-b & 1/2 < x \leq 1.\end{cases}$$

Then, by the Cauchy-Schwarz inequality,

$$\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2 = \left(\int_0^1 k_{a,b}(x)f'(x)\,\mathrm{d}x\right)^2 \leq \int_0^1 k_{a,b}^2(x)\,\mathrm{d}x \int_0^1 f'^2(x)\,\mathrm{d}x. \quad (1)$$

But

$$\begin{aligned}\int_0^1 k_{a,b}^2(x)\,\mathrm{d}x &= \int_0^{1/2}(x+a)^2\,\mathrm{d}x + \int_{1/2}^1 (x+b)^2\,\mathrm{d}x\\ &= \frac{1}{2}a^2 + \frac{1}{4}a + \frac{1}{2}b^2 + \frac{3}{4}b + \frac{1}{3}\\ &= \frac{1}{2}(a+1/4)^2 + \frac{1}{2}(b+3/4)^2 + \frac{1}{48}.\end{aligned}$$

We note that the smallest value of $\int\limits_0^1 k_{a,b}^2(x)\,\mathrm{d}x$, which produces the best inequality in (1), is $\int\limits_0^1 k_{-1/4,-3/4}^2(x)\,\mathrm{d}x = \dfrac{1}{48}$. So when we take $(a,b) = (-1/4, -3/4)$ in (1), we get $\left(\int\limits_0^1 f(x)\,\mathrm{d}x\right)^2 \leq \dfrac{1}{48}\int\limits_0^1 f'^2(x)\,\mathrm{d}x$, which concludes the proof.

545. Let $A, B \in M_n(\mathbb{R})$ such that $A^2 = -I_n$, $\det B \neq 0$, and $AB = -BA$. Prove that n is even and the sign of $\det B$ is $(-1)^{n/2}$.

Proposed by Mihai Opincariu, Brad, and Vasile Pop, Cluj-Napoca, Romania.

Solution by the authors. We note that if $X, Y \in M_n(\mathbb{R})$ and $X^2 = Y^2 = -I_n$, then X and Y are similar. Indeed, since $f(X) = 0$, where $f(x) = x^2 + 1$ has the simple roots $\pm i$, the matrix X is diagonalizable and its eigenvalues are $\pm i$. Since X is real, the conjugate eigenvalues i and $-i$ have the same multiplicity. Hence n is even and if $n = 2k$, then $X \sim iI_k \oplus -iI_k$. Similarly for Y, so we have $X \sim Y$.

In our case, we consider the matrix $P \in M_n(\mathbb{R})$, $P = \begin{pmatrix} O_k & I_k \\ -I_k & O_k \end{pmatrix}$. Then $P^2 = A^2 = -I_n$, so $P \sim A$. Since P and A are similar real matrices, they are similar over $\mathbb{R}$. Thus there is an invertible matrix $U \in M_n(\mathbb{R})$ such that $P = U^{-1}AU$.

If $C := U^{-1}BU$, then $PC := U^{-1}ABU$ and $CP := U^{-1}BAU$, so from $AB = -BA$ we deduce that $PC = -CP$. If $C = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$, then $PC = -CP$ is equivalent to $-X = T$ and $Z = T$. Thus $C = \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}$. It follows that

$$\det B = \det C = \begin{vmatrix} X & Y \\ Y & -X \end{vmatrix} = (-1)^k \begin{vmatrix} X & Y \\ -Y & X \end{vmatrix} = (-1)^k |\det(X + iY)|^2.$$

Since $\det B \neq 0$, we must have $|\det(X + iY)|^2 > 0$ and so $\operatorname{sgn}(\det B) = (-1)^k = (-1)^{n/2}$, as claimed.

Note. The formula $\begin{vmatrix} X & Y \\ -Y & X \end{vmatrix} = |\det(X + iY)|^2$ follows by taking determinants in the relation

$$\begin{pmatrix} I_k & O_k \\ -iI_k & I_k \end{pmatrix} \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \begin{pmatrix} I_k & O_k \\ iI_k & I_k \end{pmatrix} = \begin{pmatrix} X + iY & Y \\ O_k & X - iY \end{pmatrix}.$$

The determinant of the last matrix is $\det(X + iY)\det(X - iY) = \det(X + iY)\overline{\det(X + iY)} = |\det(X + iY)|^2$.

546. Let $X, Y \in M_n(\mathbb{C})$ such that $Y^2 = YX - XY$ and the rank of $X + Y$ is 1. Prove that $Y^3 = YXY = O_n$.

Proposed by Stănescu Florin, Şerban Cioculescu School, Găeşti, Romania.

Solution by the author. The relation $Y^2 = YX - XY$ from the hypothesis can also be written as $Y^2 = Y(X+Y) - (X+Y)Y$. If we put $A = X + Y$ and $B = Y$, this writes as $B^2 = BA - AB$. Also, by hypothesis, we have $\operatorname{rank} A = 1$.

We start by noting the following result.

Remark. If $M, N, P \in M_n(\mathbb{C})$ such that $MNP = 0$ and $\operatorname{rank}(N) = 1$, then, by Frobenius inequality, we have

$$\operatorname{rank}(MN) + \operatorname{rank}(NP) \leq \operatorname{rank}(MNP) + \operatorname{rank} N = 1,$$

so MN or NP has rank 0, i.e. $MN = 0$ or $NP = 0$.

We prove by induction that $B^tA - AB^t = tB^{t+1}$ for $t \geq 1$. When $t = 1$ this is just the relation $BA - AB = B^2$ we noticed above. For the induction step $t \mapsto t+1$, we multiply the relation $B^tA - AB^t = tB^{t+1}$ to the right by B and we get $B^tAB - AB^{t+1} = tB^{t+2}$. We also mutiply $BA - AB = B^2$ to the left by B^t and we get $B^{t+1}A - B^tAB = B^{t+2}$. We add the two relations and we get $B^{t+1}A - AB^{t+1} = (t+1)B^{t+2}$, so we have the induction step.

Since $\operatorname{Tr} B^tA = \operatorname{Tr} AB^t$, when we take traces in the formula $tB^{t+1} = B^tA - AB^t$, we get $t\operatorname{Tr} B^{t+1} = 0$. Thus $\operatorname{Tr} B^k = 0$ $\forall k \geq 2$. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of B, we get $\lambda_1^k + \cdots + \lambda_n^k = 0$ $\forall k \geq 2$. In particular, if $\beta_i = \lambda_i^2$, then $\beta_1^k + \cdots + \beta_n^k = 0$ $\forall k \geq 1$. By using Newton's relations, we get that $\lambda_i^2 = \beta_i = 0$, so $\lambda_i = 0$ $\forall i$. Hence the characteristic polynomial of B is X^n. By the Cayley-Hamilton theorem, we get $B^n = 0$.

We assume that $AB \neq 0$ and $BA \neq 0$. We prove, by backwards induction on t, that $B^tA = AB^t = B^{t+1} = 0$ for $1 \leq t \leq n-1$. If $t = n-1$, then $B^n = 0$, so $B^{n-1}A - AB^{n-1} = (n-1)B^n = 0$. We multiply to the right by B and we get that $0 = B^{n-1}AB - AB^n = B^{n-1}AB$. Since also $\operatorname{rank} A = 1$, by the Remark, we get that either $B^{n-1}A = 0$ or $AB = 0$. But $AB \neq 0$, by our assumption. Thus $B^{n-1}A = 0$. Since $B^{n-1}A - AB^{n-1} = 0$, we also have $AB^{n-1} = 0$. For the induction step $t+1 \mapsto t$, we have, by the induction hypothesis, that $B^{t+1}A = AB^{t+1} = B^{t+2} = 0$. Then, when we multiply the relation $tB^{t+1} = B^tA - AB^t$, to the left and to the right, by B, we get $0 = -BAB^t = B^tAB$. Since $\operatorname{rank} A = 1$, by the Remark, the relation $BAB^t = 0$ implies that $BA = 0$ or $AB^t = 0$ and the relation $B^tAB = 0$ implies that $B^tA = 0$ or $AB = 0$. Since we assumed that $AB, BA \neq 0$, we must have $B^tA = AB^t = 0$. Together with $tB^{t+1} = B^tA - AB^t$, this implies that also $B^{t+1} = 0$. This concludes the proof of the induction. In particular, when $t = 1$ we get $BA = AB = B^2 = 0$, which contradicts the assumption that $AB \neq 0$ and $BA \neq 0$. Hence we must have $AB = 0$ or $BA = 0$,

If $AB = 0$, we multiply the relation $B^2 = BA - AB$ to the right by B and we get $B^3 = BAB - AB^2 = 0$. If $BA = 0$, we multiply it to the left by B and we get $B^3 = B^2A - BAB = 0$. So, in both cases, $Y^3 = B^3 = 0$. But if $AB = 0$ or $BA = 0$, we also have $0 = BAB$, i.e. $0 = Y(X+Y)Y = YXY + Y^3 = YXY$. This concludes the proof.

Solution by Moubinool Omarjee, Paris, France. The matrix $Z := X+Y$ has rank 1, so $Z = vw^T$, where $v, w \in \mathbb{C}^n = M_{n,1}(\mathbb{C})$ are column vectors, $v, w \neq 0$.

We have $X = Z - Y$, so $YX - XY = Y(Z-Y) - (Z-Y)Y = YZ - ZY$, so the hypothesis $Y^2 = YX - XY$ writes as $Y^2 = YZ - ZY$.

Also $YXY = Y(Z-Y)Y = YZY - Y^3$, so the relations we want to prove, $Y^3 = YXY = O_n$, are equivalent to $Y^3 = YZY = O_n$.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of Y. For every $k \geq 0$ we have $Y^{k+2} = Y^k(YZ - ZY) = Y(Y^kZ) - (Y^kZ)Y$. But $Y(Y^kZ)$ and $(Y^kZ)Y$ have the same trace, so $\sum_{j=1}^{n} \lambda_j^{k+2} = \mathrm{Tr}(Y^{k+2}) = 0$ for every $k \geq 0$. Then, by a standard argument, $\lambda_1 = \cdots = \lambda_n = 0$. Hence Y is nilpotent.

We consider two cases.

(I) *v is an eigenvector of Y.* Since all eigenvalues of Y are 0, we have $Yv = 0$, which implies $YZ = Yvw^T = O_n$. This in turn implies $YZY = O_n$ and $Y^3 = Y(YZ - ZY) = Y^2Z - YZY = O_n$.

(II) *v is not an eigenvector of Y.* Since $v \neq 0$, we have that v and Yv are linearly independent. Let F be the two-dimensional space spanned by v and Yv. Then $Y^2v = YZv - ZYv = Yv(w^Tv) - v(w^TYv) = \alpha Yv - \beta v \in F$, where $\alpha = w^Tv = \langle w, v\rangle$ and $\beta = w^TYv = \langle Y^Tw, v\rangle$. (We have $\alpha, \beta \in M_{1,1}(\mathbb{C}) \cong \mathbb{C}$.) Since $Yv \in F$ and $Y(Yv) = Y^2v \in F$, we have $YF \subseteq F$. Since Y is nilpotent, it is also nilpotent on F. But $\dim_{\mathbb{C}} F = 2$, so we have $Y_{|F}^2 = 0$. In particular, $\alpha Yv - \beta v = Y^2v = 0$. Since v, Yv is a basis of F, we have $\alpha = \beta = 0$. Since $0 = \alpha = w^Tv$, we have $Z^2 = vw^Tvw^T = O_n$. We also have $Y^2Z = (Y^2v)w^T = O_n$

Since $Z^2 = Y^2Z = O_n$, when we multiply the relation $Y^2 = YZ - ZY$ to the left and right by Z, we get $ZY^2 = ZYZ$ and $O_n = -ZYZ$, which imply $ZY^2 = O_n$. Since $Y^2Z = ZY^2 = O_n$, when we multiply the relation $Y^2 = YZ - ZY$ to the left and right by Y, we get $Y^3 = -YZY$ and $Y^3 = YZY$, from which we conclude that $Y^3 = YZY = O_n$.

547. Prove that

$$\int_0^\infty \frac{|\sin x|}{1+x^2}\,\mathrm{d}x = \frac{e^2-1}{2e}\ln\left(\frac{e+1}{e-1}\right).$$

`Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.`

Solution by the author. Let us denote

$$I = \int_0^\infty \frac{|\sin x|}{1+x^2}\,\mathrm{d}x. \tag{1}$$

The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |\sin x|$, is periodic with period π and satisfies Dirichlet's conditions. Also, the function is even.

We expand the function into a Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2nx),$$

where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} |\sin x|\,\mathrm{d}x; \quad a_n = \frac{2}{\pi}\int_0^{\pi} |\sin x|\cos(2nx)\,\mathrm{d}x.$$

Calculating these integrals, we obtain

$$a_0 = \frac{2}{\pi}; \quad a_n = -\frac{4}{\pi}\cdot\frac{1}{4n^2-1}.$$

We therefore have

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{4n^2-1}.$$

Substituting this formula for $f(x)$ in (1) gives

$$\begin{aligned} I &= \frac{2}{\pi}\int_0^{\infty}\frac{1}{1+x^2}\,\mathrm{d}x - \frac{4}{\pi}\int_0^{\infty}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{(4n^2-1)(1+x^2)}\,\mathrm{d}x \\ &= 1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\int_0^{\infty}\frac{\cos(2nx)}{1+x^2}\,\mathrm{d}x. \end{aligned}$$

We now use the following identity:

$$\int_0^{\infty}\frac{\cos(mx)}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}e^{-m},\ m > 0.$$

This identity is Laplace's integral and is well known. It is easily proved, for example by using the properties of the Laplace transform.

We obtained the value of the integral I, namely

$$I = 1 - 2\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)e^{2n}}. \tag{2}$$

If

$$S = \sum_{n=0}^{\infty}\frac{1}{(4n^2-1)e^{2n}} = -1 + \sum_{n=1}^{\infty}\frac{1}{(4n^2-1)e^{2n}},$$

then $I = 1 - 2(S+1) = -1 - 2S$.

We have

$$\frac{1}{(4n^2-1)e^{2n}} = \frac{1}{2}\left[\frac{1}{(2n-1)e^{2n}} - \frac{1}{(2n+1)e^{2n}}\right].$$

Therefore,

$$S = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{1}{(2n-1)e^{2n}} - \sum_{n=0}^{\infty}\frac{1}{(2n+1)e^{2n}}\right] = \frac{1}{2}(A-B).$$

We calculate B:

$$B = \sum_{n=0}^{\infty} \frac{1}{(2n+1)e^{2n}} = \sum_{n=0}^{\infty} \int_0^1 \frac{x^{2n}}{e^{2n}}\, \mathrm{d}x = \int_0^1 \sum_{n=0}^{\infty} \left(\frac{x^2}{e^2}\right)^n \mathrm{d}x$$
$$= \int_0^1 \frac{1}{1-\frac{x^2}{e^2}}\, \mathrm{d}x \text{ (we have } 0<x<1 \text{ and } 0<\frac{x^2}{e^2}<1)$$

We obtain immediately

$$B = \frac{e}{2} \ln\left(\frac{e+1}{e-1}\right).$$

For A we note that, after a change of indices,

$$A = -1 + \sum_{n=1}^{\infty} \frac{1}{(2n-1)e^{2n}} = -1 + \sum_{n=0}^{\infty} \frac{1}{(2n+1)e^{2n+2}} = -1 + \frac{1}{e^2}B.$$

This gives

$$I = -1 - 2S = -1 + B - A = -1 + B - \left(-1 + \frac{1}{e^2}B\right) = \frac{e^2-1}{e^2}B$$
$$= \frac{e^2-1}{2e} \ln\left(\frac{e+1}{e-1}\right).$$

Thus, the problem is solved.

We received a very similar, but rather sketchy, proof from G. C. Greubel, Newport News, VA.

548. Let $A \in M_n(\mathbb{C})$ such that $(I_n - AA^*)^2 = I_n - A^*A$. Prove that $A^2A^* = A$.

Here A^* denotes the conjugate transpose of A, $A^* = \bar{A}^t$.

Proposed by Mihai Opincariu, Brad, and Vasile Pop, Cluj-Napoca, Romania.

Solution by the authors. If $X = AA^*$ and $Y = A^*A$, then the relation from the hypothesis writes as $Y = 2X - X^2$. In particular, this implies that $XY = YX$. Since X and Y are hermitian and commute, they are simultaneously diagonalizable. Let D_X and D_Y be their corresponding diagonal forms. Since $Y = 2X - X^2$, we have $D_Y = 2D_X - D_X^2$.

Now $X = AA^*$ and $Y = A^*A$ have the same characteristic polynomial, so they have the same eigenvalues (counting multiplicities). So if $D_X = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, then $D_Y = \operatorname{diag}(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)})$ for some $\sigma \in S_n$. But $D_Y = 2D_X - D_X^2$, so $\lambda_{\sigma(k)} = 2\lambda_k - \lambda_k^2$. By summing, we get $\sum_{k=1}^n \lambda_k = 2\sum_{k=1}^n \lambda_k - \sum_{k=1}^n \lambda_k^2$, i.e. $\sum_{k=1}^n \lambda_k^2 = \sum_{k=1}^n \lambda_k$.

Now $X = AA^*$ is semipositive definite, so $\lambda_k \geq 0$ for every k. Also since $2\lambda - \lambda^2 \leq 1$ for every $\lambda \in \mathbb{R}$, we have $\lambda_{\sigma(k)} = 2\lambda_k - \lambda_k^2 \leq 1\ \forall k$, so $\lambda_k \leq 1\ \forall k$.

Since $0 \le \lambda_k \le 1$, we have $\lambda_k^2 \le \lambda_k$, so $\sum_{k=1}^n \lambda_k^2 = \sum_{k=1}^n \lambda_k$ implies $\lambda_k^2 = \lambda_k$ $\forall k$. It follows that $\lambda_{\sigma(k)} = 2\lambda_k - \lambda_k^2 = \lambda_k$ $\forall k$, so $D_Y = D_X$, so $Y = X$. Then $Y = 2X - X^2$ implies $X = X^2 = XY$, i.e. $A^*A(AA^* - I_n) = X(Y - I_n) = O_n$. If $B := A(AA^* - I_n)$, then $B^*B = (AA^* - I_n)A^*A(AA^* - I_n) = O_n$, which implies $B = O_n$, i.e. $A^2A^* = A$.

549. Let $n \ge 3$ and let $a_1, \dots, a_n \in \mathbb{Z}_{\ge 0}$ be pairwise distinct. We denote by s_1, s_2, s_3 the first symmetric sums in the variables $a_1, \dots, a_n$, i.e. $s_1 = \sum_i a_i$, $s_2 = \sum_{i<j} a_i a_j$ and $s_3 = \sum_{i<j<k} a_i a_j a_k$. Prove that

$$(n-2)s_1 \left(s_2 - \frac{n(n-1)(n+1)}{12} \right) \ge 3ns_3.$$

When do we have equality?

`Proposed by Leonard Giugiuc, Traian National College, Drobeta-Turnu Severin, Romania.`

Solution by the author. First we prove the following result.

Lemma 1. *For every $n \ge 3$ we have*

(i) $$\sum_{1 \le i < j \le n-1} ij = \frac{n(n-1)(n-2)(3n-1)}{24},$$

(ii) $$\sum_{1 \le i < j < k \le n-1} ijk = \frac{n^2(n-1)^2(n-2)(n-3)}{48}.$$

Proof. By Newton's identities, we have

$$\begin{aligned}\sum_{1 \le i < j \le n-1} ij &= \frac{1}{2}\left(\left(\sum_{i=1}^{n-1} i \right)^2 - \sum_{i=1}^{n-1} i^2 \right) \\ &= \frac{1}{2}\left(\frac{n^2(n-1)^2}{4} - \frac{n(n-1)(2n-1)}{6} \right) \\ &= \frac{n(n-1)(n-2)(3n-1)}{24},\end{aligned}$$

$$\begin{aligned}\sum_{1 \le i < j < k \le n-1} ijk &= \frac{1}{3}\left(\sum_{1 \le i < j \le n-1} ij \sum_{i=1}^{n-1} i - \sum_{i=1}^{n-1} i \sum_{i=1}^{n-1} i^2 + \sum_{i=1}^{n-1} i^3 \right) \\ &= \frac{1}{3}\left(\frac{n^2(n-1)^2(n-2)(3n-1)}{48} - \frac{n^2(n-1)^2(2n-1)}{12} \right. \\ &\qquad \left. + \frac{n^2(n-1)^2}{4} \right) \\ &= \frac{n^2(n-1)^2(n-2)(n-3)}{48}.\end{aligned}$$

□

For convenience, given $m \geq 1$, let $e_0, e_1, e_3, \ldots \in \mathbb{Z}[X_1, \ldots, X_m]$ be the elementary symmetric polynomials in the variables $X_1, \ldots, X_m$. (In particular, $e_0 = 1$ and $e_i = 0$ for $i > m$.) Then $s_i = e_i(a_1, \ldots, a_n)$ and the lemma above gives the values of $e_i(1, \ldots, n-1)$ for $i = 1, 2$.

We use induction on n. For $n = 3$ we denote $(a_1, a_2, a_3) = (a, b, c)$ and we assume that $a > b > c \geq 0$. We denote $x = a - c$, $y = b - c$ and $t = c$. Hence $(a, b, c) = (t + x, t + y, t)$. We have $t \geq 0$ and $x > y \geq 1$. Let $s = x + y$ and $p = xy$. Then $s_1 = a + b + c = 3t + s$, $s_2 = ab + ac + bc = 3t^2 + 2ts + p$, and $s_3 = abc = t^3 + t^2 s + tp$. Hence the inequality we want to prove, $s_1(s_2 - 2) \geq 9s_3$, writes as

$$(3t + s)(3t^2 + 2ts + p - 2) \geq 9(t^3 + t^2 s + tp),$$

which, after reductions, is equivalent to

$$2t(s^2 - 3p - 3) + s(p - 2) \geq 0.$$

But $s(p - 2) \geq 0$, with equality iff $x = 2$, $y = 1$. And by the AM–GM inequality applied to $x \neq y$ we have $s^2 > 4p$, so $s^2 \geq 4p + 1 \geq 3p + 3$. (We have $p \geq 2$.) Thus $s^2 - 3p - 3 \geq 0$ and so we have the desired inequality, $2z(s^2 - 3p - 3) + s(p - 2) \geq 0$.

The equality holds iff $x = 2$, $y = 1$, i.e. when $(a, b, c) = (t + 2, t + 1, t)$, for some $t \geq 0$, and permutations thereof.

We now prove the induction step $n - 1 \mapsto n$. We may assume that $a_1 > \cdots > a_n \geq 0$. Let $x_i = a_i - a_n$ and $t = a_n$. We have $a_i = t + x_i$ for $i \leq n - 1$ and $a_n = t$, $x_1 > \cdots > x_{n-1} \geq 1$ and $t \geq 0$. We denote $s'_i = e_i(x_1, \ldots, x_{n-1})$. For convenience, we put $S = s'_1$, $q = s'_2$, and $r = s'_3$.

Now $s_1 = \sum_{i=1}^{n-1}(t + x_i) + t = nt + S$, while

$$\begin{aligned} 2s_2 &= 2 \sum_{1 \leq < i < j \leq n-1} (t + x_i)(t + x_j) + 2 \sum_{i=1}^{n-1} t(t + x_i) \\ &= (n-1)(n-2)t^2 + 2(n-2)St + 2q + 2(n-1)t^2 + 2St \\ &= n(n-1)t^2 + 2(n-1)St + 2q, \end{aligned}$$

and

$$\begin{aligned} 6s_3 &= 6 \sum_{1 \leq < i < j < k \leq n-1} (t + x_i)(t + x_j)(t + x_k) + 6 \sum_{1 \leq i < \leq n-1} t(t + x_i)(t + x_j) \\ &= (n-1)(n-2)(n-3)t^3 + 3(n-2)(n-3)St^2 + 6(n-3)qt + 6r \\ &\qquad + 3(n-1)(n-2)t^3 + 6(n-2)St^2 + 6qt \\ &= n(n-1)(n-2)t^3 + 3(n-1)(n-2)St^2 + 6(n-2)rt + 6r. \end{aligned}$$

The relation we want to prove writes as $(n-2)s_1(12s_2 - n(n-1)(n+1)) \geq 36s_3$. When we plug in the above formulas for s_1, s_2, s_3 in terms of S, q, r, it writes as

$$(n-2)(nt+S)(6n(n-1)t^2+12(n-1)St+12q-n(n-1)(n+1))$$
$$\geq 6n^2(n-1)(n-2)t^3+18n(n-1)(n-2)St^2+36n(n-2)qt+36nr,$$

which, after reductions, becomes

$$(n-2)t(12(n-1)S^2-24nq-n^2(n-1)(n+1))$$
$$+12(n-2)Sq-36nr-n(n-1)(n-2)(n+1)S\geq 0$$

Thus it suffices to prove the two inequalities

$$12(n-1)S^2-24nq-n^2(n-1)(n+1)\geq 0,$$
$$12(n-2)Sq-36nr-n(n-1)(n-2)(n+1)S\geq 0.$$

The first inequality is equivalent to

$$(n-1)S^2-2nq\geq \frac{n^2(n-1)(n+1)}{12}.$$

But we have

$$(n-1)S^2-2nq=(n-1)\sum_{i=1}^{n-1}x_i^2-2q=\sum_{i=1}^{n-1}x_i^2+\sum_{1\leq i<j\leq n-1}(x_i-x_j)^2.$$

(Recall $\sum_{1\leq i<j\leq n-1}(x_i-x_j)^2=(n-2)\sum_{i=1}^{n-1}x_i^2-2q$.)

As $x_1>\cdots>x_{n-1}\geq 1$, it follows that $x_i\geq n-i$ for $1\leq i\leq n-1$ and $x_i-x_j\geq j-i$ for $1\leq i<j\leq n-1$, with equality iff $(x_1,x_2\ldots,x_{n-1})=(n-1,n-2,\ldots,1)$. Thus $(n-1)S^2-2nq$ is minimum when $(x_1,x_2\ldots,x_{n-1})=(n-1,n-2,\ldots,1)$. But in this case $S=\frac{n(n-1)}{2}$ and, by Lemma (i), $q=\frac{n(n-1)(n-2)(3n-1)}{24}$. Hence the minimum of $(n-1)S^2-2nq$ is

$$(n-1)\left(\frac{n(n-1)}{2}\right)^2-2n\cdot\frac{n(n-1)(n-2)(3n-1)}{24}=\frac{n^2(n-1)(n+1)}{12}.$$

The second inequality writes as

$$12(n-2)Sq\geq 36nr+n(n-1)(n-2)(n+1)S.$$

By the induction hypothesis, we have

$$(n-3)S\left(q-\frac{n(n-1)(n-2)}{12}\right)\geq 3(n-1)r,$$

which, after multiplying by 12, can be written as

$$12(n-3)Sq\geq 36(n-1)r+n(n-1)(n-2)(n-3)S.$$

It follows that

$$12(n-2)Sq\geq\frac{36(n-1)(n-2)r}{n-3}+n(n-1)(n-2)^2S.$$

Hence it is enough to prove that

$$\frac{36(n-1)(n-2)r}{n-3}+n(n-1)(n-2)^2S\geq 36nr+n(n-1)(n-2)(n+1)S,$$

i.e. $\dfrac{72r}{n-3}\geq 3n(n-1)(n-2)S$. This can be written as $f(x_1,\ldots,x_{n-1})\geq 0$, where $f:[n-1,\infty)\times[n-2,\infty)\times\cdots\times[1,\infty)\to\mathbb{R}$ is defined by

$$f(x_1,\ldots,x_{n-1})=24\sum_{1\leq i<j\leq n-1}x_ix_jx_j-n(n-1)(n-2)(n-3)\sum_{i=1}^{n-1}x_i.$$

(Recall that $x_i\geq n-i$.)

Note that f is affine in each variable x_i and the coefficient of x_i is $24e_2(x_1,\ldots,\hat{x}_i,\ldots,x_{n-1})-n(n-1)(n-2)(n-3)$. Note also that each entry of $(x_1,\ldots,\hat{x}_i,\ldots,x_{n-1})$ is $\geq$ the corresponding entry of $(n-2,\ldots,1)$, with possible equality only if $i=1$. It follows that $e(x_1,\ldots,\hat{x}_i,\ldots,x_{n-1})\geq e_2(n-2,\ldots,1)=e_2(1,\ldots,n-2)$. But, by Lemma (ii), $24e_2(1,\ldots,n-2)-n(n-1)(n-2)(n-3)=(n-1)(n-2)(n-3)(3n-4)-n(n-1)(n-2)(n-3)>0$. Thus f is strictly increasing in each variable. It follows that

$$\begin{aligned}f(x_1,\ldots,x_{n-1})&\geq f(n-1,\ldots,1)\\&=24e_3(n-1,\ldots,1)-n(n-1)(n-2)(n-3)e_1(n-1,\ldots,1)\\&=\frac{n^2(n-1)^2(n-1)(n-3)}{2}-\frac{n^2(n-1)^2(n-1)(n-3)}{2}=0.\end{aligned}$$

(We have $e_1(n-1,\ldots,1)=e_1(1,\ldots,n-1)=\dfrac{n(n-1)}{2}$ and, by Lemma (ii), $e_3(n-1,\ldots,1)=e_3(1,\ldots,n-1)=\dfrac{n^2(n-1)^2(n-1)(n-3)}{48}$.)

This concludes the proof. The equality holds when $(x_1,\ldots,x_{n-1})=(n-1,\ldots,1)$, i.e. when $(a_1,a_2\ldots,a_n)=(t+n-1,t+n-2,\ldots,t)$ and permutations thereof. ☐

550. Let $a_1,a_2,\ldots,a_n$ be real numbers such that $a_1\geq a_2\geq\cdots\geq a_n\geq 0$ and $a_1a_2+a_2a_3+\cdots+a_na_1=n$, and let $E_n(k)=a_1^k+a_2^k+\cdots+a_n^k$.

(a) Prove that $E_5(k)\geq 5$ for $k\geq\frac{5}{4}$.

(b)* Prove or disprove that $E_7(k)\geq 7$ for $k\geq\frac{3}{2}$. (This is an open problem. At this time, the author doesn't have a solution.)

`Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania.`

Solution by the author. As is known (and easy to show), the function $f(k)=\left(\dfrac{E_n(k)}{n}\right)^{1/k}$ is increasing for $k>0$. Consequently, it suffices to prove the inequalities from (a) and (b) for $k=\frac{5}{4}$ and $k=\frac{3}{2}$, respectively.

(a) We need to show that

$$a_1^{5/4}+a_2^{5/4}+a_3^{5/4}+a_4^{5/4}+a_5^{5/4}\ge 5$$

for $a_1\ge a_2\ge a_3\ge a_4\ge a_5\ge 0$ and $a_1a_2+a_2a_3+a_3a_4+a_4a_5+a_5a_1=5$. Denote

$$x=\frac{a_1+a_2}{2},\qquad y=\frac{a_4+a_5}{2},\qquad x\ge a_3\ge y.$$

By Jensen's inequality for convex functions, we have

$$a_1^{5/4}+a_2^{5/4}\ge 2x^{5/4},\qquad a_4^{5/4}+a_5^{5/4}\ge 2y^{5/4}.$$

Also, by Bernoulli's inequality, we have

$$a_3^{5/4}=(1+(a_3-1))^{5/4}\ge 1+\frac{5}{4}(a_3-1)=\frac{5a_3-1}{4}.$$

So, it suffices to show that

$$8(x^{5/4}+y^{5/4})+5a_3\ge 21.$$

We will first show that

$$x^2+y^2+xy+a_3(x+y)\ge 5,\quad \text{that is}\quad a_3\ge\frac{5-x^2-y^2-xy}{x+y}.$$

Indeed, we have

$$\begin{aligned}4\big(x^2+y^2+xy+a_3(x+y)-5\big)&=(a_1+a_2)^2+(a_4+a_5)^2+(a_1+a_2)(a_4+a_5)\\&\quad+2a_3(a_1+a_2+a_4+a_5)-4(a_1a_2+a_2a_3+a_3a_4+a_4a_5+a_5a_1)\\&=(a_1-a_2)^2+(a_4-a_5)^2+a_1(2a_3+a_4-3a_5)+a_2(-2a_3+a_4+a_5)\\&\quad+2a_3(a_5-a_4)\\&\ge a_2(2a_3+a_4-3a_5)+a_2(-2a_3+a_4+a_5)+2a_2(-a_4+a_5)=0.\end{aligned}$$

(We have $2a_3+a_4-3a_5\ge 0$ and $a_1\ge a_2$, so $a_1(2a_3+a_4-3a_5)\ge a_2(2a_3+a_4-3a_5)$. As $-a_4+a_5\le 0$, we also have $a_3\le a_2$, so $2a_3(-a_4+a_5)\ge 2a_2(-a_4+a_5)$.)

So, we only need to show that

$$8(x^{5/4}+y^{5/4})+\frac{5(5-x^2-y^2-xy)}{x+y}\ge 21.$$

Denoting $x=s+t$, $y=s-t$, and

$$f(t)=(s+t)^{5/4}+(s-t)^{5/4},\qquad \text{for}\quad t\in[0,s],$$

we need to show that $g(t)\ge 0$, where

$$g(t)=f(t)-\frac{5t^2+15s^2+42s-25}{16s}.$$

For even j $(j\ge 2)$, we have

$$f^{(j)}(0)=2k(k-1)\cdots(k-j+1)s^{k-j}>0,$$

where $k = 5/4$. Thus, by the Maclaurin series expansion of the even function f, we have

$$f(t) = f(0) + \frac{f^{(2)}(0)t^2}{2!} + \frac{f^{(4)}(0)t^4}{4!} + \cdots \geq f(0) + \frac{f^{(2)}(0)t^2}{2!} + \frac{f^{(4)}(0)t^4}{4!}$$

$$= 2s^k + k(k-1)s^{k-2}t^2 + \frac{k(k-1)(k-2)(k-3)}{12}s^{k-4}t^4$$

$$= 2s^{5/4} + \frac{5}{16}s^{-3/4}t^2 + \frac{35}{1024}s^{-11/4}t^4 \geq 2s^{5/4} + \frac{5}{16}s^{-3/4}t^2 + \frac{1}{32}s^{-11/4}t^4.$$

Consequently, to prove that $g(t) \geq 0$, it suffices to show that

$$2s^{5/4} + \frac{5}{16}s^{-3/4}t^2 + \frac{1}{32}s^{-11/4}t^4 \geq \frac{5t^2 + 15s^2 + 42s - 25}{16s},$$

which is equivalent to

$$s^{-7/4}t^4 - 10(1 - s^{1/4})t^2 + 64s^{9/4} - 30s^2 - 84s + 50 \geq 0.$$

Substituting $r = s^{1/4}$, the inequality becomes

$$r^{-7}t^4 - 10(1-r)t^2 + 64r^9 - 30r^8 - 84r^4 + 50 \geq 0,$$

$$t^4 - 10r^7(1-r)t^2 + r^7(64r^9 - 30r^8 - 84r^4 + 50) \geq 0,$$

$$(t^2 - 5r^7 + 5r^8)^2 + r^7(39r^9 + 20r^8 - 25r^7 - 84r^4 + 50) \geq 0.$$

Since

$$39r^9 + 20r^8 - 25r^7 - 84r^4 + 50 = (r-1)^2E,$$

where

$$E = 39r^7 + 98r^6 + 132r^5 + 166r^4 + 200r^3 + 150r^2 + 100r + 50 > 0,$$

the proof is completed. The equality occurs for $a_1 = a_2 = a_3 = a_4 = a_5 = 1$.

Remark. Note that $k = 5/4$ is the smallest value of the positive exponent k such that $E_5(k) \geq 5$ for all a_i satisfying the given requirements. To show this, suppose

$$a_1 = a_2 = 1 + x, \qquad a_3 = 1 - x^2/2, \qquad a_4 = a_5 = 1 - x.$$

For $x \in [0, 1]$, we have $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq 0$ and $a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_1 = 5$, while the inequality $E_5(k) \geq 5$ is equivalent to $g_5(x) \geq 5$, where

$$g_5(x) = 2(1+x)^k + 2(1-x)^k + (1 - x^2/2)^k.$$

We have $g_5(0) = 5$, $g_5'(0) = 0$, and $g_5''(0) = k(4k-5)$. From $g_5''(0) \geq 0$, we get the necessary condition $k \geq 5/4$. Indeed, if $0 < k < 5/4$, then $g_5''(0) < 0$, and the point $x = 0$ is a local maximum of g_5. In addition, since $g_5(0) = 5$, there is a neighbourhood V of 0 such that $g_5(x) < 5$ for $x \in V \cap (0, 1]$.

(b)* We need to show that

$$a_1^{3/2} + a_2^{3/2} + \cdots + a_7^{3/2} \geq 7$$

for $a_1 \geq a_2 \geq \cdots \geq a_7 \geq 0$ and $a_1a_2 + a_2a_3 + \cdots + a_7a_1 = 7$. The computer calculations show that this inequality is true, but the method used in (a) for $n = 5$ and $k = 5/4$ fails. In addition, for $0 < k < 3/2$, there exist $a_1, a_2, \ldots, a_7$ such that $E_7(k) < 7$. To show this, suppose

$$a_1 = a_2 = a_3 = 1 + x, \qquad a_4 = 1 - 3x^2/2, \qquad a_5 = a_6 = a_7 = 1 - x.$$

For $x \in [0, 1]$, we have $a_1 \geq a_2 \geq \cdots \geq a_7 \geq 0$ and $a_1a_2+a_2a_3+\cdots+a_7a_1 = 7$, while the inequality $E_7(k) < 7$ is equivalent to $g_7(x) < 7$, where

$$g_7(x) = 3(1 + x)^k + 3(1 - x)^k + (1 - 3x^2/2)^k.$$

We have $g_7(0) = 7$, $g_7'(0) = 0$, and $g_7''(0) = 3k(2k - 3)$. Since $g_7''(0) < 0$ for $0 < k < 3/2$, the point $x = 0$ is a local maximum of g_7. In addition, since $g_7(0) = 7$, there is a neighbourhood V of 0 such that $g_7(x) < 7$ for $x \in V \cap (0, 1]$.

551. Solve in $\mathcal{M}_2(\mathbb{R})$ the equation $A^{2024} = -A^T$, where A^T denotes the transpose of A.

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania.

Solution by the authors. The solutions are the following matrices: O_2, $-I_2$, $\begin{pmatrix} a & b \\ b & -1-a \end{pmatrix}$, where $a, b \in \mathbb{R}$, with $a + a^2 + b^2 = 0$, and the *rotation matrices* $R_{\frac{1+2l}{2025}\pi}$, $l = 0, \ldots, 2024$.

If $A = \alpha I_2$ the matrix equation writes as $(\alpha^{2024} + \alpha)I_2 = O_2$ which is equivalent to $\alpha(\alpha^{2023} + 1) = \alpha^{2024} + \alpha = 0$, so to $\alpha = 0$ or $\alpha = -1$. We get $A = O_2$ or $-I_2$.

Suppose now that A is not of the form αI_2. By hypothesis, $AA^T = -A^{2025} = A^TA$. Since A and A^T commute and $A \neq \alpha I_2$ $\forall \alpha \in \mathbb{R}$, we have $A^T = \alpha A + \beta I_2$ for some $\alpha, \beta \in \mathbb{R}$. (See [1, p. 15, Theorem 1.1 (b)]). By taking the transpose, we get $A = \alpha A^T + \beta I_2$. By subtraction we obtain $(A - A^T)(1 + \alpha) = O_2$. Thus either $A = A^T$ or $\alpha = -1$.

I) We consider the case when $A = A^T$, i.e. A is real symmetric. Let λ be an eigenvalue of A. The equation $A^{2024} = -A$ implies that $\lambda^{2024} + \lambda = 0$, i.e. $\lambda(\lambda^{2023} + 1) = 0$ and since real symmetric matrices have real eigenvalues (see [1, Theorem 2.5, p. 73]) we get that $\lambda = 0$ or $\lambda = -1$. We distinguish between the cases (a) $\lambda_1 = \lambda_2 = 0$, (b) $\lambda_1 = \lambda_2 = -1$, and (c) $\lambda_1 = 0$, $\lambda_2 = -1$.

(a) $\lambda_1 = \lambda_2 = 0$. This implies, based on Cayley-Hamilton Theorem, that $A^2 = O_2$, from which we get $A = -A^{2024} = O_2$.

(b) $\lambda_1 = \lambda_2 = -1$. This shows, based on Cayley-Hamilton Theorem, that $(A + I_2)^2 = O_2$. Let $A + I_2 = B$. We have that $A = B - I_2$ and $B^2 = O_2$, so $B^k = O_2$ for $k \geq 2$. The equation $A^{2024} = -A$ implies that

$I_2 - B = -A = A^{2024} = (B - I_2)^{2024} = I_2 - 2024B$, so $A + I_2 = B = O_2$ and $A = -I_2$.

Remark. The conclusions in parts (a) and (b) also follow from the fact that real symmetric matrices, such as A, are diagonalizable, so if $\lambda_1 = \lambda_2 = \lambda$, then $A = \lambda I_2$.

(c) $\lambda_1 = 0$ and $\lambda_2 = -1$, i.e. $P_A(X) = X^2 + X$, which is equivalent to $\operatorname{Tr}(A) = -1$ and $\det A = 0$. The symmetric matrix A writes as $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Then $\operatorname{Tr}(A) = -1$ and $\det A = 0$ write as $a + c = -1$, i.e. $c = -1 - a$, and $ac - b^2 = a(-1 - a) - b^2 = 0$, i.e. $a^2 + a + b^2 = 0$. Thus the symmetric matrices satisfying $P_A(X) = X^2 + X$ have the form $A = \begin{pmatrix} a & b \\ b & -1-a \end{pmatrix}$, with $a, b \in \mathbb{R}$, $a + a^2 + b^2 = 0$. Conversely, if A is a symmetric matrix with $P_A(X) = X^2 + X = X(X + 1)$, then, by Cayley-Hamilton Theorem, $A(A+I_n) = O_2$. Since $X(X+1) \mid X(X^{2023}+1)$, this implies that $A^{2024}+A = A(A^{2023} + I_n) = O_n$, so $A^{2024} = -A = -A^T$.

II) If $\alpha = -1$, then $A = -A^T + \beta I_2$, i.e. $A + A^T = \beta I_2$ for some $\beta \in \mathbb{R}$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The equation $A + A^T = \beta I_2$ implies that $a = d = \beta/2$ and $c + b = 0$, so A has the form $A = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}$, with $a, c \in \mathbb{R}$.

By taking determinants in $A^{2024} = -A^T$, we get $(\det A)^{2024} = \det A$, i.e. $(\det A)(\det A^{2023} - 1) = 0$, so $a^2 + c^2 = \det A \in \{0, 1\}$. If $a^2 + c^2 = 0$, then $a = c = 0$, so $A = O_2$. If $a^2 + c^2 = 1$, then $a = \cos\theta$ and $b = \sin\theta$ for some unique $\theta \in [0, 2\pi)$. Hence $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Then the relation $A^{2024} = -A^T$ writes as

$$\begin{pmatrix} \cos(2024\theta) & -\sin(2024\theta) \\ \sin(2024\theta) & \cos(2024\theta) \end{pmatrix} = -\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

$$\begin{aligned} \cos(2024\theta) &= -\cos\theta = \cos(\pi - \theta), \\ \sin(2024\theta) &= \sin\theta = \sin(\pi - \theta). \end{aligned}$$

But $(\cos\alpha, \sin\alpha) = (\cos\beta, \sin\beta)$ iff $\alpha \equiv \beta \pmod{2\pi}$, so the relations above are equivalent to $2024\theta = (\pi - \theta) + 2l\pi$, i.e. $\theta = \dfrac{1+2l}{2025}\pi$ for some $l \in \mathbb{Z}$. The restriction $\theta \in [0, 2\pi)$ is equivalent to $0 \le \theta \le 2024$.

So we get the claimed solutions

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\frac{1+2l}{2025}\pi & -\sin\frac{1+2l}{2025}\pi \\ \sin\frac{1+2l}{2025}\pi & \cos\frac{1+2l}{2025}\pi \end{pmatrix}, \quad l = 0, \ldots, 2024.$$

The problem is solved.

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1. V. Pop, O. Furdui, *Square Matrices of Order 2, Theory, Applications and Problems*, Springer, Cham 2017.

We also received a solution, which is too long to include here, from Daniel Văcaru, Pitești, Romania.

552. Let $f:[0,1]\to[0,1]$ be a continuous function which is derivable on $(0,1]$, such that $f'(x)<0$ for $x\in(0,1]$, $f(1)=0$ and $f'(1)<0$. Prove that for every integer $n\geq 1$ the equation $f(x)=x^n$ has a unique solution in the interval $(0,1)$, denoted by a_n, and that $\lim\limits_{n\to\infty}\frac{n}{\ln n}(a_n-1)=-1$.

Proposed by Dumitru Popa, Department of Mathematics, Ovidius University of Constanța, Romania.

Solution by Marian-Daniel Vasile, Timișoara, Romania. From $f'(x)<0$ for $x\in(0,1]$ we get that f is decreasing on $[0,1]$.

For $n\geq 1$, let $f_n:[0,1]\to\mathbb{R}$, $f_n(x)=f(x)-x^n$. We can see that f_n is (strictly) decreasing on $[0,1]$ as a sum of two decreasing functions. Moreover f_n is continuous on $[0,1]$ and

$$f_n(1)=f(1)-1<0=f(1)<f(0)=f_n(0),$$

so, by Darboux, there is $a_n\in(0,1)$ such that $f_n(a_n)=0$ and, since f is decreasing, this a_n is unique. But $f_n(a_n)=0$ writes as $f(a_n)=a_n^n$.

We will first prove that $\lim\limits_{n\to\infty}a_n=1$. We notice that

$$f_{n+1}(x)-f_n(x)=x^n-x^{n+1}=x^n(1-x)>0 \qquad (\forall)x\in(0,1),$$

and therefore

$$f_{n+1}(a_n)>f_n(a_n)=0=f_{n+1}(a_{n+1}).$$

By the fact that f_{n+1} is strictly decreasing we see that $a_n<a_{n+1}$ for all $n\geq 1$. Then, $(a_n)_{n\geq 1}$ is convergent and we denote its limit by $l\in[0,1]$.

If $l<1$, then $\lim\limits_{n\to\infty}a_n^n=0$. Passing to limit in $f(a_n)=a_n^n$, we obtain $f(l)=0=f(1)$. Since f is strictly decreasing, we get $l=1$, which is absurd. We conclude that $\lim\limits_{n\to\infty}a_n=1$.

We note that

$$\lim_{n\to\infty}\frac{f(a_n)}{a_n-1}=\lim_{n\to\infty}\frac{f(a_n)-f(1)}{a_n-1}=f'(1) \tag{3}$$

and also

$$\lim_{n\to\infty}\frac{\ln a_n}{a_n-1}=\lim_{n\to\infty}\frac{\ln(1+(a_n-1))}{a_n-1}=1. \tag{4}$$

We have that $a_n\searrow 1$, so $f(a_n)\nearrow f(1)=0$, when \$$n\to\infty$. Hence

$$f(a_n)=a_n^n\Rightarrow \ln f(a_n)=n\ln a_n\Rightarrow \lim_{n\to\infty}n\ln a_n=\lim_{n\to\infty}\ln f(a_n)=-\infty. \tag{5}$$

Since $\ln f(a_n) = n \ln a_n$, by (3) and (5), we have that

$$\lim_{n\to\infty} \left(1 - \frac{\ln(1-a_n)}{n \ln a_n}\right) = \lim_{n\to\infty} \frac{\ln f(a_n) - \ln(1-a_n)}{n \ln a_n} = \lim_{n\to\infty} \frac{\ln\left(\frac{f(a_n)}{1-a_n}\right)}{n \ln a_n}$$
$$= \frac{\ln(-f'(1))}{-\infty} = 0.$$

so

$$\lim_{n\to\infty} \frac{\ln(1-a_n)}{n \ln a_n} = 1.$$

By multiplying with (4), we get

$$\lim_{n\to\infty} \frac{\ln(1-a_n)}{n(a_n-1)} = 1, \text{ i.e. } \lim_{n\to\infty} \frac{\frac{1}{1-a_n} \ln\left(\frac{1}{1-a_n}\right)}{n} = 1.$$

So if we denote $b_n := \dfrac{1}{1-a_n}$ for $n \geq 1$, then $\lim\limits_{n\to\infty} b_n = +\infty$. The last equation becomes

$$\lim_{n\to\infty} \frac{b_n \ln b_n}{n} = 1. \tag{6}$$

We will prove that $\lim\limits_{n\to\infty} \dfrac{b_n \ln n}{n} = 1$.

From (6) we have

$$\lim_{n\to\infty} \frac{b_n}{n} = \lim_{n\to\infty} \frac{\frac{b_n \ln b_n}{n}}{\ln b_n} = \frac{1}{+\infty} = 0. \tag{7}$$

Then, also from (6),

$$1 = \lim_{n\to\infty} \frac{b_n \ln b_n}{n} = \lim_{n\to\infty} \frac{b_n \left(\ln\left(\frac{b_n}{n}\right) + \ln n\right)}{n} = \lim_{n\to\infty} \frac{b_n}{n} \ln\left(\frac{b_n}{n}\right) + \frac{b_n \ln n}{n} \tag{8}$$

Using (7) and the fact that $\lim\limits_{x\to 0^+} x \ln x = 0$, we see that

$$\lim_{n\to\infty} \frac{b_n}{n} \ln\left(\frac{b_n}{n}\right) = 0,$$

so by (8) we conclude that

$$\lim_{n\to\infty} \frac{b_n \ln n}{n} = 1.$$

Then

$$\lim_{n\to\infty} \frac{n}{\ln n}(a_n - 1) = \lim_{n\to\infty} -\frac{n}{b_n \ln n} = -1.$$

□