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The limit of some sequences associated to log-concave functions

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Abstract. Let $f:(0,1]\to(0,\infty)$ be a log-concave derivable function such that $f(1)=1$, $f'(1)>0$. We prove that:
a) if there exists $\nu>0$ such that $\lim\limits_{x\to 0,x>0}\frac{f(x)}{x^{\nu}}\in(0,\infty)$, then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[f\left(\frac{k}{n}\right)\right]^{k}=\frac{1}{1-e^{-f'(1)}}$$

and
b) if $\lim\limits_{x\to 0,x>0}f(x)=\lambda\in(0,1)$, then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[f\left(\frac{k}{n}\right)\right]^{k}=\frac{\lambda}{1-\lambda}+\frac{1}{1-e^{-f'(1)}}.$$

Many and various concrete applications are given. For example, we prove that for all $0<\beta\le 1$, $\alpha>0$ one has

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\left[\Gamma\left(\frac{k^{\beta}}{n^{\beta}}\right)\right]^{\alpha k}}=\frac{1}{1-e^{-\alpha\beta\gamma}},$$

where Γ is the Euler gamma function and γ is the Euler constant, and if $\varphi:(-1,0]\to(0,\infty)$ is a log-concave derivable function such that $\varphi(0)=1$, $\varphi'(0)>0$, and $\lim\limits_{x\to -1,x>-1}\varphi(x)=\lambda\in(0,1)$, then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\varphi\left(\sqrt{\frac{k}{n}}-1\right)\right]^{k}=\frac{\lambda}{1-\lambda}+\frac{1}{1-e^{-\frac{\varphi'(0)}{2}}}.$$

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1. Introduction and notation

The main purpose of this paper is to prove the results stated into Abstract. These results where suggested to us by the following two limits

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{\alpha n}=\frac{1}{1-e^{-\alpha}},\ \alpha>0, \tag{1}$$

see, in historical order, [2, p. 481, problem 19, for $\alpha=1$], [1, p. 263, problem 10, for $\alpha>0$], and its natural analog

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{\alpha k}=\frac{1}{1-e^{-\alpha}},\ \alpha>0, \tag{2}$$

see [4, problem 1.6, p. 17, for $\alpha=1$ with the solution at page 157], or [8, p. 15, problems 1.44 and 1.45 (a) with the solution at pp. 272–275]. In 1996, in [5], the author of the present paper expanded the first limit (1) under the form: If $f:(0,1]\to(0,\infty)$ is a derivable function with $f(1)=1$, $f'(1)>0$, and $\ln f$ has decreasing derivative, then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[f\left(\frac{k}{n}\right)\right]^{n}=\frac{1}{1-e^{-f'(1)}}; \tag{3}$$

for a solution, see [7], for related questions see also [6, Corollary 3]. Let us mention that in [8, p. 14 with the solution at pp. 270–272] the authors reproduces our proof from [7]. In view of this result it appears as natural to ask:

Problem 1. *If* $f:(0,1]\to(0,\infty)$ *is a derivable function with* $f(1)=1$, $f'(1)>0$*, and* $\ln f$ *having decreasing derivative, then does it follow that*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[f\left(\frac{k}{n}\right)\right]^{k}=\frac{1}{1-e^{-f'(1)}}\quad ?$$

We prove that, under a natural assumption, the answer to Problem 1 is positive, see Theorem 6. However, as we will prove in Theorem 9 and in various concrete examples, the answer to Problem 1 is, in general, negative. This study was also motivated by an open problem from [3, p. 16], namely, whether

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\left[\Gamma\left(\frac{k}{n}\right)\right]^{k}}=\frac{e^{\gamma}}{e^{\gamma}-1},$$

which in the present paper is answered in the positive, see Theorem 11.

Recall that if I is a non-degenerate interval (not reduced at one point) a function $f:I\to\mathbb{R}$ is called *concave* (resp. *convex*) if for all $x,\ y\in I$, $\lambda\in[0,1]$, $h(\lambda x+(1-\lambda)y)\geq\lambda h(x)+(1-\lambda)h(y)$ (resp. $h(\lambda x+(1-\lambda)y)\leq\lambda h(x)+(1-\lambda)h(y)$). As it is well-known, the condition that a derivable function on an interval have the derivative decreasing is equivalent to the concavity of that function, see [1, Corollare 1, p. 36]. We recall that a function $f : I \to (0, \infty)$ is called *log-concave* if and only if $\ln f$ is concave. Thus if $f : I \to (0, \infty)$ is derivable, the condition that $\ln f$ to have decreasing derivative is equivalent to $\ln f$ is concave or, f log-concave.

The notations and notions used in the paper are standard, see for example [1].

2. Preliminary results

We will use throughout this paper the following two well-known results. For the sake of completeness we include their proofs.

Proposition 1. (i) *Let $g : (0, 1] \to \mathbb{R}$ be a derivable function with the derivative decreasing. Then $g(1-x) \le g(1) - xg'(1)$, $\forall x \in [0, 1)$.*
(ii) *Let $f : (0, 1] \to (0, \infty)$ be a log-concave derivable function. Then*

$$\ln f(1-x) \le \ln f(1) - x\frac{f'(1)}{f(1)}, \quad \forall x \in [0, 1).$$

Proof. (i) For $x = 0$ the inequality is true. Fix $x \in (0, 1)$. From the Lagrange theorem there exists $1 - x < c < 1$ such that $g(1) - g(1-x) = xg'(c)$. Since g' is decreasing $g'(c) \ge g'(1)$ and hence $g(1) - g(1-x) \ge xg'(1)$.
(ii) Since f is log-concave, that is $g = \ln f$ is concave, and g is derivable (f is derivable) as is well-known, g has the decreasing derivative, see [1, Corollaire 1 p. 36]. We apply (i). □

Proposition 2. *Let $h : [a, b] \to \mathbb{R}$ be a convex function. Then for all $x \in [a, b]$ we have $h(x) \le \max(h(a), h(b))$.*

Proof. Let $x \in [a, b]$. We have $x = \lambda b + (1-\lambda)a$, $0 \le \lambda = \frac{x-a}{b-a} \le 1$. Since h is convex, we get

$$\begin{aligned} h(x) &\le \lambda h(b) + (1-\lambda)h(a) \le \lambda \max(h(a), h(b)) + (1-\lambda)\max(h(a), h(b)) \\ &= \max(h(a), h(b)). \end{aligned}$$

□

3. The results

Theorem 3. *Let $f : (0, 1] \to (0, \infty)$ be a log-concave derivable function such that $f(1) = 1$, $f'(1) > 0$. Let $(\alpha_n)_{n\ge1}$ be a sequence of natural numbers such that $\lim_{n\to\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{\alpha_n^2}{n} = 0$. Then*

$$\lim_{n\to\infty} \sum_{k=n-\alpha_n}^{n} \left[f\left(\frac{k}{n}\right)\right]^k = \frac{1}{1 - e^{-f'(1)}}.$$

Proof. Let us observe that from $\lim\limits_{n\to\infty}\frac{\alpha_n^2}{n}=0$ it follows that $\lim\limits_{n\to\infty}\frac{\alpha_n}{n}=0<1$ and hence there exists $n_0\in\mathbb{N}$ such that $\frac{\alpha_n}{n}<1$, $\forall n\geq n_0$, or $n-\alpha_n\geq 1$, $\forall n\geq n_0$. The following reasoning is analogous with that we have used in the proof of the relation (3). For all $n\geq n_0$ let us denote $x_n=\sum\limits_{k=n-\alpha_n}^{n}\left[f\left(\frac{k}{n}\right)\right]^k$ and note the equality

$$x_n=\sum_{i=0}^{\alpha_n}\left[f\left(\frac{n-i}{n}\right)\right]^{n-i}=\sum_{i=0}^{\alpha_n}\left[f\left(1-\frac{i}{n}\right)\right]^{n-i}=\sum_{i=0}^{\alpha_n}e^{(n-i)\ln f\left(1-\frac{i}{n}\right)}.$$

For all $0\leq i\leq\alpha_n$, since f is a log-concave derivable function and $f(1)=1$, from Proposition 1(ii) it follows that $\ln f\left(1-\frac{i}{n}\right)\leq-\frac{i}{n}\frac{f'(1)}{f(1)}=-\frac{i}{n}f'(1)$, whence

$$\begin{aligned}(n-i)\ln f\left(1-\frac{i}{n}\right)&\leq-\frac{i(n-i)f'(1)}{n}=-if'(1)+\frac{i^2}{n}f'(1)\\&\leq-if'(1)+\frac{\alpha_n^2}{n}f'(1).\end{aligned}$$

Hence

$$x_n\leq e^{\frac{\alpha_n^2}{n}f'(1)}\cdot\sum_{i=0}^{\alpha_n}e^{-if'(1)}\leq e^{\frac{\alpha_n^2}{n}f'(1)}\cdot\sum_{i=0}^{\infty}e^{-if'(1)}.$$

Passing to the limit and using that $\lim\limits_{n\to\infty}\frac{\alpha_n^2}{n}=0$, we deduce that

$$\limsup x_n\leq\sum_{i=0}^{\infty}e^{-if'(1)}.$$

Let $m\in\mathbb{N}$. Since $\lim\limits_{n\to\infty}\alpha_n=\infty$, there exists $s\in\mathbb{N}$ such that $\forall n\geq s$ we have $\alpha_n>m$. Let us take $n\geq s$. From $m<\alpha_n$ it follows that

$$\sum_{i=0}^{m}e^{(n-i)\ln f\left(1-\frac{i}{n}\right)}\leq x_n.$$

For every $0\leq i\leq m$ from the equality

$$(n-i)\ln f\left(1-\frac{i}{n}\right)=-i\frac{\ln f\left(1-\frac{i}{n}\right)}{f\left(1-\frac{i}{n}\right)-1}\cdot\frac{f\left(1-\frac{i}{n}\right)-f(1)}{-\frac{i}{n}}\cdot\frac{n-i}{n}$$

we get $\lim\limits_{n\to\infty}(n-i)\ln f\left(1-\frac{i}{n}\right)=-if'(1)$. It follows that $\sum\limits_{i=0}^{m}e^{-if'(1)}\leq\liminf x_n$. For $m\to\infty$ we get

$$\sum_{i=0}^{\infty}e^{-if'(1)}\leq\liminf x_n.$$

Hence $\sum\limits_{i=0}^{\infty} e^{-if'(1)} \leq \liminf x_n \leq \limsup x_n \leq \sum\limits_{i=0}^{\infty} e^{-if'(1)}$, that is, $f'(1) > 0$,

$$\lim_{n\to\infty} \sum_{k=n-\alpha_n}^{n} \left[f\left(\frac{k}{n}\right)\right]^k = \sum_{i=0}^{\infty} e^{-if'(1)} = \frac{1}{1-e^{-f'(1)}}.$$

□

Proposition 4. *Let* $f:(0,1]\to(0,\infty)$ *be a log-concave derivable function such that* $f(1)=1$, $f'(1)>0$. *Let* $(\alpha_n)_{n\geq 1}$ *be a sequence of natural numbers such that* $\lim\limits_{n\to\infty}\alpha_n=\infty$, $\lim\limits_{n\to\infty}\frac{\alpha_n}{n}=0$ *and* $\lim\limits_{n\to\infty}\frac{n}{e^{2\alpha_n f'(1)}}=0$. *Then* $\lim\limits_{n\to\infty}\sum\limits_{k=\alpha_n}^{n-\alpha_n}\left[f\left(\frac{k}{n}\right)\right]^k=0$.

Proof. Because $\lim\limits_{n\to\infty}\frac{\alpha_n}{n}=0<\frac{1}{2}$, there exists $n_0\in\mathbb{N}$ such that $\frac{\alpha_n}{n}<\frac{1}{2}$, or $\alpha_n<n-\alpha_n$, $\forall n\geq n_0$. Let $n\geq n_0$. We have

$$\begin{aligned}\sum_{k=\alpha_n}^{n-\alpha_n}\left[f\left(\frac{k}{n}\right)\right]^k &= \sum_{i=\alpha_n}^{n-\alpha_n}\left[f\left(1-\frac{i}{n}\right)\right]^{n-i}=\sum_{i=\alpha_n}^{n-\alpha_n}e^{(n-i)\ln f\left(1-\frac{i}{n}\right)}\\ &\leq \sum_{i=\alpha_n}^{n-\alpha_n}e^{-\frac{i(n-i)f'(1)}{n}}\end{aligned}$$

(in the last inequality we have used Proposition 1(ii)). Since $f'(1)>0$, the function $x\mapsto -\frac{x(n-x)f'(1)}{n}$ is convex on the interval $[\alpha_n,n-\alpha_n]$ and from Proposition 2 it follows that

$$-\frac{x(n-x)f'(1)}{n}\leq -\frac{\alpha_n(n-\alpha_n)f'(1)}{n},\ \forall x\in[\alpha_n,n-\alpha_n].$$

From $\frac{n-\alpha_n}{n}>\frac{1}{2}$ it follows that $-\frac{\alpha_n(n-\alpha_n)}{n}<-\frac{\alpha_n}{2}$ and, since $f'(1)>0$, we deduce $-\frac{\alpha_n(n-\alpha_n)f'(1)}{n}<-\frac{\alpha_n f'(1)}{2}$ and hence

$$-\frac{x(n-x)f'(1)}{n}<-\frac{\alpha_n f'(1)}{2},\ \forall x\in[\alpha_n,n-\alpha_n].$$

Thus

$$\sum_{i=\alpha_n}^{n-\alpha_n}e^{-\frac{i(n-i)f'(1)}{n}}<\sum_{i=\alpha_n}^{n-\alpha_n}e^{-\frac{\alpha_n f'(1)}{2}}=\frac{n-2\alpha_n+1}{e^{\frac{\alpha_n f'(1)}{2}}}.$$

From $\lim\limits_{n\to\infty}\frac{n}{e^{\frac{\alpha_n f'(1)}{2}}}=0$ and the squeeze theorem we get the limit from the statement. □

Proposition 5. *Let* $f:(0,1]\to(0,\infty)$ *be a function with the property that there exists* $\nu>0$ *such that* $\lim\limits_{x\to 0,x>0}\frac{f(x)}{x^\nu}\in(0,\infty)$. *Let* $(\alpha_n)_{n\geq 1}$ *be a sequence* *of natural numbers such that* $\lim\limits_{n\to\infty}\alpha_n=\infty$, $\lim\limits_{n\to\infty}\frac{\alpha_n}{n}=0$, $\lim\limits_{n\to\infty}\frac{\alpha_n}{n^{2\nu}}=0$, $\lim\limits_{n\to\infty}\frac{\ln\alpha_n}{\ln n}=b<1$. *Then* $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{\alpha_n}\left[f\left(\frac{k}{n}\right)\right]^k=0$.

Proof. We first prove that

$$\lim_{n\to\infty}\sum_{k=2}^{\alpha_n}\left[f\left(\frac{k}{n}\right)\right]^k=0. \tag{4}$$

From $\lim\limits_{x\to0,x>0}\frac{f(x)}{x^\nu}=\lambda<2\lambda$ $(\lambda>0)$ it follows that there exists $\delta>0$ such that

$$\forall 0<x<\delta \text{ we have } \frac{f(x)}{x^\nu}<2\lambda,\ f(x)<2\lambda x^\nu. \tag{5}$$

By the hypotheses,

$$\lim_{n\to\infty}\left(\frac{2\ln(2\lambda)+2\nu\ln\nu}{\alpha_n\ln n}-\frac{2\nu}{\alpha_n}-\frac{\ln(2\lambda)}{\ln n}-\frac{\nu\ln\alpha_n}{\ln n}+\nu\right)=\nu(1-b)>0,$$

hence there exists $n_0\in\mathbb{N}$ such that $\frac{2\ln(2\lambda)+2\nu\ln\nu}{\alpha_n\ln n}-\frac{2\nu}{\alpha_n}-\frac{\ln(2\lambda)}{\ln n}-\frac{\nu\ln\alpha_n}{\ln n}+\nu>0$, $\forall n\geq n_0$ or equivalently,

$$2\ln(2\lambda)+2\nu\ln\nu-2\nu\ln n>\alpha_n\ln(2\lambda)+\nu\alpha_n\ln\alpha_n-\nu\alpha_n\ln n, \forall n\geq n_0. \tag{6}$$

Since $\lim\limits_{n\to\infty}\frac{\alpha_n}{n}=0$ there exists $n_1\in\mathbb{N}$ such that $\forall n\geq n_1$ we have $\frac{\alpha_n}{n}<\delta$. Let $n\geq\max(n_0,n_1)$. For all $2\leq k\leq\alpha_n$ we have $0<\frac{k}{n}\leq\frac{\alpha_n}{n}<\delta$ and, by (5), $f\left(\frac{k}{n}\right)<2\lambda\left(\frac{k}{n}\right)^\nu$, which implies that

$$k\ln f\left(\frac{k}{n}\right)<k\ln(2\lambda)+\nu k\ln k-\nu k\ln n.$$

Since $\nu>0$, the function $x\mapsto x\ln(2\lambda)+\nu x\ln x-\nu x\ln n$ is convex on the interval $[2,\alpha_n]$ and from Proposition 2 it follows that for all $2\leq k\leq\alpha_n$

$$\begin{aligned}&k\ln(2\lambda)+\nu k\ln k-\nu k\ln n\\ &\leq\max(2\ln(2\lambda)+2\nu\ln\nu-2\nu\ln n,\alpha_n\ln(2\lambda)+\nu\alpha_n\ln\alpha_n-\nu\alpha_n\ln n)\\ &=2\ln(2\lambda)+2\nu\ln\nu-2\nu\ln n\end{aligned}$$

by the relation (6). Hence $k\ln f\left(\frac{k}{n}\right)<2\ln(2\lambda)+2\nu\ln\nu-2\nu\ln n$, that is, $\left[f\left(\frac{k}{n}\right)\right]^k<\frac{4\lambda^2\nu^{2\nu}}{n^{2\nu}}$. We deduce that

$$0<\sum_{k=2}^{\alpha_n}\left[f\left(\frac{k}{n}\right)\right]^k\leq\frac{4\lambda^2\nu^{2\nu}(\alpha_n-1)}{n^{2\nu}}.$$

From $\lim\limits_{n\to\infty}\frac{\alpha_n}{n^{2\nu}}=0$ and the squeeze theorem we get the limit (4). From $\lim\limits_{x\to0,x>0}\frac{f(x)}{x^\nu}=\lambda$ and $\nu>0$ we deduce $\lim\limits_{x\to0,x>0}f(x)=0$. Then $\lim\limits_{n\to\infty}f\left(\frac{1}{n}\right)=0$ and from (4) we get the limit from the statement. □

Now we prove the first basic result of this paper.

Theorem 6. *Let* $f:(0,1]\rightarrow(0,\infty)$ *be a log-concave derivable function such that* $f(1)=1$, $f'(1)>0$, *and there exists* $\nu>0$ *such that* $\lim\limits_{x\rightarrow 0,x>0}\frac{f(x)}{x^{\nu}}\in(0,\infty)$. *Then*

$$\lim_{n\rightarrow\infty}\sum_{k=1}^{n}\left[f\left(\frac{k}{n}\right)\right]^{k}=\frac{1}{1-e^{-f'(1)}}.$$

Proof. Let $0<b<1$ and consider $\alpha_n=\lfloor n^b\rfloor$ (the integer part), where $0<b<\min\left(\frac{1}{2},2\nu\right)$. Then $\lim\limits_{n\rightarrow\infty}\alpha_n=\infty$, $\lim\limits_{n\rightarrow\infty}\frac{\ln\alpha_n}{\ln n}=b$, $\lim\limits_{n\rightarrow\infty}\frac{\alpha_n}{\sqrt{n}}=0$, $\lim\limits_{n\rightarrow\infty}\frac{\alpha_n}{n^{2\nu}}=0$. From $\lim\limits_{n\rightarrow\infty}\alpha_n=\infty$, $\lim\limits_{n\rightarrow\infty}\frac{\alpha_n}{n}=0<\frac{1}{2}$ we deduce that there exists $n_0\in\mathbb{N}$ such that $2\leq\alpha_n<n-\alpha_n$, $\forall n\geq n_0$. For all $n\geq n_0$ let us note the decomposition

$$\sum_{k=1}^{n}\left[f\left(\frac{k}{n}\right)\right]^{k}=\sum_{k=1}^{\alpha_n-1}\left[f\left(\frac{k}{n}\right)\right]^{k}+\sum_{k=\alpha_n}^{n-\alpha_n-1}\left[f\left(\frac{k}{n}\right)\right]^{k}+\sum_{k=n-\alpha_n}^{n}\left[f\left(\frac{k}{n}\right)\right]^{k}. \tag{7}$$

Since for all $\beta>0$, $\lim\limits_{n\rightarrow\infty}\frac{n}{e^{2\alpha_n\beta}}=0$, in particular, $\lim\limits_{n\rightarrow\infty}\frac{n}{e^{\frac{\alpha_n f'(1)}{2}}}=0$ and $0<\sum\limits_{k=1}^{\alpha_n-1}\left[f\left(\frac{k}{n}\right)\right]^{k}<\sum\limits_{k=1}^{\alpha_n}\left[f\left(\frac{k}{n}\right)\right]^{k}$, from Proposition 5 we deduce that

$$\lim_{n\rightarrow\infty}\sum_{k=1}^{\alpha_n-1}\left[f\left(\frac{k}{n}\right)\right]^{k}=0. \tag{8}$$

From $0<\sum\limits_{k=\alpha_n}^{n-\alpha_n-1}\left[f\left(\frac{k}{n}\right)\right]^{k}<\sum\limits_{k=\alpha_n}^{n-\alpha_n}\left[f\left(\frac{k}{n}\right)\right]^{k}$ and Proposition 4 we deduce that

$$\lim_{n\rightarrow\infty}\sum_{k=\alpha_n}^{n-\alpha_n-1}\left[f\left(\frac{k}{n}\right)\right]^{k}=0. \tag{9}$$

From Theorem 3 we deduce that

$$\lim_{n\rightarrow\infty}\sum_{k=n-\alpha_n}^{n}\left[f\left(\frac{k}{n}\right)\right]^{k}=\frac{1}{1-e^{-f'(1)}}. \tag{10}$$

From the relations (8), (9), (10), and (7) we get the limit from the statement. □

Corollary 7. *Let* $g:(0,1]\to(0,\infty)$ *be a log-concave derivable function such that* $g(1)=1$, $g'(1)>0$, *and there exists* $\theta>0$ *such that* $\lim\limits_{x\to 0,x>0}\frac{g(x)}{x^\theta}\in(0,\infty)$. *Then, for all* $0<\beta\le 1$, $\alpha>0$,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[g\left(\frac{k^\beta}{n^\beta}\right)\right]^{\alpha k}=\frac{1}{1-e^{-\alpha\beta g'(1)}}.$$

Proof. Let $f:(0,1]\to(0,\infty)$ be defined by $f(x)=\left[g\left(x^\beta\right)\right]^\alpha$. Then f is derivable and $\frac{f'(x)}{f(x)}=\frac{\alpha\beta}{x^{1-\beta}}\cdot\frac{g'\left(x^\beta\right)}{g\left(x^\beta\right)}$. Since $0<\beta\le 1$ and $\frac{g'}{g}$ is decreasing (because g is log-concave), it follows that $\frac{f'}{f}$ is decreasing, that is, f is log-concave. Moreover, $f(1)=1$, $f'(1)=\alpha\beta g'(1)>0$, and $\lim\limits_{x\to 0,x>0}\frac{f(x)}{x^{\alpha\beta\theta}}=\lim\limits_{x\to 0,x>0}\left[\frac{g\left(x^\beta\right)}{x^{\beta\theta}}\right]^\alpha=\lim\limits_{t\to 0,t>0}\left[\frac{g(t)}{t^\theta}\right]^\alpha\in(0,\infty)$. By Theorem 6 we have $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left[f\left(\frac{k}{n}\right)\right]^k=\frac{1}{1-e^{-f'(1)}}$. This and a simple calculation yield the limit from the statement. □

For the proof of the second basic result we need

Proposition 8. *Let* $f:(0,1]\to(0,\infty)$ *be such that* $\lim\limits_{x\to 0,x>0}f(x)=\lambda\in(0,1)$. *Let* $(\alpha_n)_{n\ge 1}$ *be a sequence of natural numbers such that* $\lim\limits_{n\to\infty}\alpha_n=\infty$ *and* $\lim\limits_{n\to\infty}\frac{\alpha_n}{n}=0$. *Then*

$$\lim_{n\to\infty}\sum_{k=1}^{\alpha_n-1}\left[f\left(\frac{k}{n}\right)\right]^k=\frac{\lambda}{1-\lambda}.$$

Proof. Let $0<\varepsilon<\min(\lambda,1-\lambda)$. There exists $\delta_\varepsilon>0$ such that $\forall 0<x<\delta_\varepsilon$ we have $|f(x)-f(0)|<\varepsilon$. Since $\lim\limits_{n\to\infty}\frac{\alpha_n}{n}=0$, there exists $n_\varepsilon\in\mathbb{N}$ such that $0<\frac{\alpha_n}{n}<\delta_\varepsilon$, $\forall n\ge n_\varepsilon$. Let $n\ge n_\varepsilon$. For all $1\le k\le\alpha_n$ we have $0<\frac{k}{n}\le\frac{\alpha_n}{n}<\delta_\varepsilon$ and hence $\left|f\left(\frac{k}{n}\right)-\lambda\right|<\varepsilon$, that is, $0<a:=\lambda-\varepsilon<f\left(\frac{k}{n}\right)<\lambda+\varepsilon=:b<1$. We deduce that $a^k<\left[f\left(\frac{k}{n}\right)\right]^k<b^k$, from where $\sum\limits_{k=1}^{\alpha_n}a^k<\sum\limits_{k=1}^{\alpha_n}\left[f\left(\frac{k}{n}\right)\right]^k<\sum\limits_{k=1}^{\alpha_n}b^k$, that is

$$\frac{a\left(1-a^{\alpha_n}\right)}{1-a}\le\sum_{k=1}^{\alpha_n}\left[f\left(\frac{k}{n}\right)\right]^k\le\frac{b\left(1-b^{\alpha_n}\right)}{1-b}.$$

Since $a, b \in (0,1)$ and $\lim\limits_{n\to\infty} \alpha_n = \infty$, we deduce that

$$\begin{aligned}\frac{\lambda-\varepsilon}{1-\lambda+\varepsilon} &= \frac{a}{1-a} \leq \liminf \sum_{k=1}^{\alpha_n} \left[f\left(\frac{k}{n}\right)\right]^k \leq \limsup \sum_{k=1}^{\alpha_n} \left[f\left(\frac{k}{n}\right)\right]^k \\ &\leq \frac{b}{1-b} = \frac{\lambda+\varepsilon}{1-\lambda-\varepsilon}.\end{aligned}$$

For $\varepsilon \to 0$, $\varepsilon > 0$, we get

$$\frac{\lambda}{1-\lambda} \leq \liminf \sum_{k=1}^{\alpha_n} \left[f\left(\frac{k}{n}\right)\right]^k \leq \limsup \sum_{k=1}^{\alpha_n} \left[f\left(\frac{k}{n}\right)\right]^k \leq \frac{\lambda}{1-\lambda},$$

that is, $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\alpha_n} \left[f\left(\frac{k}{n}\right)\right]^k = \frac{\lambda}{1-\lambda}$. Since $\lim\limits_{n\to\infty} \left[f\left(\frac{\alpha_n}{n}\right)\right]^{\alpha_n} = \lim\limits_{n\to\infty} e^{\alpha_n \ln f\left(\frac{\alpha_n}{n}\right)} = 0$ (as $\ln\lambda < 0$), the proof is finished. □

Now we prove the second basic result of this paper. It shows that, in general, the answer to Problem 1 is negative.

Theorem 9. *Let $f : (0,1] \to (0,\infty)$ be a log-concave derivable function such that $f(1) = 1$, $f'(1) > 0$, and $\lim\limits_{x\to 0, x>0} f(x) = \lambda \in (0,1)$. Then*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[f\left(\frac{k}{n}\right)\right]^k = \frac{\lambda}{1-\lambda} + \frac{1}{1-e^{-f'(1)}}.$$

Proof. Let $0 < b < 1$ and consider as above $\alpha_n = \lfloor n^b \rfloor$, where $0 < b < \frac{1}{2}$. Then $\lim\limits_{n\to\infty} \alpha_n = \infty$, $\lim\limits_{n\to\infty} \frac{\alpha_n^2}{n} = 0$, and $\lim\limits_{n\to\infty} \frac{n}{e^{2\alpha_n \beta}} = 0$ for all $\beta > 0$. From $\lim\limits_{n\to\infty} \alpha_n = \infty$ and $\lim\limits_{n\to\infty} \frac{\alpha_n}{n} = 0 < \frac{1}{2}$ we deduce that there exists $n_0 \in \mathbb{N}$ such that $2 \leq \alpha_n < n - \alpha_n$, $\forall n \geq n_0$. For all $n \geq n_0$ let us note the decomposition

$$\begin{aligned}\sum_{k=1}^{n} \left[f\left(\frac{k}{n}\right)\right]^k &= \sum_{k=1}^{\alpha_n - 1} \left[f\left(\frac{k}{n}\right)\right]^k + \sum_{k=\alpha_n}^{n-\alpha_n-1} \left[f\left(\frac{k}{n}\right)\right]^k \\ &\quad + \sum_{k=n-\alpha_n}^{n} \left[f\left(\frac{k}{n}\right)\right]^k. \qquad (11)\end{aligned}$$

By Proposition 8 we have

$$\lim_{n\to\infty} \sum_{k=1}^{\alpha_n - 1} \left[f\left(\frac{k}{n}\right)\right]^k = \frac{\lambda}{1-\lambda}. \qquad (12)$$

From $0 < \sum\limits_{k=\alpha_n}^{n-\alpha_n-1} \left[f\left(\frac{k}{n}\right)\right]^k < \sum\limits_{k=\alpha_n}^{n-\alpha_n} \left[f\left(\frac{k}{n}\right)\right]^k$ and Proposition 4 we deduce that

$$\lim_{n\to\infty} \sum_{k=\alpha_n}^{n-\alpha_n-1} \left[f\left(\frac{k}{n}\right)\right]^k = 0, \qquad (13)$$

while Theorem 3 gives

$$\lim_{n\to\infty}\sum_{k=n-\alpha_n}^{n}\left[f\left(\frac{k}{n}\right)\right]^{k}=\frac{1}{1-e^{-f'(1)}}. \tag{14}$$

From the relations (12), (13), (14), and (11) we get the limit from the statement. □

Corollary 10. *Let $g:(0,1]\to(0,\infty)$ be a log-concave derivable function such that $g(1)=1$, $g'(1)>0$, and $\lim\limits_{x\to 0,x>0} g(x)=\lambda\in(0,1)$. Then for all $0<\beta\le 1$, $\alpha>0$,*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[g\left(\frac{k^{\beta}}{n^{\beta}}\right)\right]^{\alpha k}=\frac{\lambda^{\alpha}}{1-\lambda^{\alpha}}+\frac{1}{1-e^{-\alpha\beta g'(1)}}.$$

Proof. Let $f:(0,1]\to(0,\infty)$ be defined by $f(x)=\left[g\left(x^{\beta}\right)\right]^{\alpha}$. Then f is log-concave derivable, $f(1)=1$, $f'(1)=\alpha\beta g'(1)>0$, and $\lim\limits_{x\to 0,x>0} f(x)=\lim\limits_{x\to 0,x>0}\left[g\left(x^{\beta}\right)\right]^{\alpha}=\lim\limits_{t\to 0,t>0}[g(t)]^{\alpha}=\lambda^{\alpha}\in(0,1)$. From Theorem 9 we get

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[f\left(\frac{k}{n}\right)\right]^{k}=\frac{\lambda^{\alpha}}{1-\lambda^{\alpha}}+\frac{1}{1-e^{-f'(1)}}$$

and by simple calculation, the limit from the statement. □

4. THE FIRST TYPE OF EXAMPLES

In this section, as application of Theorem 6, we give various examples. In the sequel we denote by γ the Euler constant and $\Gamma:(0,\infty)\to(0,\infty)$ is the Euler Gamma function, that is, $\Gamma(a)=\int_0^{\infty}x^{a-1}e^{-x}\,\mathrm{d}x$. We recall that Γ is log-convex, that is, $\ln\Gamma$ is convex, or equivalently $\ln\frac{1}{\Gamma}=-\ln\Gamma$ is log-concave. As it is well-known, Γ is derivable and $\Gamma'(1)=-\gamma$, see [1, Chapitre VII]. From $a\Gamma(a)=\Gamma(a+1)$, $\forall a>0$, we deduce $\lim\limits_{a\to 0,a>0} a\Gamma(a)=\lim\limits_{a\to 0,a>0}\Gamma(a+1)=\Gamma(1)=1$.

Theorem 11. *For all $0<\beta\le 1$, $\alpha>0$,*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\left[\Gamma\left(\frac{k^{\beta}}{n^{\beta}}\right)\right]^{\alpha k}}=\frac{1}{1-e^{-\alpha\beta\gamma}}.$$

Proof. Let $g:(0,1]\to(0,\infty)$ be defined by $g(x)=\frac{1}{\Gamma(x)}$. Then f is log-concave and derivable with $g'(x)=-\frac{\Gamma'(x)}{[\Gamma(x)]^2}$, $g'(1)=-\frac{\Gamma'(1)}{\Gamma(1)}=\gamma$. Moreover $\lim\limits_{x\to 0,x>0}\frac{g(x)}{x}=\lim\limits_{x\to 0,x>0}\frac{1}{x\Gamma(x)}=1$. From Corollary 7 we get the limit from the statement. □

Let us mention that Theorem 11 gives a positive answer for $\alpha = \beta = 1$ to the open problem stated in Introduction, see also [3, p. 16].

Proposition 12. *For all* $\alpha > 0$, $\beta > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\left(\frac{k}{n}\right)^{\alpha k}}{\left[\Gamma\left(\frac{k}{n}\right)\right]^{\beta k}} = \frac{1}{1 - e^{-(\alpha+\beta\gamma)}}.$$

Proof. Let $f : (0,1] \to (0,\infty)$ be defined by $f(x) = \frac{x^\alpha}{[\Gamma(x)]^\beta}$. Then $\ln f(x) = \alpha \ln x + \beta \ln \frac{1}{\Gamma(x)}$ is concave (sum of two concave functions). Moreover, $\frac{f'(x)}{f(x)} = \frac{\alpha}{x} - \frac{\beta\Gamma'(x)}{\Gamma(x)}$, $f'(1) = \alpha - \beta\Gamma'(1) = \alpha + \beta\gamma$, $\lim\limits_{x\to 0, x>0} \frac{g(x)}{x^{\alpha+\beta}} = \lim\limits_{x\to 0, x>0} \frac{1}{[x\Gamma(x)]^\beta} = 1$. From Theorem 6 we get the limit from the statement. □

The next result contains two of the possible extensions of the limit (2) from Introduction.

Proposition 13. *For all* $p \in \mathbb{N}$ *and* $\alpha > 0$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{k(k+n)(k+2n)\cdots(k+(p-1)n)}{p!n^p}\right]^{\alpha k} = \frac{1}{1 - e^{-\alpha\left(1+\frac{1}{2}+\cdots+\frac{1}{p}\right)}};$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{k(k+n)(2k+n)\cdots((p-1)k+n)}{p!n^p}\right]^{\alpha k} = \frac{1}{1 - e^{-\alpha p+\alpha\left(1+\frac{1}{2}+\cdots+\frac{1}{p}\right)}}.$$

Proof. For the first limit let $g : (0,1] \to (0,\infty)$ be defined by $g(x) = \frac{x(x+1)(x+2)\cdots(x+p-1)}{p!}$. Then g is derivable, $g(1) = 1$, $\frac{g'(x)}{g(x)} = \frac{1}{x} + \frac{1}{x+1} + \cdots + \frac{1}{x+p-1}$, hence g is log-concave and $\lim\limits_{x\to 0, x>0} \frac{g(x)}{x} = 1$. For the second limit let $g : (0,1] \to (0,\infty)$ be defined by $g(x) = \frac{x(x+1)(2x+1)\cdots((p-1)x+1)}{p!}$. Then g is derivable, $g(1) = 1$, $\frac{g'(x)}{g(x)} = \sum\limits_{k=1}^{p-1} \frac{k}{kx+1}$, hence g is log-concave and $\lim\limits_{x\to 0, x>0} \frac{g(x)}{x} = 1$. In both cases we apply Corollary 7 with $\beta = 1$. □

Proposition 14. *For all* $p \in \mathbb{N}$, $0 < \beta \le 1$, $\alpha > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{\ln\left(1+\frac{k^\beta}{n^\beta}\right)}{\ln 2} \frac{\ln\left(2+\frac{k^\beta}{n^\beta}\right)}{\ln 3} \cdots \frac{\ln\left(p+\frac{k^\beta}{n^\beta}\right)}{\ln(p+1)}\right]^{\alpha k} = \frac{1}{1 - e^{-\alpha\beta S}},$$

where $S = \sum\limits_{i=2}^{p+1} \frac{1}{i \ln i}$.

Proof. Let $g:(0,1]\to(0,\infty)$ be defined by $g(x)=\frac{\ln(1+x)\ln(2+x)\cdots\ln(p+x)}{(\ln 2)(\ln 3)\cdots(\ln(p+1))}$. Then $\lim\limits_{x\to 0,x>0}\frac{g(x)}{x}=\frac{1}{\ln(p+1)}$, $\frac{g'(x)}{g(x)}=\sum\limits_{i=1}^{p}\frac{1}{(i+x)\ln(i+x)}$ is decreasing, hence g is log-concave. From Corollary 7 we get the limit from the statement. $\square$

Proposition 15. *Let $\varphi:(0,1]\to(0,\infty)$ be a log-concave derivable function such that $\varphi(1)=1$, $\varphi'(1)>0$ and there exists $\nu>0$ such that $\lim\limits_{x\to 0,x>0}\frac{\varphi(x)}{x^{\nu}}\in(0,\infty)$. Then*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\varphi\left(\frac{\ln\left(1+\frac{k}{n}\right)}{\ln 2}\right)\right]^{k}=\frac{1}{1-e^{-\frac{\varphi'(1)}{2\ln 2}}}.$$

In particular, $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left[\ln\left(\frac{\ln 2+\ln\left(1+\frac{k}{n}\right)}{\ln^{2}2}\right)\right]^{k}=\frac{1}{1-e^{-\frac{1}{4\ln^{2}2}}}$.

Proof. Let $f:(0,1]\to(0,\infty)$ be defined by $f(x)=\varphi\left(\frac{\ln(1+x)}{\ln 2}\right)$. Then $f(1)=\varphi(1)=1$, $\frac{f'(x)}{f(x)}=\frac{\varphi'\left(\frac{\ln(1+x)}{\ln 2}\right)}{\varphi\left(\frac{\ln(1+x)}{\ln 2}\right)}\cdot\frac{1}{(x+1)\ln 2}$ is decreasing, hence, f is log-concave and $\lim\limits_{x\to 0,x>0}\frac{f(x)}{x^{\nu}}=\frac{1}{(\ln 2)^{\nu}}\lim\limits_{x\to 0,x>0}\frac{\varphi(t)}{t^{\nu}}\in(0,\infty)$. We apply Theorem 6. For the second limit we take $\varphi(x)=\frac{\ln(1+x)}{\ln 2}$. $\square$

Proposition 16. *Let $\varphi:(0,1]\to(0,\infty)$ be a log-concave derivable function such that $\varphi(1)=1$, $\varphi'(1)>0$, and there exists $\nu>0$ such that $\lim\limits_{x\to 0,x>0}\frac{\varphi(x)}{x^{\nu}}\in(0,\infty)$. Then*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\varphi\left(\frac{\sqrt{k+n}-\sqrt{n}}{\left(\sqrt{2}-1\right)\sqrt{n}}\right)\right]^{k}=\frac{1}{1-e^{-\frac{\varphi'(1)}{2\left(2-\sqrt{2}\right)}}}.$$

In particular, $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left[\frac{\ln\left(1+\frac{\sqrt{k+n}-\sqrt{n}}{\left(\sqrt{2}-1\right)\sqrt{n}}\right)}{\ln 2}\right]^{k}=\frac{1}{1-e^{-\frac{1}{4\left(2-\sqrt{2}\right)\ln 2}}}$.

Proof. Let $f:(0,1]\to(0,\infty)$ be defined by $f(x)=\varphi\left(\frac{\sqrt{x+1}-1}{\sqrt{2}-1}\right)$. Then $f(1)=\varphi(1)=1$, $\frac{f'(x)}{f(x)}=\frac{\varphi'\left(\frac{\sqrt{x+1}-1}{\sqrt{2}-1}\right)}{\varphi\left(\frac{\sqrt{x+1}-1}{\sqrt{2}-1}\right)}\cdot\frac{1}{2\sqrt{x+1}\left(\sqrt{2}-1\right)}$ is decreasing, so that f is log-concave and $\lim\limits_{x\to 0,x>0}\frac{f(x)}{x^{\nu}}=\frac{1}{2^{\nu}\left(\sqrt{2}-1\right)^{\nu}}\lim\limits_{x\to 0,x>0}\frac{\varphi(t)}{t^{\nu}}\in(0,\infty)$. We apply Theorem 6. For the second limit we take $\varphi(x)=\frac{\ln(1+x)}{\ln 2}$. $\square$

5. The second type of examples

In this section, we give various examples as application of Theorem 9. These examples show, in particular, that the answer to Problem 1 is, in general, negative.

Proposition 17. *Let $\varphi : \left[\frac{1}{2}, 1\right] \to (0, \infty)$ be a log-concave derivable function such that $\varphi(1) = 1$, $\varphi'(1) > 0$, and $\varphi\left(\frac{1}{2}\right) \in (0, 1)$. Then*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[\varphi\left(\frac{k+n}{2n}\right)\right]^k = \frac{\varphi\left(\frac{1}{2}\right)}{1-\varphi\left(\frac{1}{2}\right)} + \frac{1}{1-e^{-\frac{\varphi'(1)}{2}}}.$$

In particular, $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left(\frac{k+n}{2n}\right)^k = \frac{2\sqrt{e}-1}{\sqrt{e}-1}$.

Proof. Let us define $f : (0, 1] \to (0, \infty)$ by $f(x) = \varphi\left(\frac{x+1}{2}\right)$. Then $f(1) = \varphi(1) = 1$, $\frac{f'(x)}{f(x)} = \frac{\varphi'\left(\frac{x+1}{2}\right)}{2\varphi\left(\frac{x+1}{2}\right)}$ is decreasing, hence f is log-concave, $f'(1) = \frac{\varphi'(1)}{2}$, and $\lim\limits_{x\to 0, x>0} f(x) = \varphi\left(\frac{1}{2}\right) = \lambda \in (0, 1)$. From Theorem 9 we get the limit from the statement. For the second limit we take $\varphi(x) = x$. $\square$

Proposition 18. *For all $p \in \mathbb{N}$ we have*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{(k+n)(k+2n)\cdots(k+pn)}{(p+1)!n^p}\right]^k = \frac{1}{p} + \frac{1}{1-e^{-\left(\frac{1}{2}+\cdots+\frac{1}{p+1}\right)}};$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{(k+n)(2k+n)\cdots(pk+n)}{(p+1)!n^p}\right]^k = \frac{1}{(p+1)!-1} + \frac{1}{1-e^{-\left(p-\frac{1}{2}-\cdots-\frac{1}{p+1}\right)}}.$$

Proof. For the first limit let $f : (0, 1] \to (0, \infty)$ be defined by $f(x) = \frac{(x+1)(x+2)\cdots(x+p)}{(p+1)!}$. Then $f(1) = 1$, $\frac{f'(x)}{f(x)} = \sum\limits_{i=1}^{p} \frac{1}{x+i}$ is decreasing, hence f is log-concave and $\lim\limits_{x\to 0, x>0} f(x) = \frac{1}{p+1} \in (0, 1)$. For the second limit let $f : (0, 1] \to (0, \infty)$ be defined by $f(x) = \frac{(x+1)(2x+1)\cdots(px+1)}{(p+1)!}$. Then $f(1) = 1$, $\frac{f'(x)}{f(x)} = \sum\limits_{i=1}^{p} \frac{i}{ix+1}$ is decreasing, so that f is log-concave and $\lim\limits_{x\to 0, x>0} f(x) = \frac{1}{(p+1)!} \in (0, 1)$. In both cases we apply Theorem 9. $\square$

Proposition 19. *Let $\varphi : (-1, 0] \to (0, \infty)$ be a log-concave derivable function such that $\varphi(0) = 1$, $\varphi'(0) > 0$, and $\lim\limits_{x\to -1, x>-1} \varphi(x) = \lambda \in (0, 1)$. Then*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[\varphi\left(\sqrt{\frac{k}{n}} - 1\right)\right]^k = \frac{\lambda}{1-\lambda} + \frac{1}{1-e^{-\frac{\varphi'(0)}{2}}}.$$

In particular, $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} e^{k\left(\sqrt{\frac{k}{n}}-1\right)} = \frac{e+\sqrt{e}+1}{e-1}$.

Proof. Let us define $f:(0,1]\to(0,\infty)$ by $f(x)=\varphi(\sqrt{x}-1)$. Then $f(1)=\varphi(0)=1$, $\frac{f'(x)}{f(x)}=\frac{1}{2\sqrt{x}}\frac{\varphi'(\sqrt{x}-1)}{\varphi(\sqrt{x}-1)}$ is decreasing (as a product of two strictly positive decreasing functions), hence, f is log-concave and $\lim\limits_{x\to 0, x>0} f(x)=\lambda$. From Theorem 9 we get the first limit from the statement. For the second one we take $\varphi(x)=e^x$. □

Proposition 20. *Let* $\varphi:\left(\frac{1}{2},1\right]\to(0,\infty)$ *be a log-concave derivable function such that* $\varphi(1)=1$, $\varphi'(1)>0$, *and* $\lim\limits_{x\to\frac{1}{2}, x>\frac{1}{2}}\varphi(x)=\lambda\in(0,1)$. *Then*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\varphi\left(\frac{2k+n}{k+2n}\right)\right]^{k}=\frac{\lambda}{1-\lambda}+\frac{1}{1-e^{-\frac{\varphi'(1)}{3}}}.$$

In particular, $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\frac{2k+n}{k+2n}\right)^{k}=\frac{2\sqrt[3]{e}-1}{\sqrt[3]{e}-1}$.

Proof. Let us define $f:(0,1]\to(0,\infty)$ by $f(x)=\varphi\left(\frac{2x+1}{x+2}\right)$. Then $f(1)=\varphi(1)=1$, $\frac{f'(x)}{f(x)}=\frac{\varphi'\left(\frac{2x+1}{x+2}\right)}{\varphi\left(\frac{2x+1}{x+2}\right)}\cdot\frac{3}{(2x+1)(x+2)}$ is decreasing as a product of two positive decreasing functions, hence, f is log-concave and $\lim\limits_{x\to 0, x>0} f(x)=\lambda$. From Theorem 9 we get the first limit from the statement. For the second one we take $\varphi(x)=x$. □

Proposition 21. *Let* $\varphi:(-\ln 2,0]\to(0,\infty)$ *be a log-concave derivable function such that* $\varphi(0)=1$, $\varphi'(0)>0$, *and* $\lim\limits_{x\to-\ln 2, x>-\ln 2}\varphi(x)=\lambda\in(0,1)$. *Then for all* $0<\beta\le 1$, $\alpha>0$,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\varphi\left(\ln\frac{k^{\beta}+n^{\beta}}{2n^{\beta}}\right)\right]^{\alpha k}=\frac{\lambda^{\alpha}}{1-\lambda^{\alpha}}+\frac{1}{1-e^{-\frac{\alpha\beta\varphi'(0)}{2}}}.$$

In particular, $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(1+\ln\frac{k^{\beta}+n^{\beta}}{2n^{\beta}}\right)^{k}=\frac{(1-\ln 2)^{\alpha}}{1-(1-\ln 2)^{\alpha}}+\frac{1}{1-e^{-\frac{\alpha\beta}{2}}}$.

Proof. Let $g:(0,1]\to(0,\infty)$ be defined by $g(x)=\varphi\left(\ln\frac{x+1}{2}\right)$. Then $g(1)=\varphi(0)=1$, $\frac{g'(x)}{g(x)}=\frac{\varphi'\left(\ln\frac{x+1}{2}\right)}{\varphi\left(\ln\frac{x+1}{2}\right)}\cdot\frac{1}{x+1}$ is decreasing, hence g is log-concave and $\lim\limits_{x\to 0, x>0} g(x)=\lambda\in(0,1)$. From Corollary 10 we get the limit from the statement.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\varphi\left(\ln\frac{k^{\beta}+n^{\beta}}{2n^{\beta}}\right)\right]^{\alpha k}=\frac{\lambda^{\alpha}}{1-\lambda^{\alpha}}+\frac{1}{1-e^{-\frac{\alpha\beta\varphi'(0)}{2}}}.$$

For the second equality claimed in the statement we take $\varphi(t)=1+t$. □

Proposition 22. *For all $\alpha>0$*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\left[\Gamma\left(\frac{k+n}{2n}\right)\right]^{\alpha k}}=\frac{1}{\sqrt{\pi^{\alpha}}-1}+\frac{1}{1-e^{-\frac{\alpha\gamma}{2}}};$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\left[\Gamma\left(\frac{2k+n}{k+2n}\right)\right]^{\alpha k}}=\frac{1}{\sqrt{\pi^{\alpha}}-1}+\frac{1}{1-e^{-\frac{\alpha\gamma}{3}}}.$$

Proof. For the first limit let $\varphi:\left[\frac{1}{2},1\right]\to(0,\infty)$ be defined by $\varphi(x)=\frac{1}{[\Gamma(x)]^{\alpha}}$. Then φ is log concave $\frac{\varphi'(x)}{\varphi(x)}=-\frac{\alpha\Gamma'(x)}{\Gamma(x)}$, $\varphi'(1)=\alpha\gamma$, and $\varphi\left(\frac{1}{2}\right)=\lambda=\frac{1}{\left[\Gamma\left(\frac{1}{2}\right)\right]^{\alpha}}=\frac{1}{\sqrt{\pi^{\alpha}}}\in(0,1)$, $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$. We apply Proposition 17. For the second limit we apply Proposition 20 for $\varphi:\left(\frac{1}{2},1\right]\to(0,\infty)$, $\varphi(x)=\frac{1}{[\Gamma(x)]^{\alpha}}$.
□

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Two short proofs of a notable symmetric inequality

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Abstract. In this paper we give two short solutions to the notable inequality

$$\frac{1}{a_1^2+1}+\frac{1}{a_2^2+1}+\cdots+\frac{1}{a_n^2+1}\ge\frac{n}{2},$$

which holds for any nonnegative real numbers $a_1, a_2, \dots, a_n$ satisfying $\sum_{1\le i<j\le n} a_ia_j=\frac{n(n-1)}{2}$. For $n\ge 3$, the equality occurs when $a_1=a_2=\cdots=a_n=1$, and also when $a_1=a_2=\cdots=a_{n-1}=\sqrt{\frac{n}{n-2}}$ and $a_n=0$ (or any cyclic permutation).

Keywords: Nonnegative variables, symmetric constraint and inequality, minimum value

MSC: 26D10, 26D15

A proof of the inequality

$$\frac{1}{a_1^2+1}+\frac{1}{a_2^2+1}+\cdots+\frac{1}{a_n^2+1}\ge\frac{n}{2}$$

for nonnegative real numbers $a_1, a_2, \dots, a_n$, under the constraint

$$\sum_{1\le i<j\le n} a_ia_j=\frac{n(n-1)}{2},$$

is given in [2] for $n\le 8$, and in [3] for any integer $n\ge 3$. In this paper, we give two simpler and shorter solutions than the one in [3], which uses the method of Lagrange multipliers. Note that the inequality was proposed and proved for $n=3$ in 2005 [1]. Later, in 2013, Henrique Vaz posted it for $n=4$ on the website Art of Problem Solving [4].

1. First solution

First we need the following lemma:

Lemma 1. *Let a and b be positive real constants, and let $x\ge y\ge 0$ such that*

$$xy+a(x+y)=b.$$

Then, the expression

$$E=\frac{1}{x^2+1}+\frac{1}{y^2+1}$$

has the minimum value for $y=0$ or $x=y$.

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Proof. Let $s = x + y$ and $p = xy$. We need to show that if

$$0 \le 4p \le s^2$$

and

$$p + as = b,$$

then the expression

$$E = \frac{s^2 - 2p + 2}{s^2 + (p-1)^2}$$

has the minimum value for $p = 0$ (when $y = 0$) or $4p = s^2$ (when $x = y$). From

$$b = p + as \ge p + 2a\sqrt{p}\ ,$$

we get

$$p \le p_1 = (\sqrt{a^2 + b} - a)^2,$$

with equality for $4p = s^2$. Consider further the cases $p \ge 1$ and $0 \le p \le 1$.

Case 1: $p \ge 1$. Since

$$b = p + as \ge p + 2a\sqrt{p} \ge 1 + 2a,$$

this case is possible only when

$$b \ge 1 + 2a.$$

We will show that E has the minimum value for $4p = s^2$. Indeed, from

$$E - \frac{2}{p+1} = \frac{(s^2 - 4p)(p-1)}{(p+1)[s^2 + (p-1)^2]}\ ,$$

we get $E \ge \dfrac{2}{p+1}$, therefore

$$E \ge \frac{2}{p_1 + 1}\ ,$$

with equality for $4p = s^2$.

Case 2: $0 \le p \le 1$. Since

$$E = 1 + \frac{1 - p^2}{s^2 + (p-1)^2} = 1 + F(p),$$

where

$$F(p) = \frac{a^2(1 - p^2)}{(a^2 + 1)p^2 - 2(a^2 + b)p + a^2 + b^2},$$

the expression E has the minimum value when $F(p)$ has the minimum value. We will show next that $F(p)$ has the minimum value when $p = 0$ or $4p = s^2$ or $p = 1$. Since $F(p) \ge 0$ for $1 \le p \le 1$, $F(p)$ has the minimum value 0 if p can take the value 1, i.e. if

$$b \ge 1 + 2a.$$

Indeed, the equality $p = 1$ implies

$$b = p + as \geq p + 2a\sqrt{p} = 1 + 2a.$$

Consider next that

$$b \leq 1 + 2a.$$

From

$$\sqrt{p_1} - 1 = \sqrt{a^2 + b} - a - 1 = \frac{b - 1 - 2a}{\sqrt{a^2 + b} + a + 1} \leq 0,$$

it follows that $p_1 \leq 1$, therefore

$$0 \leq p \leq p_1 \leq 1.$$

Denoting by m $(m \geq 0)$ the minimum value of $F(p)$ for $0 \leq p \leq p_1$, we have

$$F(p) \geq m,$$

with equality for at least a value of $p \in [0, p_1]$. Write the inequality $F(p) \geq m$ as

$$F_1(p) \geq 0,$$

where

$$F_1(p) = -[(m + 1)a^2 + m]p^2 + 2m(a^2 + b)p - (m - 1)a^2 - mb^2.$$

Since $F_1(p)$ is concave, the inequality $F_1(p) \geq 0$ holds for $0 \leq p \leq p_1$ if and only if $F_1(0) \geq 0$ and $F_1(p_1) \geq 0$. In addition, we have $F_1(p) = 0$ ($F(p)$ has the minimum value m) for $p = 0$ or for $p = p_1$ (when $4p = s^2$).

To finish the proof of Lemma 1, we need to show that E does not have the minimum value when $xy = 1$ and $x \neq y$. Since $xy = 1$ entails $E = 1$ and

$$b = xy + a(x + y) > xy + 2a\sqrt{xy} = 1 + 2a,$$

it suffices to show that $E < 1$ for $b > 1 + 2a$ and $x = y$. Indeed, for $b > 1 + 2a$ and $x = y$, from the constraint $xy + a(x + y) = b$ we get

$$x = y = \sqrt{a^2 + b} - a > \sqrt{a^2 + 1 + 2a} - a = 1,$$

therefore

$$E = \frac{1}{x^2 + 1} + \frac{1}{y^2 + 1} = \frac{2}{x^2 + 1} < 1.$$

□

Now, to prove the original inequality, we use the following theorem:

Theorem 2. *Let $n \geq 3$, and let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that*

$$\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n - 1)}{2}.$$

If a_k and a_m are variable numbers and all other numbers are fixed, then the expression

$$F(a_1, a_2, \dots, a_n) = \frac{1}{a_1^2+1} + \frac{1}{a_2^2+1} + \cdots + \frac{1}{a_n^2+1}$$

has the minimum value when $a_k = a_m$ or $a_k a_m = 0$.

Proof. Without loss of generality, assuming that $a_k = a_1$, $a_m = a_2$, and $a_1 \geq a_2$, the expression $F(a_1, a_2, \dots, a_n)$ has the minimum value when

$$E(a_1, a_2) = \frac{1}{a_1^2+1} + \frac{1}{a_2^2+1}$$

has the minimum value. Denoting

$$x = a_1, \qquad y = a_2,$$

$$a = \sum_{i=3}^{n} a_i, \qquad b = \frac{n(n-1)}{2} - \sum_{3 \leq i < j \leq n} a_i a_j,$$

we have $a \geq 0$, $b \geq 0$, $x \geq y \geq 0$, and $xy + a(x+y) = b$. There are three cases to consider: 1) $a = 0$; 2) $b = 0$, $a > 0$; 3) $a, b > 0$.

Case 1: $a = 0$. Since $a_3 = \cdots = a_n = 0$ and $a_1 a_2 = \dfrac{n(n-1)}{2} > 1$, we have

$$\begin{aligned} E(a_1, a_2) &= 1 - \frac{a_1^2 a_2^2 - 1}{a_1^2 + a_2^2 + a_1^2 a_2^2 + 1} \geq 1 - \frac{a_1^2 a_2^2 - 1}{2a_1 a_2 + a_1^2 a_2^2 + 1} \\ &= 1 - \frac{a_1 a_2 - 1}{a_1 a_2 + 1} = \frac{4}{n^2 - n + 2}. \end{aligned}$$

Therefore, the expression $E(a_1, a_2)$ has the minimum value when $a_1^2 + a_2^2 = 2a_1 a_2$, hence when $a_1 = a_2$.

Case 2: $b = 0$, $a > 0$. We have $xy + a(x+y) = b = 0$, which holds only when $x = y = 0$, hence $a_1 = a_2 = 0$.

Case 3: $a, b > 0$. By Lemma 1, the expression $E(a_1, a_2)$ has the minimum value when $a_1 = a_2$ or $a_2 = 0$.

□

Based on Theorem 2, we can prove the original inequality

$$F(a_1, a_2, \dots, a_n) \geq \frac{n}{2}.$$

For $n = 2$, the inequality is an identity. Consider further $n \geq 3$. By Theorem 2, it suffices to consider the cases when $a_1 = \cdots = a_j := x$ and $a_{j+1} = \cdots = a_n = 0$, where $j \in \{2, \dots, n\}$. So, we need to show that

$$j(j-1)x^2 = n(n-1)$$

implies

$$\frac{j}{x^2+1}+n-j\geq\frac{n}{2},$$

which is equivalent to

$$(n-2j)x^2+n\geq 0.$$

Indeed, we have

$$(n-2j)x^2+n=\frac{n(n-1)(n-2j)}{j(j-1)}+n=\frac{n(n-j)(n-j-1)}{j(j-1)}\geq 0.$$

The proof is completed. For $n\geq 3$, the equality occurs when $a_1=a_2=\cdots=a_n=1$, and also when $a_1=a_2=\cdots=a_{n-1}=\sqrt{\dfrac{n}{n-2}}$ and $a_n=0$ (or any cyclic permutation).

2. Second solution

We will use the induction method. For $n=2$, the inequality is an identity. Assume now that the statement holds for $n\geq 2$ nonnegative real numbers a_i and show that it also holds for $n+1$ nonnegative numbers a_i, that is, if

$$\sum_{1\leq i<j\leq n+1}a_ia_j=\frac{n(n+1)}{2},$$

then

$$\sum_{i=1}^{n+1}\frac{1}{a_i^2+1}\geq\frac{n+1}{2}.$$

Without loss of generality, assume that $a_{n+1}=\min\{a_1,a_2,\dots,a_n\}$. We claim that this assumption implies

$$\sum_{1\leq i<j\leq n}a_ia_j\geq\frac{n(n-1)}{2},$$

hence

$$\sum_{1\leq i<j\leq n}a_ia_j=\frac{n(n-1)t^2}{2},\qquad t\geq 1.$$

To prove this claim, we denote a_{n+1} by y, and write the desired inequality as follows:

$$(n+1)\sum_{1\leq i<j\leq n}a_ia_j\geq(n-1)\sum_{1\leq i<j\leq n+1}a_ia_j,$$

$$(n+1)\sum_{1\leq i<j\leq n}a_ia_j\geq(n-1)\left(\sum_{1\leq i<j\leq n}a_ia_j+y\sum_{i=1}^{n}a_i\right),$$

$$2\sum_{1\le i<j\le n} a_ia_j \ge (n-1)y\sum_{i=1}^{n} a_i.$$

Using the substitutions $a_i = x_i + y$ for $i = 1, 2, \ldots, n$, we have all $x_i \ge 0$ and

$$2\sum_{1\le i<j\le n} a_ia_j - (n-1)y\sum_{i=1}^{n} a_i = 2\sum_{1\le i<j\le n} (x_i+y)(x_j+y) - (n-1)y\sum_{i=1}^{n}(x_i+y)$$

$$= 2\sum_{1\le i<j\le n} x_ix_j + (n-1)y\sum_{i=1}^{n} x_i \ge 0.$$

Next, from the known inequality

$$a_1^2 + a_2^2 + \cdots + a_n^2 \ge \frac{1}{n}(a_1 + a_2 + \cdots + a_n)^2,$$

we get

$$(a_1 + a_2 + \cdots + a_n)^2 - n(n-1)t^2 \ge \frac{1}{n}(a_1 + a_2 + \cdots + a_n)^2,$$

therefore

$$a_1 + a_2 + \cdots + a_n \ge nt.$$

Since

$$\begin{aligned}\frac{n(n+1)}{2} &= \sum_{1\le i<j\le n+1} a_ia_j = \sum_{1\le i<j\le n} a_ia_j + (a_1 + a_2 + \cdots + a_n)a_{n+1}\\ &\ge \frac{n(n-1)t^2}{2} + nta_{n+1},\end{aligned}$$

we obtain

$$a_{n+1} \le \frac{T}{2t}, \quad \text{where} \quad T = n + 1 - (n-1)t^2.$$

Let us define the nonnegative real numbers $b_i = \dfrac{a_i}{t}$ for $i = 1, 2, \ldots, n$. Since

$$\sum_{1\le i<j\le n} b_ib_j = \frac{1}{t^2}\sum_{1\le i<j\le n} a_ia_j = \frac{n(n-1)}{2},$$

by the induction hypothesis we have

$$\sum_{i=1}^{n} \frac{1}{b_i^2 + 1} \ge \frac{n}{2},$$

hence

$$\sum_{i=1}^{n} \frac{1}{a_i^2 + t^2} \ge \frac{n}{2t^2}.$$

By the Cauchy-Schwarz inequality, we have

$$[(a_i^2 + 1) + (t^2 - 1)]\left[\frac{(t^2+1)^2}{a_i^2 + 1} + (t^2 - 1)\right] \ge [(t^2 + 1) + (t^2 - 1)]^2,$$

which is equivalent to

$$\frac{(t^2+1)^2}{a_i^2+1}+t^2-1\geq\frac{4t^4}{a_i^2+t^2}.$$

By summing these inequalities for all $i\leq n$, we obtain

$$(t^2+1)^2\sum_{i=1}^{n}\frac{1}{a_i^2+1}+n(t^2-1)\geq 4t^2\sum_{i=1}^{n}\frac{1}{a_i^2+t^2}\geq 2nt^2,$$

hence

$$\sum_{i=1}^{n}\frac{1}{a_i^2+1}\geq\frac{n}{t^2+1}.$$

Finally, we have

$$\begin{aligned}\sum_{i=1}^{n+1}\frac{1}{a_i^2+1}&=\sum_{i=1}^{n}\frac{1}{a_i^2+1}+\frac{1}{a_{n+1}^2+1}\geq\frac{n}{t^2+1}+\frac{1}{a_{n+1}^2+1}\\&\geq\frac{n}{t^2+1}+\frac{1}{T^2/(4t^2)+1}\\&=\frac{n+1}{2}-\frac{(n^2-1)[(n-1)t^6-(3n-1)t^4+(3n+1)t^2-n-1]}{2(t^2+1)(T^2+4t^2)}\\&=\frac{n+1}{2}-\frac{(n^2-1)(t^2-1)^2[(n-1)t^2-n-1]}{2(t^2+1)(T^2+4t^2)}\\&=\frac{n+1}{2}+\frac{(n^2-1)(t^2-1)^2T}{2(t^2+1)(T^2+4t^2)}\geq\frac{n+1}{2}.\end{aligned}$$

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An alternating sum with three consecutive harmonic numbers

NANDAN SAI DASIREDDY[1)]

Abstract. In this paper we give a closed form expression for the following alternating series

$$\sum_{n=1}^{\infty}(-1)^n\frac{H_nH_{n+1}H_{n+2}}{n\,(n+1)\,(n+2)},$$

solving the second part of open problem 3.105 in the recent book *Sharpening mathematical analysis skills* by Ovidiu Furdui and Alina Sîntămărian. Our proof involves the use of some identities due to Anthony Sofo.

Keywords: Classical harmonic numbers, alternating linear harmonic sums, nonlinear harmonic sums, Riemann zeta function, Dirichlet's eta function, polylogarithm function

MSC: 40A25, 11M06

1. INTRODUCTION

Furdui and Sîntămărian considered the following problem of evaluating an alternating series involving consecutive harmonic numbers as an open problem in [4, p. 119], to which we will provide a solution in this paper.

$$\sum_{n=1}^{\infty}(-1)^n\frac{H_nH_{n+1}H_{n+2}}{n\,(n+1)\,(n+2)}. \tag{1}$$

Throughout this paper, H_n denotes the nth classical harmonic number defined by $H_n=\sum_{k=1}^{n}\frac{1}{k}$, $\lfloor x\rfloor$ denotes the floor function, which is defined for $x\in\mathbb{R}$ by $\lfloor x\rfloor=\max\{k\in\mathbb{Z}\,|\,k\le x\}$, $\zeta(s)$ denotes the Riemann zeta function, which is defined by $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$, $\Re(s)>1$, $\mathrm{Li}_n(x)$ is the polylogarithm function defined for $|x|\le 1$ by $\mathrm{Li}_n(x)=\sum_{k=1}^{\infty}\frac{x^k}{k^n}$, $n\in\mathbb{N}$, $n\ge 2$, and $\eta(z)$, $z\in\mathbb{C}$, denotes the alternating zeta function (also known as Dirichlet's eta function, or Euler's eta function), which is defined by $\eta(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^z}$, $\Re(z)>0$, with closed-form expressions for $\eta(1)$, $\eta(2)$, $\eta(3)$ and $\eta(4)$ being given in [1, p. 811]:

$$\eta(1)=\ln 2,\quad \eta(2)=\frac{1}{2}\zeta(2),\quad \eta(3)=\frac{3}{4}\zeta(3)\quad \eta(4)=\frac{7}{8}\zeta(4).$$

To evaluate (1) we shall establish some lemmas.

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Lemma 1. *The following identity holds*

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n+2} = \frac{1}{2}\ln^2 2 - 2\ln 2 + 1.$$

Proof. In the proof of Lemma 3 from the recent article [5], Sofo has obtained the following relation for $r \geq 2$

$$-\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n+r} - \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n+r-1} = \frac{(1+(-1)^r)\ln 2}{r-1} - \frac{(-1)^{r+1}}{r-1}\left(H_{\lfloor \frac{r-1}{2} \rfloor} - H_{r-1}\right).$$

Letting $r = 2$ on both sides, we obtain that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n+2} - \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n+1} = 2\ln 2 - 1,$$

where we used that $H_0 = 0$. The alternating harmonic sum $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n H_n}{n+1}$ was evaluated in the same article [5, Lemma 3] to $-\frac{1}{2}\ln^2 2$, giving us the desired equality. □

Lemma 2. *The following identity is valid*

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(n+2)^2} = \frac{1}{8}\zeta(3) - 2\ln 2 - \frac{1}{2}\zeta(2) + 2.$$

Proof. In [6, Proof of Lemma 4] Sofo has obtained the following relation for $s \geq 2$

$$-\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(n+s)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(n+s-1)^2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+s-1)^2}. \tag{2}$$

Setting $s = 2$ on both sides of (2), we get that

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(n+2)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(n+1)^2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)^2}.$$

In Equation (1.9) from the same article [6,] the alternating harmonic sum $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n H_n}{(n+1)^2}$ was evaluated to $-\dfrac{1}{8}\zeta(3)$, giving us that the following equalities hold:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(n+2)^2} &= -\frac{1}{8}\zeta(3)-\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)^2}\\ &= -\frac{1}{8}\zeta(3)-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^2}\\ &= -\frac{1}{8}\zeta(3)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}+\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}-1+\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i^2}-1\\ &= -\frac{1}{8}\zeta(3)+2\eta(1)+\eta(2)-2\\ &= -\frac{1}{8}\zeta(3)+2\ln 2+\frac{1}{2}\zeta(2)-2. \qquad \square\end{aligned}$$

Lemma 3. *The following identity holds*

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(n+1)^3} = -2\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{15}{8}\zeta(4)-\frac{1}{12}\ln^4 2-\frac{7}{4}\zeta(3)\ln 2+\frac{1}{2}\zeta(2)\ln^2 2.$$

Proof. We have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(n+1)^3} &= \sum_{n=1}^{\infty}\frac{(-1)^n H_{n+1}}{(n+1)^3}-\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^4}\\ &= \sum_{j=1}^{\infty}\frac{(-1)^{j-1} H_j}{j^3}-\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j^4}\\ &= \sum_{j=1}^{\infty}\frac{(-1)^{j-1} H_j}{j^3}-\eta(4)=\sum_{j=1}^{\infty}\frac{(-1)^{j-1} H_j}{j^3}-\frac{7}{8}\zeta(4).\end{aligned}$$

In [2, p. 32], Flajolet and Salvy have listed the following alternating linear harmonic sum

$$\sum_{j=1}^{\infty}\frac{(-1)^{j-1} H_j}{j^3} = -2\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{11}{4}\zeta(4)-\frac{1}{12}\ln^4 2-\frac{7}{4}\zeta(3)\ln 2+\frac{1}{2}\zeta(2)\ln^2 2$$

from which it follows that

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(n+1)^3} = -2\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{15}{8}\zeta(4)-\frac{1}{12}\ln^4 2-\frac{7}{4}\zeta(3)\ln 2+\frac{1}{2}\zeta(2)\ln^2 2. \qquad \square$$

Lemma 4. *The following identity holds*

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{H_{n+2}}{n+2}\right)^2 = 2\operatorname{Li}_4\left(\frac{1}{2}\right)-\frac{41}{16}\zeta(4)+\frac{1}{12}\ln^4 2+\frac{7}{4}\zeta(3)\ln 2-\frac{1}{2}\zeta(2)\ln^2 2+\frac{7}{16}.$$

Proof. We have $\sum_{n=1}^{\infty}(-1)^n\left(\frac{H_{n+2}}{n+2}\right)^2=\sum_{q=1}^{\infty}(-1)^q\,\frac{H_q^2}{q^2}+\frac{7}{16}$.

In [8, p. 16], the nonlinear harmonic sum $\sum_{q=1}^{\infty}\frac{(-1)^q\,H_q^2}{q^2}$ was evaluated to $2\,\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{41}{16}\zeta(4)+\frac{1}{12}\ln^4 2+\frac{7}{4}\zeta(3)\ln 2-\frac{1}{2}\zeta(2)\ln^2 2$. Hence, the desired equality follows. ☐

Lemma 5. *The following identity holds*

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{(n+1)\,(n+2)}+\frac{1}{(n+2)^2}\right)^2=\zeta(2)+\frac{3}{2}\zeta(3)-\frac{7}{8}\zeta(4)-\frac{41}{16}.$$

Proof. Using partial fraction decomposition, it is found that the sum in the left-hand side can be rewritten successively as follows:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(-1)^n}{((n+1)\,(n+2))^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+2)^4}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)\,(n+2)^3}\\
=\;&\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+2)^2}-2\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+2)^3}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+2)^4}\\
=\;&\left(\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p^2}-1\right)+\left(\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p^2}-\frac{3}{4}\right)+2\left(\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p^3}-\frac{7}{8}\right)\\
&-\left(\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p^4}-\frac{15}{16}\right)\\
=\;&2\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p^2}+2\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p^3}-\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p^4}-\frac{41}{16}\\
=\;&2\eta\,(2)+2\eta\,(3)-\eta\,(4)-\frac{41}{16}=\zeta(2)+\frac{3}{2}\zeta(3)-\frac{7}{8}\zeta(4)-\frac{41}{16}.
\end{aligned}$$

☐

Lemma 6. *The following identity holds*

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^n\,H_{n+2}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)}{(n+2)^2}&=\zeta(2)-\ln^2 2+2\ln 2+\frac{5}{8}\zeta(3)+2\,\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{41}{16}\\
&\quad-\frac{11}{4}\zeta(4)+\frac{1}{12}\ln^4 2+\frac{7}{4}\zeta(3)\ln 2-\frac{1}{2}\zeta(2)\ln^2 2.
\end{aligned}$$

Proof. By partial fraction decomposition, the left-hand side can be brought to the following forms:

$$
\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^n H_{n+2} \left(\frac{1}{(n+1)(n+2)^2} + \frac{1}{(n+2)^3} \right) \\
= \quad & \sum_{n=1}^{\infty} (-1)^n H_{n+2} \left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+2)^2} + \frac{1}{(n+2)^3} \right) \\
= \quad & \sum_{n=1}^{\infty} (-1)^n H_{n+2} \left(\frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{(n+2)^2} + \frac{1}{(n+2)^3} \right) \\
= \quad & \sum_{n=1}^{\infty} \frac{(-1)^n H_{n+2}}{n+1} - \sum_{n=1}^{\infty} \frac{(-1)^n H_{n+2}}{n+2} - \sum_{n=1}^{\infty} \frac{(-1)^n H_{n+2}}{(n+2)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n H_{n+2}}{(n+2)^3} \\
= \quad & \sum_{n=1}^{\infty} \frac{(-1)^n H_{n+1}}{n+1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)(n+2)} - \sum_{n=1}^{\infty} \frac{(-1)^n H_{n+2}}{n+2} \\
& \qquad - \sum_{n=1}^{\infty} \frac{(-1)^n H_{n+2}}{(n+2)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n H_{n+2}}{(n+2)^3} \\
= \quad & - \left(\sum_{m=1}^{\infty} \frac{(-1)^m H_m}{m} + 1 \right) + 2 \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} - \frac{3}{2} - \left(\sum_{m=1}^{\infty} \frac{(-1)^m H_m}{m} + \frac{1}{4} \right) \\
& \qquad - \left(\sum_{m=1}^{\infty} \frac{(-1)^m H_m}{m^2} + \frac{5}{8} \right) + \sum_{m=1}^{\infty} \frac{(-1)^m H_m}{m^3} + \frac{13}{16} \\
= \quad & -2 \sum_{m=1}^{\infty} \frac{(-1)^m H_m}{m} + 2\eta(1) - \sum_{m=1}^{\infty} \frac{(-1)^m H_m}{m^2} + \sum_{m=1}^{\infty} \frac{(-1)^m H_m}{m^3} - \frac{41}{16} \\
= \quad & -2 \sum_{m=1}^{\infty} \frac{(-1)^m H_m}{m} + 2\ln 2 - \sum_{m=1}^{\infty} \frac{(-1)^m H_m}{m^2} + \sum_{m=1}^{\infty} \frac{(-1)^m H_m}{m^3} - \frac{41}{16}.
\end{aligned}
$$

In [5, Equation (1.9)] the alternating harmonic sum $\sum_{m=1}^{\infty} \frac{(-1)^m H_m}{m}$ was evaluated to $-\frac{1}{2}\zeta(2) + \frac{1}{2}\ln^2 2$ and in [6, Equation (1.9)] the harmonic sum $\sum_{m=1}^{\infty} \frac{(-1)^m H_m}{m^2}$ was evaluated to $-\frac{5}{8}\zeta(3)$. It follows, from the previous formulae, that $\sum_{n=1}^{\infty} \frac{(-1)^n H_{n+2} \left(\frac{1}{n+1} + \frac{1}{n+2} \right)}{(n+2)^2} = \zeta(2) - \ln^2 2 + 2\ln 2 + \frac{5}{8}\zeta(3) + 2\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{11}{4}\zeta(4) + \frac{1}{12}\ln^4 2 + \frac{7}{4}\zeta(3)\ln 2 - \frac{1}{2}\zeta(2)\ln^2 2 - \frac{41}{16}$. □

Lemma 7. *The following formula holds*

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{(n+2)^2} = -2\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{33}{16}\zeta(4)+\frac{1}{4}\zeta(3)-\zeta(2)-\frac{1}{12}\ln^4 2+\frac{1}{2}\zeta(2)\ln^2 2 - \frac{7}{4}\zeta(3)\ln 2+2\ln^2 2-4\ln 2+3.$$

Proof. The sum in the left-hand side can be rewritten as follows:

$$\sum_{n=1}^{\infty}\frac{(-1)^n\left(H_{n+2}-\left(\frac{1}{n+1}+\frac{1}{n+2}\right)\right)^2}{(n+2)^2}$$

$$=\sum_{n=1}^{\infty}(-1)^n\left(\frac{H_{n+2}}{n+2}\right)^2+\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)^2}\right)^2 -2\sum_{n=1}^{\infty}\frac{(-1)^n H_{n+2}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)}{(n+2)^2}$$

$$=2\operatorname{Li}_4\left(\frac{1}{2}\right)-\frac{41}{16}\zeta(4)+\frac{1}{12}\ln^4 2+\frac{7}{4}\zeta(3)\ln 2-\frac{1}{2}\zeta(2)\ln^2 2+\frac{7}{16}+\zeta(2)+\frac{3}{2}\zeta(3)-\frac{7}{8}\zeta(4)-\frac{41}{16}$$
$$-2\zeta(2)+2\ln^2 2-4\ln 2-\frac{5}{4}\zeta(3)-4\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{11}{2}\zeta(4)-\frac{1}{6}\ln^4 2-\frac{7}{2}\zeta(3)\ln 2+\zeta(2)\ln^2 2+\frac{41}{8}$$
$$=-2\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{33}{16}\zeta(4)+\frac{1}{4}\zeta(3)-\zeta(2)-\frac{1}{12}\ln^4 2+\frac{1}{2}\zeta(2)\ln^2 2-\frac{7}{4}\zeta(3)\ln 2+2\ln^2 2-4\ln 2+3.$$

□

Lemma 8. *The following identity holds*

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{(n+1)^2} = 2\operatorname{Li}_4\left(\frac{1}{2}\right)-\frac{33}{16}\zeta(4)+\frac{1}{12}\ln^4 2+\frac{7}{4}\zeta(3)\ln 2-\frac{1}{2}\zeta(2)\ln^2 2.$$

Proof. The desired sum is given as follows:

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_{n+1}^2}{(n+1)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^4}-2\sum_{n=1}^{\infty}\frac{(-1)^n H_{n+1}}{(n+1)^3}$$
$$=\sum_{l=1}^{\infty}\frac{(-1)^{l-1}H_l^2}{l^2}+\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l^4}-2\sum_{l=1}^{\infty}\frac{(-1)^{l-1}H_l}{l^3}$$
$$=\sum_{l=1}^{\infty}\frac{(-1)^{l-1}H_l^2}{l^2}+\eta(4)-2\sum_{l=1}^{\infty}\frac{(-1)^{l-1}H_l}{l^3}$$
$$=\sum_{l=1}^{\infty}\frac{(-1)^{l-1}H_l^2}{l^2}+\frac{7}{8}\zeta(4)-2\sum_{l=1}^{\infty}\frac{(-1)^{l-1}H_l}{l^3}$$

$$= -2\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{41}{16}\zeta(4)-\frac{1}{12}\ln^4 2-\frac{7}{4}\zeta(3)\ln 2+\frac{1}{2}\zeta(2)\ln^2 2+\frac{7}{8}\zeta\left(4\right)+4\operatorname{Li}_4\left(\frac{1}{2}\right)$$
$$-\frac{11}{2}\zeta(4)+\frac{1}{6}\ln^4 2+\frac{7}{2}\zeta(3)\ln 2-\zeta(2)\ln^2 2$$
$$= 2\operatorname{Li}_4\left(\frac{1}{2}\right)-\frac{33}{16}\zeta(4)+\frac{1}{12}\ln^4 2+\frac{7}{4}\zeta(3)\ln 2-\frac{1}{2}\zeta(2)\ln^2 2.$$

□

Lemma 9. *The following identity holds*

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n+2} = -\frac{1}{4}\zeta(3)-\frac{1}{3}\ln^3 2+\frac{1}{2}\zeta(2)\ln 2-\frac{1}{2}\zeta(2)+2\ln^2 2-2\ln 2+1.$$

Proof. In the recent article [5, Proof of Lemma 4] Sofo has obtained the following relation for $t \geq 2$

$$-\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n+t} = \sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n+t-1} - 2\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n\,(n+t-1)} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\,(n+t-1)}. \tag{3}$$

Plugging in $t = 2$ on both sides of (3) we get that the opposite of the desired sum is

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n+1} - 2\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n\,(n+1)} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\,(n+1)}$$
$$= \sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n+1} - 2\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n} + 2\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n+1}$$
$$-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} - 1$$
$$= \sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n+1} + \zeta(2) - 2\ln^2 2 - \eta\left(2\right) + 2\eta\left(1\right) - 1$$
$$= \sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n+1} + \frac{1}{2}\zeta(2) - 2\ln^2 2 + 2\ln 2 - 1.$$

In [3, p. 217] Mező has evaluated the following alternating nonlinear harmonic sum

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n+1} = \frac{1}{4}\zeta(3)+\frac{1}{3}\ln^3 2-\frac{1}{2}\zeta(2)\ln 2,$$

which combined to the above calculations show that the desired result holds and the lemma is proved. □

Lemma 10. *The following identity holds*

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n\,(n+1)^3\,(n+2)}+\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n\,(n+1)^2\,(n+2)^2}=2\operatorname{Li}_4\left(\frac{1}{2}\right)-\frac{15}{8}\zeta(4)+\frac{1}{12}\ln^4 2$$

$$+\frac{7}{4}\zeta(3)\ln 2-\frac{1}{2}\zeta(2)\ln^2 2+\frac{1}{16}\zeta\,(3)-\frac{1}{8}\zeta\,(2)+\frac{5}{2}\ln 2-\frac{7}{4}.$$

Proof. By partial fraction decomposition, the left-hand side is found to be

$$-\frac{3}{4}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n+2}+\frac{3}{4}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n}-\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(n+1)^2}$$

$$-\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(n+1)^3}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(n+2)^2}$$

$$=-\frac{3}{4}\left(\frac{1}{2}\ln^2 2-2\ln 2+1\right)-\frac{1}{2}\left(\frac{1}{8}\zeta\,(3)-2\ln 2-\frac{1}{2}\zeta\,(2)+2\right)+2\operatorname{Li}_4\left(\frac{1}{2}\right)$$

$$-\frac{15}{8}\zeta(4)+\frac{1}{12}\ln^4 2+\frac{7}{4}\zeta(3)\ln 2-\frac{1}{2}\zeta(2)\ln^2 2+\frac{1}{8}\zeta\,(3)+\frac{3}{4}\left(-\frac{1}{2}\zeta(2)+\frac{1}{2}\ln^2 2\right)$$

$$=2\operatorname{Li}_4\left(\frac{1}{2}\right)-\frac{15}{8}\zeta(4)+\frac{1}{12}\ln^4 2+\frac{7}{4}\zeta(3)\ln 2-\frac{1}{2}\zeta(2)\ln^2 2+\frac{1}{16}\zeta\,(3)+\frac{1}{4}\zeta\,(2)-\frac{3}{8}\ln^2 2$$

$$+\frac{5}{2}\ln 2-\frac{7}{4}+\frac{3}{4}\left(-\frac{1}{2}\zeta(2)+\frac{1}{2}\ln^2 2\right)$$

$$=2\operatorname{Li}_4\left(\frac{1}{2}\right)-\frac{15}{8}\zeta(4)+\frac{1}{12}\ln^4 2+\frac{7}{4}\zeta(3)\ln 2-\frac{1}{2}\zeta(2)\ln^2 2+\frac{1}{16}\zeta\,(3)-\frac{1}{8}\zeta\,(2)+\frac{5}{2}\ln 2-\frac{7}{4}.$$

□

Lemma 11. *The following identity holds*

$$2\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n\,(n+1)^2\,(n+2)}+\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n\,(n+1)\,(n+2)^2}=-5\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{165}{32}\zeta\,(4)-\zeta(3)$$

$$-\frac{3}{8}\zeta(2)-\frac{2}{3}\ln^3 2-\frac{5}{24}\ln^4 2+\frac{1}{2}\ln^2 2+\zeta(2)\ln 2-\frac{35}{8}\zeta(3)\ln 2+\frac{5}{4}\zeta(2)\ln^2 2-\frac{3}{2}\ln 2+\frac{5}{4}.$$

Proof. By partial fraction decomposition, we find that the left-hand side is

$$\frac{5}{4}\sum_{n=1}^{\infty}(-1)^n\frac{H_n^2}{n}-\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n+1}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n+2}$$

$$-2\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{(n+1)^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{(n+2)^2}$$

$$=\frac{5}{4}\sum_{n=1}^{\infty}(-1)^n\frac{H_n^2}{n}-\left(\frac{1}{4}\zeta(3)+\frac{1}{3}\ln^3 2-\frac{1}{2}\zeta(2)\ln 2\right)$$

$$
\begin{aligned}
&-\frac{1}{4}\left(-\frac{1}{4}\zeta(3)-\frac{1}{3}\ln^3 2+\frac{1}{2}\zeta(2)\ln 2-\frac{1}{2}\zeta(2)+2\ln^2 2-2\ln 2+1\right)-4\operatorname{Li}_4\left(\frac{1}{2}\right)\\
&-2\left(-\frac{33}{16}\zeta(4)+\frac{1}{12}\ln^4 2+\frac{7}{4}\zeta(3)\ln 2-\frac{1}{2}\zeta(2)\ln^2 2\right)-\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{33}{32}\zeta(4)\\
&+\frac{1}{2}\left(\frac{1}{4}\zeta(3)-\zeta(2)-\frac{1}{12}\ln^4 2+\frac{1}{2}\zeta(2)\ln^2 2-\frac{7}{4}\zeta(3)\ln 2+2\ln^2 2-4\ln 2+3\right)\\
&=\frac{5}{4}\sum_{n=1}^{\infty}(-1)^n\frac{H_n^2}{n}-5\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{165}{32}\zeta(4)-\frac{1}{16}\zeta(3)-\frac{3}{8}\zeta(2)-\frac{1}{4}\ln^3 2-\frac{5}{24}\ln^4 2\\
&\qquad+\frac{1}{2}\ln^2 2+\frac{3}{8}\zeta(2)\ln 2-\frac{35}{8}\zeta(3)\ln 2+\frac{5}{4}\zeta(2)\ln^2 2-\frac{3}{2}\ln 2+\frac{5}{4}.
\end{aligned}
$$

Since $\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n}=-\frac{3}{4}\zeta(3)-\frac{1}{3}\ln^3 2+\frac{1}{2}\zeta(2)\ln 2$ (see [5, Equation (1.10)]) we get that the left-hand side of the desired equality is equal to

$$
\begin{aligned}
&\frac{5}{4}\left(-\frac{3}{4}\zeta(3)-\frac{1}{3}\ln^3 2+\frac{1}{2}\zeta(2)\ln 2\right)-5\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{165}{32}\zeta(4)-\frac{1}{16}\zeta(3)-\frac{3}{8}\zeta(2)\\
&-\frac{1}{4}\ln^3 2-\frac{5}{24}\ln^4 2+\frac{1}{2}\ln^2 2+\frac{3}{8}\zeta(2)\ln 2-\frac{35}{8}\zeta(3)\ln 2+\frac{5}{4}\zeta(2)\ln^2 2-\frac{3}{2}\ln 2+\frac{5}{4}\\
&\quad=-5\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{165}{32}\zeta(4)-\zeta(3)-\frac{3}{8}\zeta(2)-\frac{2}{3}\ln^3 2-\frac{5}{24}\ln^4 2+\frac{1}{2}\ln^2 2\\
&\qquad+\zeta(2)\ln 2-\frac{35}{8}\zeta(3)\ln 2+\frac{5}{4}\zeta(2)\ln^2 2-\frac{3}{2}\ln 2+\frac{5}{4}.
\end{aligned}
$$

□

Now we are ready to state the main result of this paper.

Theorem 12. *The following identity holds*

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^n H_nH_{n+1}H_{n+2}}{n(n+1)(n+2)}&=-3\operatorname{Li}_4\left(\frac{1}{2}\right)+\frac{5}{2}\zeta(4)-\frac{15}{8}\zeta(3)-\zeta(2)+\frac{3}{8}\ln^4 2\\
&\quad-\frac{5}{3}\ln^3 2+2\ln^2 2-\frac{3}{8}\zeta(3)\ln 2-\frac{3}{4}\zeta(2)\ln^2 2+\frac{5}{2}\zeta(2)\ln 2.
\end{aligned}
$$

Proof. The left-hand side is rewritten successively as follows:

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(-1)^n\left(H_n^2+\frac{H_n}{n+1}\right)\left(H_n+\frac{1}{n+1}+\frac{1}{n+2}\right)}{n(n+1)(n+2)}\\
&=\sum_{n=1}^{\infty}\frac{(-1)^n H_n^3}{n(n+1)(n+2)}+2\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n(n+1)^2(n+2)}+\sum_{n=1}^{\infty}\frac{(-1)^n H_n^2}{n(n+1)(n+2)^2}\\
&\qquad+\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n(n+1)^3(n+2)}+\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n(n+1)^2(n+2)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{(-1)^n H_n^3}{n\,(n+1)\,(n+2)} - 5\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{165}{32}\zeta\,(4) - \zeta(3) - \frac{3}{8}\zeta(2) - \frac{2}{3}\ln^3 2 \\
&-\frac{5}{24}\ln^4 2+\frac{1}{2}\ln^2 2+\zeta(2)\ln 2-\frac{35}{8}\zeta(3)\ln 2+\frac{5}{4}\zeta(2)\ln^2 2+\frac{5}{4}-\frac{3}{2}\ln 2+2\,\mathrm{Li}_4\left(\frac{1}{2}\right) \\
&-\frac{15}{8}\zeta(4) + \frac{1}{12}\ln^4 2 + \frac{7}{4}\zeta(3)\ln 2 - \frac{1}{2}\zeta(2)\ln^2 2 + \frac{1}{16}\zeta\,(3) - \frac{1}{8}\zeta\,(2) + \frac{5}{2}\ln 2 - \frac{7}{4} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n H_n^3}{n\,(n+1)\,(n+2)} - 3\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{105}{32}\zeta\,(4) - \frac{15}{16}\zeta(3) - \frac{1}{2}\zeta(2) - \frac{1}{8}\ln^4 2 - \frac{2}{3}\ln^3 2 \\
&\qquad + \frac{1}{2}\ln^2 2 + \ln 2 - \frac{21}{8}\zeta(3)\ln 2 + \frac{3}{4}\zeta(2)\ln^2 2 + \zeta(2)\ln 2 - \frac{1}{2}.
\end{aligned}
$$

In [7, Remark 2.1, p. 12], Sofo has evaluated the following alternating nonlinear harmonic sum

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n H_n^3}{n\,(n+1)\,(n+2)} &= -\frac{25}{32}\zeta(4) - \frac{15}{16}\zeta(3) - \frac{1}{2}\zeta(2) + \frac{1}{2}\ln^4 2 - \ln^3 2 + \frac{3}{2}\ln^2 2 \\
&\qquad - \ln 2 + \frac{9}{4}\zeta(3)\ln 2 - \frac{3}{2}\zeta(2)\ln^2 2 + \frac{3}{2}\zeta(2)\ln 2 + \frac{1}{2}.
\end{aligned}
$$

It follows that the desired sum $\sum_{n=1}^{\infty} \frac{(-1)^n H_n H_{n+1} H_{n+2}}{n\,(n+1)\,(n+2)}$ equals

$$
\begin{aligned}
&-\frac{25}{32}\zeta(4) - \frac{15}{16}\zeta(3) - \frac{1}{2}\zeta(2) + \frac{1}{2}\ln^4 2 - \ln^3 2 + \frac{3}{2}\ln^2 2 - \ln 2 + \frac{9}{4}\zeta(3)\ln 2 \\
&-\frac{3}{2}\zeta(2)\ln^2 2 + \frac{3}{2}\zeta(2)\ln 2 + \frac{1}{2} - 3\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{105}{32}\zeta\,(4) - \frac{15}{16}\zeta(3) - \frac{1}{2}\zeta(2) \\
&-\frac{1}{8}\ln^4 2 - \frac{2}{3}\ln^3 2 + \frac{1}{2}\ln^2 2 + \ln 2 - \frac{21}{8}\zeta(3)\ln 2 + \frac{3}{4}\zeta(2)\ln^2 2 + \zeta(2)\ln 2 - \frac{1}{2} \\
&= -3\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{5}{2}\zeta\,(4) - \frac{15}{8}\zeta(3) - \zeta(2) + \frac{3}{8}\ln^4 2 - \frac{5}{3}\ln^3 2 + 2\ln^2 2 - \frac{3}{8}\zeta(3)\ln 2 \\
&\qquad - \frac{3}{8}\zeta(3)\ln 2 - \frac{3}{4}\zeta(2)\ln^2 2 + \frac{5}{2}\zeta(2)\ln 2,
\end{aligned}
$$

and the theorem is proved. □

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18th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2024

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Abstract. The 18th South Eastern European Mathematical Olympiad for University Students (SEEMOUS 2024) took place on April 9-14, 2024, in Iași, Romania. We present the competition problems and their solutions, as given by the authors. We also include alternative solutions provided by members of the jury or by contestants.

Keywords: 15A21, 40A05, 26A24, 26A42.

MSC:

The 18th South Eastern European Mathematical Competition for University Students with International Participation (SEEMOUS 2024) was hosted between 9th and 14th of April in Iași by the Department of Mathematics and Informatics of Gheorghe Asachi Technical University, with the support of the Mathematical Society of South-Eastern Europe (MASSEE) and of the Romanian Mathematical Society. This competition is addressed to students in the first or second year of undergraduate studies, from universities in countries that are members of the MASSEE, or from invited countries that are not affiliated to MASSEE.

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A number of 79 students participated in the contest, representing 24 universities from France, Greece, North Macedonia, Romania, and Turkmenistan. The jury awarded 9 gold medals, 19 silver medals and 27 bronze medals. No contestant obtained the maximum possible score. The student Horia Mercan from National University of Science and Technology Politehnica Bucharest, Romania, obtained the highest score of the contest, and won the title of Absolute Winner of the competition. University of Bucharest, Romania, won the title of Best University.

We present the competition problems and their solutions as given by the corresponding authors, together with alternative solutions provided by members of the jury or by the contestants.

Problem 1. Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_{n+1} = x_n - \frac{x_n^2}{\sqrt{n}}$ for all $n \geq 1$, and $x_1 \in (0,1)$. Find the values of $\alpha \in \mathbb{R}$ for which the series $\sum\limits_{n=1}^{\infty} x_n^{\alpha}$ is convergent.

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Author's solution. By induction, we can easily deduce that $x_n \in (0,1)$ for all $n \geq 1$. Next, from $0 < \dfrac{x_n}{\sqrt{n}} < \dfrac{1}{\sqrt{n}}$ for all $n \geq 1$, it follows that $\lim\limits_{n\to\infty} \dfrac{x_n}{\sqrt{n}} = 0$. Since $1 - \dfrac{x_{n+1}}{x_n} = \dfrac{x_n - x_{n+1}}{x_n} = \dfrac{x_n}{\sqrt{n}}$ for all $n \geq 1$, we deduce that $\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = 1$.

Now let $n \geq 1$. By the recurrence relation we have

$$\frac{1}{x_{n+1}} - \frac{1}{x_n} = \frac{x_n - x_{n+1}}{x_n x_{n+1}} = \frac{x_n}{x_{n+1}} \cdot \frac{1}{\sqrt{n}},$$

which implies that

$$\lim_{n\to\infty} \frac{\frac{1}{x_{n+1}} - \frac{1}{x_n}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = 1.$$

Since $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n-1}}\right) = \infty$, it follows by the Stolz-Cesàro lemma that

$$\lim_{n\to\infty} \frac{\frac{1}{x_n}}{1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n-1}}} = 1.$$

Also, by the same lemma,

$$\lim_{n\to\infty} \frac{\sqrt{n}}{1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n-1}}} = \lim_{n\to\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{2}.$$

Combining the previous two limits, we obtain $\lim_{n\to\infty} \frac{\frac{1}{x_n}}{\sqrt{n}} = 2$, hence $\lim_{n\to\infty} \frac{x_n^\alpha}{\frac{1}{n^{\frac{\alpha}{2}}}} = 2^{-\alpha}$. By the comparison criterion for positive series it follows that $\sum_{n=1}^{\infty} x_n^\alpha$ is convergent if and only if $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{\alpha}{2}}}$ is convergent, that is, if and only if $\frac{\alpha}{2} > 1$, which finally leads to $\alpha > 2$.

Alternative solution. This follows the ideas from a solution given by the jury. Deduce, as above, that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$, hence $\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = 1$, so there exists $n_1 \in \mathbb{N}$ such that $\frac{1}{2} < \frac{x_n}{x_{n+1}} < 2$ for all $n \geq n_1$. Next, using $\frac{1}{x_{n+1}} - \frac{1}{x_n} = \frac{x_n}{x_{n+1}} \cdot \frac{1}{\sqrt{n}}$, we obtain that for all $n \geq n_1$,

$$\frac{1}{2\sqrt{n}} < \frac{1}{x_{n+1}} - \frac{1}{x_n} < \frac{2}{\sqrt{n}}. \tag{1}$$

Using also the fact that the sequence $\left(1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n}\right)$ converges, one can find some constants $c_1, c_2 \in \mathbb{R}$ and $n_2 \geq n_1$ such that, for all $n \geq n_2$,

$$c_1 + 2\sqrt{n} \leq \frac{1}{\sqrt{n_2}} + \cdots + \frac{1}{\sqrt{n}} \leq c_2 + 2\sqrt{n}. \tag{2}$$

Taking the sum for $k = n_2, \ldots, n$ in relation (1), and using (2), we get for all $n \geq n_2$ that

$$\frac{c_1}{2} + \sqrt{n} \leq \frac{1}{2}\sum_{k=n_2}^{n} \frac{1}{\sqrt{k}} < \frac{1}{x_{n+1}} - \frac{1}{x_{n_2}} < 2\sum_{k=n_2}^{n} \frac{1}{\sqrt{k}} \leq 2c_2 + 4\sqrt{n}.$$

Dividing the previous relation by $\sqrt{n+1}$, since $\lim_{n\to\infty} \left(\frac{c_1}{2} + \sqrt{n}\right) \cdot \frac{1}{\sqrt{n+1}} = 1$, $\lim_{n\to\infty} \frac{1}{x_{n_2} \cdot \sqrt{n+1}} = 0$, and $\lim_{n\to\infty} (2c_2 + 4\sqrt{n}) \cdot \frac{1}{\sqrt{n+1}} = 4$, it follows that there exist some positive constants $k_1, k_2 > 0$ and $n_3 \geq n_2$ such that, for all $n \geq n_3$,

$$k_1 \leq \frac{1}{x_{n+1} \cdot \sqrt{n+1}} \leq k_2,$$

hence the sequence $\left(\frac{x_n}{\frac{1}{\sqrt{n}}}\right)_{n\geq 1}$ is bounded from above and from below by positive numbers. This is enough to guarantee that the series $\sum\limits_{n=1}^{\infty} x_n^{\alpha}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{\alpha}{2}}}$ have the same nature, so the conclusion follows as above.

Remark. The author's solution proves that $x_n \sim \frac{1}{2\sqrt{n}}$ (i.e., $\lim\limits_{n\to\infty} \frac{x_n}{\frac{1}{2\sqrt{n}}} = 1$), while the alternative solution limits the argument to showing that $(x_n)_{n\geq 1}$ can be squeezed between two sequences $\left(\frac{a}{\sqrt{n}}\right)_{n\geq 1}$ and $\left(\frac{b}{\sqrt{n}}\right)_{n\geq 1}$, for some $a, b > 0$.

An (incomplete) argument which shows that $x_n \sim \frac{1}{2\sqrt{n}}$ can be given by assuming (without proof) that $x_n \sim \frac{c}{n^k}$ for some positive numbers k and c. Then, by the recurrence relation, we have that $x_n - x_{n+1} = \frac{x_n^2}{\sqrt{n}} \sim \frac{c^2}{n^{2k+\frac{1}{2}}}$, while $x_n - x_{n+1} \sim c\left(\frac{1}{n^k} - \frac{1}{(n+1)^k}\right) = \frac{c}{n^k}\left(1 - \left(1 - \frac{1}{n+1}\right)^k\right) \sim \frac{c}{n^k} \cdot \frac{k}{n+1} \sim \frac{ck}{n^{k+1}}$. It follows that $2k + \frac{1}{2} = k+1$ and $c^2 = ck$, hence $c = k = \frac{1}{2}$.

Although this problem was considered to be easy by the jury, only 13 *contestants solved it completely. The ideas of the contestants mainly followed the author's solution.*

Problem 2. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ two real, symmetric matrices with nonnegative eigenvalues. Prove that $A^3 + B^3 = (A+B)^3$ if and only if $AB = O_n$.

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Author's solution. If $AB = O_n$, then

$$AB = O_n = (AB)^T = B^T A^T = BA.$$

Therefore A and B commute and

$$(A+B)^3 = A^3 + B^3 + 3AB(A+B) = A^3 + B^3.$$

Assume now that $A^3 + B^3 = (A+B)^3$. Since the trace operator is linear and invariant under cyclic permutations, it follows that

$$\mathrm{Tr}(ABA) + \mathrm{Tr}(BAB) = 0. \tag{3}$$

We recall that a real, symmetric matrix M has nonnegative eigenvalues $\lambda_1, \ldots, \lambda_n$, i.e., M is positive semidefinite, if and only if M can be decomposed as a product $M = Q^T Q$ for some real matrix Q. Moreover, if for such a matrix $\operatorname{Tr} M = 0$, then $M = O_n$. Let $U, V \in \mathcal{M}_n(\mathbb{R})$ such that $A = U^T U$ and $B = V^T V$. Then, using the symmetry of A and B we get

$$ABA = AV^T VA = (VA)^T (VA) \quad \text{and} \quad BAB = BU^T UB = (UB)^T (UB),$$

so $\operatorname{Tr}(ABA) \geq 0$ and $\operatorname{Tr}(BAB) \geq 0$. From (3) it follows that we must have $\operatorname{Tr}(ABA) = \operatorname{Tr}(BAB) = 0$ and therefore $ABA = BAB = O_n$.

In particular, for every $x \in \mathbb{R}^n$ we have

$$\|VAx\|^2 = x^T (VA)^T (VA) x = x^T ABAx = 0,$$

so $VA = O_n$. Again, for every $x \in \mathbb{R}^n$

$$\|ABx\|^2 = x^T (AB)^T (AB) x = x^T V^T (VA) ABx = 0$$

and, finally, we find $AB = O_n$.

Alternative solution. This is based on the solution given by Marian Panțiruc. The matrix A is a real, positive semi-definite matrix, so $\operatorname{Tr} A \geq 0$ and $\operatorname{Tr} A = 0$ if and only if $A = O_n$. Denoting the usual scalar product over $\mathbb{R}^n$ by

$$\langle x, y \rangle = x^T y = y^T x, \quad \text{for all } x, y \in \mathbb{R}^n,$$

we have $\langle Ax, x \rangle \geq 0$ for every $x \in \mathbb{R}^n$ and $\langle Ax, x \rangle = 0$ if and only if $x \in \operatorname{Ker} A$. The same goes for B.

We observe that BAB is symmetric and

$$\langle BABx, x \rangle = \langle ABx, Bx \rangle \geq 0, \quad \text{for all } x \in \mathbb{R}^n,$$

which implies that BAB is also a positive semi-definite matrix. Similarly, ABA is symmetric and positive semi-definite.

Then, if $(A + B)^3 = A^3 + B^3$, as in the author's solution we obtain $\operatorname{Tr}(ABA) = \operatorname{Tr}(BAB) = 0$, so $BAB = O_n$. Because

$$\langle BABx, x \rangle = \langle ABx, Bx \rangle = 0, \quad \text{for all } x \in \mathbb{R}^n,$$

we conclude that $Bx \in \operatorname{Ker} A$, for every $x \in \mathbb{R}^n$, i.e., $AB = O_n$.

This problem had 13 complete solutions given by the contestants and generated the greatest total number of points in the competition.

Problem 3. For every $n \geq 1$ define x_n by

$$x_n = \int_0^1 \ln(1 + x + x^2 + \cdots + x^n) \cdot \ln \frac{1}{1 - x} \, \mathrm{d}x.$$

(a) Show that x_n is finite for every $n \geq 1$ and $\lim\limits_{n \to \infty} x_n = 2$.

(b) Calculate $\lim\limits_{n \to \infty} \dfrac{n}{\ln n} (2 - x_n)$.

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Author's solution. (a) For all $n \geq 1$ and $x \in [0,1)$,

$$\frac{1}{1-x} \geq 1 \quad \text{and} \quad 0 \leq \ln(1+x+x^2+\cdots+x^n)\cdot\ln\frac{1}{1-x} \leq \ln n \cdot \ln\frac{1}{1-x}.$$

Since $\displaystyle\int_0^1 \ln\frac{1}{1-x}\,\mathrm{d}x$ is convergent (to 1, by a direct computation), it follows that x_n is finite.

Next, the sequence of functions $f_n(x) = \ln(1+x+x^2+\cdots+x^n)\cdot\ln\dfrac{1}{1-x}$ satisfies:

$$0 \leq f_n(x) \leq f_{n+1}(x), \text{ for all } x \in [0,1) \text{ and } n \geq 1,$$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(\ln\frac{1-x^{n+1}}{1-x}\cdot\ln\frac{1}{1-x}\right) = \ln^2\frac{1}{1-x}, \text{ for all } x \in [0,1).$$

It follows by *the Lebesgue–Beppo–Levi theorem* (of *monotone convergence*) that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\int_0^1 f_n(x)\,\mathrm{d}x = \int_0^1 \ln^2\frac{1}{1-x}\,\mathrm{d}x = 2$$

(the last equality follows by an elementary computation).

(b) From (a),

$$2-x_n = \int_0^1\left(\ln^2\frac{1}{1-x} - \ln\frac{1-x^{n+1}}{1-x}\cdot\ln\frac{1}{1-x}\right)\mathrm{d}x = \int_0^1 \ln(1-x^{n+1})\cdot\ln(1-x)\,\mathrm{d}x$$

and with the change of variable $y = x^{n+1}$, we obtain that

$$2 - x_n = \frac{1}{n+1}\int_0^1 \ln(1-y)\cdot\ln\left(1-y^{\frac{1}{n+1}}\right)\cdot y^{\frac{1}{n+1}-1}\,\mathrm{d}y.$$

By shifting the index, for convenience, it follows that

$$\begin{aligned}\lim_{n\to\infty}\frac{n}{\ln n}(2-x_n) &= \lim_{n\to\infty}\frac{n-1}{\ln(n-1)}(2-x_{n-1}) = \lim_{n\to\infty}\frac{n-1}{n}\cdot\lim_{n\to\infty}\frac{\ln n}{\ln(n-1)}\\ &\quad\cdot\lim_{n\to\infty}\frac{1}{\ln n}\int_0^1 \ln(1-y)\cdot\ln\left(1-y^{\frac{1}{n}}\right)\cdot y^{\frac{1}{n}-1}\,\mathrm{d}y\\ &= \lim_{n\to\infty}\int_0^1 \frac{\ln(1-y)}{y}\cdot\frac{y^{\frac{1}{n}}\ln(1-y^{\frac{1}{n}})}{\ln n}\,\mathrm{d}y.\end{aligned}$$

We want to verify the conditions in the *Lebesgue dominated convergence theorem*, so consider

$$g_n(y) = \frac{\ln(1-y)}{y}\cdot\frac{y^{\frac{1}{n}}\ln(1-y^{\frac{1}{n}})}{\ln n}, \text{ for } y \in (0,1), \text{ and } n \geq 2.$$

The pointwise convergence follows in a standard manner: we start from

$$\lim_{n\to\infty}\frac{y^{\frac{1}{n}}-1}{\frac{1}{n}} = \ln y, \text{ hence } \lim_{n\to\infty} n\left(1-y^{\frac{1}{n}}\right) = \ln\frac{1}{y} > 0,$$

which leads to

$$\lim_{n\to\infty}\left(\ln\left(1-y^{\frac{1}{n}}\right)+\ln n\right)=\ln\left(\ln\frac{1}{y}\right).$$

Then

$$\begin{aligned}\lim_{n\to\infty} g_n(y) &= \frac{\ln(1-y)}{y}\cdot\lim_{n\to\infty} y^{\frac{1}{n}}\cdot\lim_{n\to\infty}\frac{\ln\left(1-y^{\frac{1}{n}}\right)}{\ln n}\\ &= \frac{\ln(1-y)}{y}\cdot\lim_{n\to\infty}\left(\frac{\ln\left(1-y^{\frac{1}{n}}\right)+\ln n}{\ln n}-1\right)\\ &= \frac{\ln(1-y)}{y}\left(\ln\left(\ln\frac{1}{y}\right)\cdot\frac{1}{\infty}-1\right)\\ &= -\frac{\ln(1-y)}{y},\ \text{for all } y\in(0,1).\end{aligned}$$

To check the domination condition, let $g(t)=-\ln(1-t)=\ln\frac{1}{1-t}$, for $t\in[0,1)$. Note that g is positive. Since $0\le y^{\frac{1}{n}}\le 1$, it follows that

$$0\le g_n(y)\le\frac{\ln(1-y)}{y}\cdot\frac{\ln\left(1-y^{\frac{1}{n}}\right)}{\ln n}=\frac{g(y)}{y}\cdot\frac{g\left(y^{\frac{1}{n}}\right)}{\ln n},\ \text{for all } n\ge 2,\ y\in(0,1). \tag{4}$$

From

$$g(t)-g(t^n)=\ln\frac{1-t^n}{1-t}=\ln(1+t+\dots+t^{n-1})\le\ln n,\ \text{for all } t\in(0,1),\ n\ge 1,$$

it follows that $g\left(y^{\frac{1}{n}}\right)-g(y)\le\ln n$, hence

$$\frac{g\left(y^{\frac{1}{n}}\right)}{\ln n}\le 1+\frac{g(y)}{\ln n}\le 1+g(y),\ \text{for all } n\ge 3. \tag{5}$$

Combining (4) and (5) and replacing g, we finally obtain

$$0\le g_n(y)\le\frac{\ln^2(1-y)-\ln(1-y)}{y},\ \text{for all } n\ge 3,\ y\in(0,1).$$

It is an elementary exercise to check that $\int_0^1\frac{\ln^2(1-y)-\ln(1-y)}{y}\,\mathrm{d}y$ is convergent, which concludes the proof of the domination condition and establishes that

$$L=\lim_{n\to\infty}\frac{n}{\ln n}(2-x_n)=-\int_0^1\frac{\ln(1-y)}{y}\,\mathrm{d}y=\frac{\pi^2}{6},$$

where the last equality is a well-known result, that can be obtained by integrating the Maclaurin series of $-\dfrac{\ln(1-y)}{y}$ and then using Euler's identity $\sum\limits_{n\geq 1}\dfrac{1}{n^2}=\dfrac{\pi^2}{6}$.

Alternative solution. This solution, given by Mircea Rus, uses a different approach to the computation of the limit $\lim\limits_{n\to\infty}\dfrac{n}{\ln n}\displaystyle\int_0^1 \ln(1-x^n)\cdot\ln(1-x)\,\mathrm{d}x$, which is the answer to (b), as seen from the previous solution. This approach was suggested in an incomplete solution by the contestant Adrian–Nicolae Ariton, from National University of Science and Technology Politehnica Bucharest, Romania, whose intuition led to the correct result. We try here to fill in the gaps in the contestant's solution with some alternative arguments and simplifications.

For every $n\geq 2$, denote $a_n=\dfrac{n}{\ln n}\displaystyle\int_0^1 \ln(1-x^n)\cdot\ln(1-x)\,\mathrm{d}x$. Now, fix $n\geq 2$. We have $\ln(1-x^n)\cdot\ln(1-x)=\displaystyle\sum_{k=1}^{\infty}\frac{x^{kn}}{k}\ln\frac{1}{1-x}$ for all $x\in[0,1)$ from the Maclaurin power series expansion of $\ln(1-x^n)$. Using the monotone convergence theorem for the sequence of partial sums of the above series, we obtain that

$$a_n=\frac{n}{\ln n}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^1 x^{nk}\ln\frac{1}{1-x}\,\mathrm{d}x\right).$$

It follows, by an elementary computation, that for every $m\in\mathbb{N}$

$$\begin{aligned}\int_0^1 x^m\ln\frac{1}{1-x}\,\mathrm{d}x&=\frac{1}{m+1}\int_0^1\left(x^{m+1}-1\right)'\cdot\ln\frac{1}{1-x}\,\mathrm{d}x\\&=\frac{1}{m+1}\left(x^{m+1}-1\right)\ln\frac{1}{1-x}\bigg|_0^{1-0}+\frac{1}{m+1}\int_0^1\frac{x^{m+1}-1}{x-1}\,\mathrm{d}x\\&=\frac{1}{m+1}\int_0^1(1+x+\cdots+x^m)\,\mathrm{d}x=\frac{H_{m+1}}{m+1},\end{aligned}$$

where $H_{m+1}=1+\dfrac{1}{2}+\cdots+\dfrac{1}{m+1}$. It is easy to show that the sequence defined by $c_m=H_{m+1}-\ln m$ is decreasing and with positive values (it is convergent to the Euler–Mascheroni constant γ). This leads to

$$a_n=\frac{n}{\ln n}\sum_{k=1}^{\infty}\frac{1}{k}\cdot\frac{H_{nk+1}}{nk+1}=\sum_{k=1}^{\infty}\frac{1}{k^2}\cdot\frac{nk}{nk+1}\cdot\frac{c_{nk}+\ln nk}{\ln n}.$$

Using Euler's identity $\sum\limits_{k\geq 1}\dfrac{1}{k^2}=\dfrac{\pi^2}{6}$, we claim that $\lim\limits_{n\to\infty}a_n=\dfrac{\pi^2}{6}$.

Indeed,

$$\begin{aligned}\left|a_n - \frac{\pi^2}{6}\right| &= \left|\sum_{k=1}^{\infty} \frac{1}{k^2}\left(\frac{nk}{nk+1}\cdot\frac{c_{nk}+\ln nk}{\ln n} - 1\right)\right| \\ &\le \sum_{k=1}^{\infty}\frac{1}{k^2}\left|\frac{nk}{nk+1}\cdot\frac{c_{nk}+\ln nk}{\ln n} - 1\right|\end{aligned}$$

and

$$\begin{aligned}\left|\frac{nk}{nk+1}\cdot\frac{c_{nk}+\ln nk}{\ln n} - 1\right| &\le \frac{nk}{nk+1}\left|\frac{c_{nk}+\ln nk}{\ln n} - 1\right| + \left|\frac{nk}{nk+1} - 1\right| \\ &\le \frac{c_{nk}+\ln k}{\ln n} + \frac{1}{nk+1} \le \frac{c_2+\ln k}{\ln n} + \frac{1}{nk},\end{aligned}$$

for all $k \ge 1$, so

$$\left|a_n - \frac{\pi^2}{6}\right| \le \sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{c_2+\ln k}{\ln n} + \frac{1}{nk}\right) = \frac{1}{\ln n}\left(c_2\sum_{k=1}^{\infty}\frac{1}{k^2} + \sum_{k=1}^{\infty}\frac{\ln k}{k^2}\right) + \frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k^3}.$$

Because the three series in the last expression are all convergent, we can conclude that $\left|a_n - \frac{\pi^2}{6}\right| \to 0$ (as $n \to \infty$), so the claim is proven.

This problem proved to be the most difficult of the contest, a maximum of 7 *from* 10 *possible points being obtained by only* 2 *contestants.*

Problem 4. Let $n \in \mathbb{N}$, $n \ge 2$. Find all the values $k \in \mathbb{N}$, $k \ge 1$, for which the following statement holds:

$$\text{“If } A \in \mathcal{M}_n(\mathbb{C}) \text{ is such that } A^kA^* = A, \text{ then } A = A^*.\text{”} \qquad (*)$$

(Here, $A^* = \overline{A}^t$ denotes the transpose conjugate of A.)

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Authors' solution. First, we limit the range of the possible values for k, by choosing $A = \varepsilon I_n$, with suitable $\varepsilon \in \mathbb{C}$, $|\varepsilon| = 1$, such that the implication in (*) is false, so we ask that $A^kA^* = A$, but $A \neq A^*$. Then $\varepsilon I_n = A = A^kA^* = \varepsilon^k\overline{\varepsilon}I_n = \varepsilon^{k-1}I_n$ and $\varepsilon I_n = A \neq A^* = \overline{\varepsilon}I_n$, which are equivalent to $\varepsilon^{k-2} = 1$ and $\varepsilon \notin \mathbb{R}$. Consequently, if $k = 2$, then let $\varepsilon = i$ and if $k \ge 5$, we can take $\varepsilon = \cos\frac{2\pi}{k-2} + i\sin\frac{2\pi}{k-2} \notin \mathbb{R}$ (since $\frac{2\pi}{k-2} \in (0,\pi)$).

This means that $k \in \{1,3,4\}$. We prove next that the statement (*) is true for these values of k.

For $k = 1$, if $AA^* = A$, then $A^* = (AA^*)^* = (A^*)^*A^* = AA^* = A$, so (*) is true.

For $k \in \{3,4\}$, we provide two methods.

First method. $A^kA^* = A$ implies that $\operatorname{rank} A = \operatorname{rank}\left(A^kA^*\right) \le \operatorname{rank} A^k \le \operatorname{rank} A$, so $\operatorname{rank} A^k = \operatorname{rank} A = \operatorname{rank} A^*$. By the rank–nullity theorem, it follows that $\dim \operatorname{Ker} A^k = \dim \operatorname{Ker} A = \dim \operatorname{Ker} A^*$. Since $\operatorname{Ker} A^* \subseteq \operatorname{Ker} A$ (by $A^kA^* = A$) and $\operatorname{Ker} A \subseteq \operatorname{Ker} A^k$, we obtain

$$\operatorname{Ker} A^* = \operatorname{Ker} A^k = \operatorname{Ker} A. \tag{6}$$

Next, $A^kA^*A^{k-1} = AA^{k-1} = A^k$, so $A^k\left(A^*A^{k-1} - I_n\right) = O_n$, then we deduce that $A^*\left(A^*A^{k-1} - I_n\right) = O_n$, by (6), hence

$$\left(A^*\right)^2 A^{k-1} = A^*. \tag{7}$$

For $k = 3$, (7) becomes $\left(A^*\right)^2 A^2 = A^*$, so we have $A = \left(\left(A^*\right)^2 A^2\right)^* = \left(A^*\right)^2 A^2 = A^*$, which means that the statement (*) is true.

For $k = 4$, (7) becomes $\left(A^*\right)^2 A^3 = A^*$, so $\left(A^*\right)^2 A^4A^* = \left(A^*\right)^2 A^3 \cdot AA^* = A^*AA^*$. At the same time, $\left(A^*\right)^2 A^4A^* = \left(A^*\right)^2 A$, so $\left(A^*\right)^2 A = A^*AA^*$, which leads to $\left(A^*\right)^2 A^2 = (A^*A)^2$. With $B = A^*A - AA^*$, we have $B^* = B$ and

$$\operatorname{Tr} BB^* = \operatorname{Tr} B^2 = \operatorname{Tr}\left(A^*A - AA^*\right)^2 = 2\left(\operatorname{Tr}\left(A^*A\right)^2 - \operatorname{Tr}\left(\left(A^*\right)^2 A^2\right)\right) = 0,$$

hence $B = O_n$. This proves that $A^*A = AA^*$ (i.e., A is normal), so A is unitarily diagonalizable, $A = U^*DU$, $D = \operatorname{diag}\left(\lambda_1, \lambda_2, \ldots, \lambda_n\right)$ with $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$, $U \in \mathcal{M}_n(\mathbb{C})$ with $U^{-1} = U^*$. Then $A^* = U^*\overline{D}U$, and $A^4A^* = A$ becomes $D^4\overline{D} = D$, which means that $\lambda_i^4\overline{\lambda_i} = \lambda_i$, for all $i = 1, 2, \ldots, n$. It follows that $\lambda_i \in \{-1, 0, 1\}$, for all $i = 1, 2, \ldots, n$, so $\overline{D} = D$, therefore $A^* = A$, which means that the statement (*) is true.

Second method. We continue from relation (6) (from the first method). It is true in general, for any matrix $A \in \mathcal{M}_n(\mathbb{C})$, that $\operatorname{Ker} A^* \perp \operatorname{Im} A$. (Indeed, if $Y \in \operatorname{Ker} A^*$ and $Z = AX \in \operatorname{Im} A$, then $\langle Z, Y\rangle = \langle AX, Y\rangle = \langle X, A^*Y\rangle = \langle X, O\rangle = 0$.) Then, by (1), it follows that $\operatorname{Ker} A \perp \operatorname{Im} A$, so $\mathbb{C}^n = \operatorname{Ker} A \oplus \operatorname{Im} A$.

Consider an orthonormal basis in $\operatorname{Ker} A$ and an orthonormal basis in $\operatorname{Im} A$, which together give an orthonormal basis in $\mathbb{C}^n$, such that $A = U^*A_1U$, where $A_1 = \begin{bmatrix} B & O \\ O & O \end{bmatrix}$ with $B \in \mathcal{M}_m(\mathbb{C})$ invertible, and $U \in \mathcal{M}_n(\mathbb{C})$ with $U^{-1} = U^*$. Then the relation $A^kA^* = A$ becomes $B^kB^* = B$, hence $B^* = \left(B^{-1}\right)^{k-1}$. From the Cayley-Hamilton theorem, it follows that $B^{-1} = f(B)$ for some polynomial f of degree at most $n-1$, so $B^* = (f(B))^{k-1}$, which leads to $B^*B = BB^*$ (that is, B is normal). Just like in the previous approach, B is unitarily diagonalizable, $B = V^*DV$, $D = \operatorname{diag}\left(\lambda_1, \lambda_2, \ldots, \lambda_m\right)$ with $\lambda_1, \lambda_2, \ldots, \lambda_m \neq 0$, $V \in \mathcal{M}_m(\mathbb{C})$ with $V^{-1} = V^*$. Then $B^* = V^*\overline{D}V$, and the relation $B^kB^* = B$ becomes $D^k\overline{D} = D$, which leads to $\lambda_i^{k-1}\overline{\lambda_i} = 1$, for all i. It follows that $|\lambda_i| = 1$ and $\lambda_i^{k-2} = 1$, for all i. When $k = 3$ or $k = 4$,

then $\lambda_i \in \{-1, 1\}$ for all i, so $\overline{D} = D$, therefore $B^* = B$, then $A^* = A$, which means that the statement (*) is true.
Conclusion: $k \in \{1, 3, 4\}$.

Alternative solution. One other solution, proposed by Marian Panțiruc, uses an idea that is similar to the one in the second method above. More precisely, the proof is based on the claim that *every matrix* that verify the relation $A^k A^* = A$ for some $k \in \mathbb{N}^*$ is a *normal matrix*, i.e. $AA^* = A^*A$. We postpone proving the claim to analyze its effects. If A is normal, then A and A^* are simultaneously unitarily diagonalizable, so $A = A^*$ if and only if all its eigenvalues are real numbers. Then, the problem is equivalent to finding those values of $k \in \mathbb{N}^*$ for which the equation $\lambda^k \bar{\lambda} = \lambda$ has *only* real solutions. We obtain $\lambda = 0$ (which is real) or $|\lambda| = 1$ and $\lambda^{k-2} = 1$. If $k - 2 \geq 3$, the last equation has at least two complex roots, and if $k = 2$, any non-zero complex number (of modulus 1) is a solution, so we can only have $k \in \{1, 3, 4\}$.

Let us now prove that A is a normal matrix.

If A is invertible, then A^* is invertible and from (*) we have $A^* = (A^{k-1})^{-1} = (A^{-1})^{k-1}$. But, using Cayley-Hamilton theorem, A^{-1} is a polynomial of the matrix A, hence so is $A^* = (A^{-1})^{k-1}$ and because a matrix commutes with any of its powers, we get $AA^* = A^*A$, i.e. A is normal.

Assume now A is not invertible. Because $A^k A^* = A$ it follows that $\operatorname{Ker} A^* \subset \operatorname{Ker} A$ and since we always have $\operatorname{def} A = \operatorname{def} A^*$ we obtain $\operatorname{Ker} A = \operatorname{Ker} A^*$. (Here $\operatorname{def} A$ stands for dimension of the nullity of matrix A.) Moreover, the algebraic multiplicity of 0, denoted by $a(0)$, equals the geometric multiplicity of 0, here denoted by $g(0)$. Indeed, if, on contrary, $a(0) > g(0) = \operatorname{def} A = \operatorname{def} A^*$, then there exists some $v \in \mathbb{C}^n$ such that

$$Au \neq 0 \quad \text{and} \quad A^2 u = 0.$$

But then $A^* u \neq 0$ and $A^*(A^* u) = 0$ on account that $\operatorname{Ker} A = \operatorname{Ker} A^*$, and it follows that

$$0 = \left\langle A^k A^*(A^* u), u \right\rangle = \langle A(A^* u), u \rangle = \langle A^* u, A^* u \rangle = \|A^* u\|^2,$$

thus obtaining $A^* u = 0$, which leads to $Au = 0$, a contradiction.

Consider now $\mathcal{B} = \{u_1, \ldots, u_p, u_{p+1}, \ldots, u_n\}$ an orthonormal basis in $\mathbb{C}^n$ such that $\{u_1, \ldots, u_p\}$ is a basis in $\operatorname{Ker} A$ and denote by S the matrix having these vectors as columns. Clearly,

$$S^* = S^{-1} \quad \text{and} \quad Au_1 = \cdots = Au_p = 0.$$

Let $q \in \{p + 1, \ldots, n\}$ and consider $\alpha_{1q}, \ldots, \alpha_{pq}, \alpha_{p+1,q}, \ldots, \alpha_{nq}$ the coordinates of Au_q with respect to $\mathcal{B}$:

$$Au_q = \alpha_{1q} u_1 + \cdots + \alpha_{pq} u_p + \alpha_{p+1,q} u_{p+1} + \cdots + \alpha_{nq} u_n = S \begin{pmatrix} \alpha_{1q} \\ \vdots \\ \alpha_{nq} \end{pmatrix}.$$

Then, using the scalar product in $\mathbb{C}^n$, the orthonormality of $\mathcal{B}$, and the equality $\operatorname{Ker} A = \operatorname{Ker} A^*$ we obtain for every $i \in \{1, \dots, p\}$

$$\alpha_{iq} = \langle Au_q, u_i \rangle = \langle u_q, A^* u_i \rangle = \langle u_q, 0 \rangle = 0.$$

Then,

$$\begin{aligned} AS &= [Au_1, \dots, Au_p, Au_{p+1}, \dots, Au_n] = [0, \dots, 0, Au_{p+1}, \dots, Au_n] \\ &= S \begin{pmatrix} O_p & O_{p,n-p} \\ O_{n-p,p} & B \end{pmatrix}, \end{aligned}$$

where $B = (\alpha_{ij})$, $i, j = p+1, \dots, n$, and we actually have

$$S^* A S = \begin{pmatrix} O_p & O_{p,n-p} \\ O_{n-p,p} & B \end{pmatrix}.$$

The matrix B cannot have 0 as eigenvalue, so it is invertible (of order $n-p$). The initial relation $A^k A^* = A$ leads to $B^k B^* = B$. From the first part it follows that B is normal ($BB^* = B^* B$) and then by a straightforward computation A is normal, completing the proof.

Alternative solution. Another solution, proposed by the contestant Balkan Jepbarov from Magtymguly Turkmen State University, Turkmenistan, uses Schur's triangularization theorem and the assumptions of the problem to prove that A is normal, and then, by the same observations as in the beginning of the previous solution, to get the conclusion. Let us briefly describe the contestant's solution. Using Schur's triangularization theorem, there exists an orthogonal matrix Q such that $A = Q^* T Q$, where T is upper triangular. It follows that $A^* = Q^* T^* Q$, then $A^k = Q^* T^k Q$, hence the assumption $A^k A^* = A$ implies $T^k T^* = T$. Denote

$$T = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ 0 & t_{22} & \dots & \dots \\ \dots & \dots & t_{n-1,n-1} & t_{n-1,n} \\ 0 & \dots & 0 & t_{nn} \end{pmatrix},$$

hence

$$T^* = \begin{pmatrix} \overline{t_{11}} & 0 & \dots & 0 \\ \overline{t_{12}} & \overline{t_{22}} & \dots & \dots \\ \dots & \dots & \dots & 0 \\ \overline{t_{1n}} & \dots & \overline{t_{n-1,n}} & \overline{t_{nn}} \end{pmatrix}, \quad T^k = \begin{pmatrix} t_{11}^k & c_{12} & \dots & c_{1n} \\ 0 & t_{22}^k & \dots & \dots \\ \dots & \dots & \dots & c_{n-1,n} \\ 0 & \dots & 0 & t_{nn}^k \end{pmatrix}.$$

We have

$$\begin{pmatrix} t_{11}^k & c_{12} & \dots & c_{1n} \\ 0 & t_{22}^k & \dots & \dots \\ \dots & \dots & \dots & c_{n-1,n} \\ 0 & \dots & 0 & t_{nn}^k \end{pmatrix} \cdot \begin{pmatrix} \overline{t_{11}} & 0 & \dots & 0 \\ \overline{t_{12}} & \overline{t_{22}} & \dots & \dots \\ \dots & \dots & \dots & 0 \\ \overline{t_{1n}} & \dots & \dots & \overline{t_{nn}} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ 0 & t_{22} & \dots & \dots \\ \dots & \dots & \dots & t_{n-1,n} \\ 0 & \dots & 0 & t_{nn} \end{pmatrix},$$

hence we deduce

$$\left(T^kT^*\right)_{ni} = t_{nn}^k\overline{t_{in}} = (T)_{ni} = t_{ni} = 0, \quad i = \overline{1, n-1}.$$

If $t_{nn} \neq 0$, then $t_{in} = 0$, for $i = \overline{1, n-1}$. If $t_{nn} = 0$, then

$$t_{in} = \left(T^kT^*\right)_{in} = c_{in}\overline{t_{nn}} = 0.$$

Hence, $t_{in} = 0$ for $i = \overline{1, n-1}$, so

$$\begin{pmatrix} t_{11}^k & c_{12} & \dots & c_{1n} \\ 0 & t_{22}^k & \dots & \dots \\ \dots & \dots & \dots & c_{n-1,n} \\ 0 & \dots & 0 & t_{nn}^k \end{pmatrix} \cdot \begin{pmatrix} \overline{t_{11}} & 0 & \dots & 0 \\ \overline{t_{12}} & \overline{t_{22}} & \dots & \dots \\ \dots & \dots & \dots & 0 \\ \overline{t_{1n}} & \dots & \overline{t_{n-1,n}} & \overline{t_{nn}} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & \dots & 0 \\ 0 & t_{22} & \dots & \dots \\ \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & t_{nn} \end{pmatrix},$$

Next, we have that

$$\left(T^kT^*\right)_{n-1,i} = t_{n-1,n-1}^k\overline{t_{in-1}} = (T)_{n-1,i} = t_{n-1,i} = 0, \quad i = \overline{1, n-2}.$$

As above, by considering the two cases $t_{n-1,n-1} \neq 0$ and $t_{n-1,n-1} = 0$ and similar method as above, one deduces that $t_{i,n-1} = 0$ for $i = \overline{1, n-2}$. In the same fashion, one proves that $t_{ij} = 0$, for any $j > i$ and $i = \overline{1, j-1}$. We deduce then that T is a diagonal matrix, so A is normal.

Remark. We can summarize the conclusions of this problem as follows. If $k \in \mathbb{N}$, $k \geq 1$, and $A \in \mathcal{M}_n(\mathbb{C})$ are such that $A^kA^* = A$ then:

- If $k = 1$ or $k = 3$, the matrix A is a hermitian projection.
- If $k = 4$, the matrix A is hermitian and A^2 is a (hermitian) projection.
- If $k = 2$, the matrix A is unitarily similar to a diagonal matrix having the modulus of every diagonal entry 0 or 1. A is not necessarily hermitian. If every non-zero eigenvalue of A is a root of unity, then there exists $m \in \mathbb{N}$, $m \geq 2$, such that A^m is a hermitian projection. Otherwise, $A^p \neq A^q$ for all $p, q \in \mathbb{N}$, $p \neq q$.
- If $k \geq 5$, then A^{k-2} is a hermitian projection, A is not necessarily hermitian.

We leave to the interested reader the analysis of the converses of the claims stated before.

Although it was considered the most difficult problem by the jury, 6 *complete solutions were found by the contestants.*

PROBLEMS

Authors should submit proposed problems to `gmaproblems@rms.unibuc.ro`. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the authorsquare will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of November 2024**.

PROPOSED PROBLEMS

553. Let $ABCD$ be an isosceles tetrahedron with centroid G. Let M, N be two points such that $\overrightarrow{NG} = 3\overrightarrow{GM}$. Prove that

$$NA + NB + NC + ND \geq MA + MB + MC + MD.$$

Proposed by Leonard Giugiuc, Drobeta Turnu-Severin, Romania.

99**554.** Let $n \in \mathbb{N}$, $n \geq 2$.

(a) Prove that $\det(A^2 - B^2)(C^2 - B^2) \geq 0$ for all $A, B, C \in \mathcal{M}_n(\mathbb{R})$ with $AB = BC$.

(b) Find all values $k \geq 1$ such that $\det(A^k - B^2)(C^k - B^2) \geq 0$ holds for all $A, B, C \in \mathcal{M}_n(\mathbb{R})$ with $AB = BC$.

Proposed by Mihai Opincariu, Brad, Romania, and Vasile Pop, Technical University of Cluj-Napoca, Romania.

555. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative such that $f(1) = 0$ and $f'(1) = 1$. Prove that there exists $c \in (0,1)$ such that

$$f(c) = f'(c) \int_0^c f(x)\,\mathrm{d}x.$$

Proposed by Cezar Lupu, Beijing Institute of Mathematical Sciences and Applications (BIMSA) and Tsinghua University, Beijing, P. R. China.

556. For given $n \geq 3$, prove that $k = 2n - 3$ is the smallest positive constant such that

$$\frac{1}{a_1 + k} + \frac{1}{a_2 + k} + \cdots + \frac{1}{a_n + k} \leq \frac{n}{1 + k}$$

holds for any nonnegative real numbers $a_1, \ldots, a_n$ such that at most one of them is > 1 and $\sum\limits_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}$.

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania.

557. Find the differentiable functions $f : (0,\infty) \to \mathbb{R}$ that satisfy the identity:

$$f'(x) = x \cdot f\left(\frac{1}{x}\right)$$

for all $x \in (0,\infty)$.

Proposed by Dorian Popa, Technical University of Cluj-Napoca, Romania.

558. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 x^k f(x)\,\mathrm{d}x = 0 \quad \text{for} \quad 0 \le k \le n-1$$

and

$$\int_0^1 x^n f(x)\,\mathrm{d}x = 1.$$

Prove that

$$\int_0^1 f^2(x)\,\mathrm{d}x \ge (2n+1)\binom{2n}{n}^2.$$

Proposed by Cezar Lupu, Beijing Institute of Mathematical Sciences and Applications (BIMSA) and Tsinghua University, Beijing, P. R. China.

559. Let $f : [0,1] \to [-1,1]$ be a continuous function, with finite derivative in 0 and $f(0) = 1$. Find $\lim\limits_{n\to\infty} \displaystyle\int_0^1 f^n(x^n)\,\mathrm{d}x$.

Proposed by Mircea Rus, Technical University of Cluj-Napoca, Romania.

560. Let $(x_n)_{n\ge1}$ be the sequence defined by $x_1 \in (0,1)$ and $x_{n+1} = x_n - \frac{x_n^2}{2^n}$ for all $n \ge 1$. Prove that the sequence $(x_n)_{n\ge1}$ is convergent to a limit $C > 0$ and moreover,

$$\lim_{n\to\infty} 8^{n-1}\left(x_n - C - \frac{C^2}{2^{n-1}} - \frac{C^3}{3\cdot 4^{n-2}}\right) = \frac{12C^4 + 32C^3}{21}.$$

Proposed by Dumitru Popa, Ovidius University of Constanța, Romania.

561. Calculate

$$\sum_{n=1}^{\infty}\left[n^2\left(\frac{1}{n^3} - \frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} - \cdots\right) - \frac{1}{2n}\right].$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania.

562. For any matrix M, let $M^* = \overline{M}^t$ denote the transpose conjugate of M. The matrix M is called *anti-Hermitian* if $M^* = -M$.

Prove that if $A \in \mathcal{M}_n(\mathbb{C})$ is invertible and anti-Hermitian, then the function

$$f : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C}), \quad f(X) = AX - XA^2, \quad X \in \mathcal{M}_n(\mathbb{C})$$

is bijective.

`Proposed by Mihai Opincariu, Brad, Romania, and Vasile Pop, Technical University of Cluj-Napoca, Romania`

SOLUTIONS

536. Let p be a prime number, $\mathbb{F}_p$ the field with p elements, and $n \geq 1$ an integer. If $f \in \mathbb{F}_p[X]$ is the polynomial $X^p - X \in \mathbb{F}_p[X]$ composed with itself n times, determine the splitting field of f over $\mathbb{F}_p$.

`Proposed by Tudor Păişanu, École Polytechnique, Paris, France.`

Solution by the author. For m nonnegative integer, define f_m as the polynomial $X^p - X$ composed with itself m times. Then we have $f_{m+1}(X) = f_m(X)^p - f_m(X) = f_m(X^p) - f_m(X)$. (For every $g \in \mathbb{F}_p[X]$ we have $g(X)^p = g(X^p)$.)

Notice that $f_{m+1}(X)' = pX^{p-1}f'_m(X^p) - f'_m(X) = -f'_m(X)$. Since $f_0(X) = X$, so $f'_0(X) = 1$, by induction, $f'_m(X) = (-1)^m$. In particular, f_m and f'_m are coprime, so f_m has $\deg(f_m) = p^m$ distinct roots in the algebraic closure $\bar{\mathbb{F}}_p$.

We'll view all field extensions of $\mathbb{F}_p$ as vector spaces over it. Consider $D : \bar{\mathbb{F}}_p \to \bar{\mathbb{F}}_p$ the linear operator on $\bar{\mathbb{F}}_p$ given by $x \mapsto x^p$, i.e. the Frobenius automorphism. Note that $(D-1)f_m(\alpha) = f_m(\alpha^p) - f_m(\alpha) = f_{m+1}(\alpha)$. Then, by induction, $(D-1)^m\alpha = (D-1)f_0(\alpha) = f_m(\alpha)$ for any $\alpha \in \bar{\mathbb{F}}_p$, and thus the set of roots of f_m is $\ker(D-1)^m$, for any $m \geq 1$. Therefore $\ker(D-1)^m$ has p^m elements, so $\dim\ker(D-1)^m = m$.

Thus we can construct a flag, i.e. a linearly independent sequence $(\alpha_m)_m \subseteq \bar{\mathbb{F}}_p$ such that $\ker(D-1)^m = \langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle$ for all $m \geq 1$. As $\ker(D-1) = \mathbb{F}_p$, without loss of generality, take $\alpha_1 = 1$.

For any $n \geq 1$, $\alpha_1, \ldots, \alpha_n$ are roots of f_n that generate all the others, so $L_n = \mathbb{F}_p(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the splitting field of f_n. Consider $(m_n)_n$ the sequence of natural numbers for which $L_n = \mathbb{F}_{p^{m_n}}$. The key observation is that for all $x \in \bar{\mathbb{F}}_p$, $x \in L_n \iff x^{p^{m_n}} = x \iff x \in \ker(D^{m_n} - I)$. Thus, $L_n = \ker(D^{m_n} - I)$.

Let p^{q_n} be the smallest power of p larger than or equal to n. I claim that $m_n = p^{q_n}$ for all $n \geq 1$, which will be proven by induction.

For $n = 1$, $L_1 = \mathbb{F}_p(\alpha_1) = \mathbb{F}_p$ so $m_1 = 1 = p^{q_1}$. Now, suppose that for some $n \geq 1$, $m_n = p^{q^n}$ and thus $L_n = \ker(D^{p^{q_n}} - I) = \ker(D-1)^{p^{q_n}}$. We analyse two cases:

1. $n + 1 \leq p^{q_n}$, i.e. $q_{n+1} = q_n$ In this case, $\alpha_{n+1} \in \ker(D-1)^{n+1} \subseteq \ker(D-1)^{p^{q_n}} = L_n$, so $L_{n+1} = L_n(\alpha_{n+1}) = L_n$. Therefore, $m_{n+1} = p^{q_{n+1}}$.

2. $n+1 > p^{q_n}$, i.e. $q_{n+1} = q_n + 1$ In this case, $L_n = \ker(D-1)^{p^{q_n}} \subsetneq \ker(D-1)^{n+1}$. If $\alpha_{n+1} \in L_n$, then $\ker(D-1)^{n+1} = \langle a_1, \dots, a_{n+1}\rangle \subseteq L_n$, false. Therefore $\alpha = \alpha_{n+1} \notin L_n$. We need to find the minimal polynomial of α over L_n.

We have $(D-1)^{n+1}\alpha = 0$, so $\beta = \alpha^p - \alpha = (D-1)\alpha$ is a root of f_n. (We have $f_n(\beta)(D-1)^n\beta = (D-1)^{n+1}(\alpha) = 0$.) It is not hard to check that the p roots of the polynomial $X^p - X - \beta \in L_n[X]$ are $\alpha, \alpha+1, \dots, \alpha+(p-1)$. (For every $i \in \mathbb{F}_p$ we have $i^p = i$, so $(\alpha+i)^p - (\alpha+i) - \beta = (\alpha^p + i^p) - (\alpha+i) - \beta = \alpha^p - \alpha - \beta = 0$.)

Hence, the Galois conjugates of α, i.e., the roots of its minimal polynomial over L_n, are of the form $\alpha, \alpha+i_1, \dots, \alpha+i_l$, for some $i_1, \dots, i_l \in \mathbb{F}_p$ with $1 \le l \le p-1$. Their sum, $(l+1)\alpha + (i_1 + \dots + i_l)$ is in L_n, so that $(l+1)\alpha \in L_n$. As $\alpha \notin L_n$, we find $l \equiv -1 \pmod{p}$, i.e. $l = p-1$. Hence, $[L_{n+1} : L_n] = [L_n(\alpha) : L_n] = l + 1 = p$.

As such, $m_{n+1} = [L_{n+1} : L_n][L_n : \mathbb{F}_p] = p^{1+q_n} = p^{q_{n+1}}$, and the induction is complete. We thus obtain that the splitting field of f_n is $\mathbb{F}_{p^{p^{\lceil \log_p n \rceil}}}$.

Editor's note. We have that the set of roots of f_n is $\ker(D-1)^n$ and $\ker(D-1)^n \subsetneq \ker(D-1)^{n+1}$. More generally, $\ker(D-1)^{n'} \subsetneq \ker(D-1)^n$ if $n' < n$. From here one can proceed as follows.

Recall that $\mathbb{F}_{p^m}$ is the set of all $\alpha \in \bar{\mathbb{F}}_p$ satisfying $0 = \alpha^{p^m} - \alpha = (D^m - 1)\alpha$, i.e. $\mathbb{F}_{p^m} = \ker(D^m - 1)$. On the other hand the set of roots of f_n is $\ker(D-1)^n$. Hence f_n splits in $\mathbb{F}_{p^m}$ iff $\ker\big((D-1)^n\big) \subseteq \ker(D^m - 1)$.

We have $\gcd((X-1)^n, X^m - 1) = (X-1)^{n'}$ for some $n' \le n$. Then the condition $\ker(D-1)^n \subseteq \ker(D^m - 1)$ is equivalent to $\ker(D-1)^n = \ker(D-1)^n \cap \ker(D^m - 1) = \ker(D-1)^{n'}$. (If $P, Q \in \mathbb{F}_p[X]$, then $\ker P(D) \cap \ker Q(D) = \ker R(D)$, where $R = \gcd(P, Q)$.) But this is equivalent to $n = n'$. (Otherwise $n' < n$, so $\ker(D-1)^{n'} \subsetneq \ker(D-1)^n$.) Hence f_n splits in $\mathbb{F}_{p^m}$ iff $\gcd((X-1)^n, X^m - 1) = (X-1)^n$, i.e. iff $(X-1)^n \mid X^m - 1$.

Let $m = p^k l$, with $p \nmid l$. Then $X^m - 1 = (X^{p^k} - 1)P(X) = (X-1)^{p^k}P(X)$, where $P(X) = X^{p^k(l-1)} + \dots + X^{p^k} + 1$. Since $P(1) = l \ne 0$ in $\mathbb{F}_p$, we have $X - 1 \nmid P(X)$, so the largest power of $X-1$ dividing $X^m - 1$ is $(X-1)^{p^k}$. Therefore $(X-1)^n \mid X^m - 1$ iff $n \le p^k$, i.e. iff $k \ge \lceil \log_p n \rceil$, which is equivalent to $p^{\lceil \log_p n \rceil} \mid m$. Hence the smallest m such that f_n splits in $\mathbb{F}_{p^m}$ is $m = p^{\lceil \log_p n \rceil}$. Thus the splitting field of f_n is $\mathbb{F}_{p^{p^{\lceil \log_p n \rceil}}}$.

537. Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be such that $A^2 = A$ and $B^2 = B$. Prove that

$$\operatorname{Im}(AB - BA) = \operatorname{Im}(A + B - I_n) \cap \operatorname{Im}(A - B),$$

where $\operatorname{Im} M = \{MX \mid X \in \mathcal{M}_{n,1}(\mathbb{C})\}$ for every $M \in \mathcal{M}_n(\mathbb{C})$.

`Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania.`

Solution by the author. We have

$$(A - B)(A + B - I_n) = AB - BA, \tag{1}$$

$$(A + B - I_n)(A - B) = -(AB - BA). \tag{2}$$

From (1) we get

$$\mathrm{Im}(AB - BA) \subset \mathrm{Im}(A - B)$$

and from (2) we get

$$\mathrm{Im}(AB - BA) \subset \mathrm{Im}(A + B - I_n).$$

Hence

$$\mathrm{Im}(AB - BA) \subset \mathrm{Im}(A + B + I_n) \cap \mathrm{Im}(A - B).$$

For the reverse inclusion, let $X \in \mathrm{Im}(A + B - I_n) \cap \mathrm{Im}(A - B)$. This means that

$$X = (A + B - I_n)Y \text{ and } X = (A - B)Z, \text{ with } X, Y, Z \in \mathcal{M}_{n,1}(\mathbb{C}).$$

We therefore have

$$(A - B)X = (AB - BA)Y$$

and

$$(A + B - I_n)X = -(AB - BA)Z.$$

By adding, we get

$$(2A - I_n)X = (AB - BA)(Y - Z). \tag{3}$$

We multiply (3) by $2A - I_n$ at left and we obtain

$$(2A - I_n)^2 X = (2A - I_n)(AB - BA)(Y - Z)$$

$$\Leftrightarrow X = (AB - BA)(2A - I_n)(Z - Y),$$

so that

$$X = (AB - BA)U, \text{ where } U = (2A - I_n)(Z - Y).$$

Thus $X \in \mathrm{Im}(AB - BA)$.

Remarks. At the end of the proof we used the relations $(2A - I_n)^2 = I_n$ and $(2A - I_n)(AB - BA) = -(AB - BA)(2A - I_n)$, which are easy consequences of $A^2 = A$.

We also used the inclusions $\mathrm{Im}\,(CD) \subset \mathrm{Im}\,C$ and $\mathrm{Ker}\,D \subset \mathrm{Ker}\,(CD)$ valid for all $C, D \in \mathcal{M}_n(\mathbb{C})$.

We also received a solution from Moubinool Omarjee, Lycée Henri IV, Paris, France. The proof of the $\mathrm{Im}(AB - BA) \subset \mathrm{Im}(A + B - I) \cap \mathrm{Im}(A - B)$ inclusion is the same as in the author's solution.

For the reverse inclusion, he uses the Grassmann formula $\dim(F+G) = \dim F + \dim G - \dim(F\cap G)$, with $F = \mathrm{Im}(A+B-I)$ and $G = \mathrm{Im}(A-B)$, and he gets

$$\begin{aligned}&\dim(\mathrm{Im}(A+B-I)\cap\mathrm{Im}(A-B))\\ &= \mathrm{rank}(A+B-I)+\mathrm{rank}(A-B)-\dim(\mathrm{Im}(A+B-I)+\mathrm{Im}(A-B)).\end{aligned}$$

Since $\mathrm{Im}(X+Y)\subset \mathrm{Im}X+\mathrm{Im}Y$, we have $\mathrm{Im}(2A-I)\subset\mathrm{Im}(A+B-I)+\mathrm{Im}(A-B)$. After taking dimensions, we get $\mathrm{rank}(2A-I)\le\dim(\mathrm{Im}(A+B-I)+\mathrm{Im}(A-B))$. It follows that

$$\dim(\mathrm{Im}(A+B-I)\cap\mathrm{Im}(A-B))\le\mathrm{rank}(A+B-I)+\mathrm{rank}(A-B)-\mathrm{rank}(2A-I).$$

But $A^2=A$, so A is similar to $\begin{pmatrix}I_r & 0\\ 0 & 0\end{pmatrix}$, with $r=\mathrm{rank}A$. Then $2A-I$ is similar to $\begin{pmatrix}I_r & 0\\ 0 & -I_{n-r}\end{pmatrix}$, so it is invertible, i.e $\mathrm{rank}(2A-I)=n$.

We get $\dim(\mathrm{Im}(A+B-I)\cap\mathrm{Im}(A-B))\le\mathrm{rank}(A+B-I)+\mathrm{rank}(A-B)-n$. But, by Theorem 2.7 in [1], since A and B are idempotents, we have $\mathrm{rank}(A+B-I)+\mathrm{rank}(A-B)-n=\mathrm{rank}(AB-BA)$. Hence $\dim(\mathrm{Im}(A+B-I)\cap\mathrm{Im}(A-B))\le\mathrm{rank}(AB-BA)$. Since also $\mathrm{Im}(AB-BA)\subset\mathrm{Im}(A+B-I)\cap\mathrm{Im}(A-B)$, we must have $\mathrm{Im}(AB-BA)=\mathrm{Im}(A+B-I)\cap\mathrm{Im}(A-B)$. ☐

References

[1] Y. Tian, G. Styan, Rank equalities for idempotent and involutory matrices, *Linear Algebra Appl.* **335** (2001), 101–117.

538. Let $n\ge 1$ be an integer and let $a_1,\dots,a_{2n}\in\mathbb{Z}$ be pairwise distinct. Prove that

$$\sum_{i=1}^{2n}a_i^2+\left(\sum_{i=1}^{2n}a_i\right)^2\ge\frac{n(n+1)(2n+1)}{3}.$$

When do we have equality?

`Proposed by Leonard Giugiuc, Traian National College, Drobeta-Turnu Severin, Romania.`

Solution by the author. First note that the sequence of integers ordered by absolute value begins with

$$|0|<|-1|=|1|<\cdots<|-(n-1)|=|n-1|<|-n|=|n|.$$

It follows that

$$\sum_{i=1}^{2n}a_i^2\ge 0^2+2(1^2+\cdots+(n-1)^2)+n^2=\frac{n(n-1)(2n-1)}{3}+n^2$$

and this minimum is attained iff $\{a_1,\dots,a_{2n}\}=\{0,\pm1,\dots,\pm(n-1)\}\cup\{n\}$ or $\{0,\pm1,\dots,\pm(n-1)\}\cup\{-n\}$.

Let $\beta = \frac{n(n-1)(2n-1)}{3}$ and $\gamma = \frac{n(n+1)(2n+1)}{3} = \beta + 2n^2$. So we have $\sum_{i=1}^{2n} a_i^2 \geq \beta + n^2$ and we we want to prove that $\sum_{i=1}^{2n} a_i^2 + \left(\sum_{i=1}^{2n} a_i\right)^2 \geq \gamma$.

Note that if $|\sum_{i=1}^{2n} a_i| \geq n$ then $\sum_{i=1}^{2n} a_i^2 + \left(\sum_{i=1}^{2n} a_i\right)^2 \geq \beta + n^2 + n^2 = \gamma$, so we are done. Hence we may assume that $|\sum_{i=1}^{2n} a_i| \leq n-1$.

WLOG we may assume that $a_1 > \cdots > a_{2n}$. Since $|\sum_{i=1}^{2n} a_i| \leq n-1$, we cannot have $a_{2n} > -n$, since this would imply $\sum_{i=1}^{2n} a_i \geq (1-n) + (2-n) + \cdots + (n-2) + (n-1) + n = n$. And we cannot have $a_1 < n$, since this would imply $\sum_{i=1}^{2n} a_i \leq (n-1) + (n-2) + \cdots + (2-n) + (1-n) + (-n) = -n$. Thus $a_1 \geq n$ and $a_{2n} \leq -n$.

We now prove our statement by induction. If $n = 1$, then $\gamma = 2$. Since $a_1 \geq 1$ and $a_2 \leq -1$, we have $a_1^2 + a_2^2 + (a_1 + a_2)^2 \geq a_1^2 + a_2^2 \geq 2 = \gamma$, so we are done.

We now assume that $n > 1$ and we prove the induction step $n - 1 \to n$. As seen above, we may assume that $a_1 \geq n$, $a_{2n} \leq -n$ and, if $m = \sum_{i=1}^{2n} a_i$, then $|m| \leq n-1$. We put $T = \sum_{i=2}^{2n-1} a_i$, $x = a_1 - n \geq 0$, and $y = -a_{2n} - n \geq 0$. Then $a_1 = n + x$, $a_{2n} = -n - y$, and $m = T + a_1 + a_{2n} = T + x - y$.

The relation we want to prove writes as $\sum_{i=1}^{2n} a_i^2 + m^2 \geq \gamma$, i.e.

$$\sum_{i=2}^{2n-1} a_i^2 + a_1^2 + a_{2n}^2 + m^2 \geq \gamma = \beta + 2n^2.$$

But the induction hypothesis applied to $a_2, \ldots, a_{2n-1}$ gives $\sum_{i=2}^{2n-1} a_i^2 + T^2 \geq \beta$. Hence it suffices to prove that it holds

$$a_1^2 + a_{2n}^2 + m^2 - T^2 \geq 2n^2, \quad \text{i.e.} \quad (n+x)^2 + (-n-y)^2 + (T+x-y)^2 - T^2 \geq 2n^2,$$

which, after reductions, becomes

$$x^2 + y^2 + 2n(x+y) + (x-y)^2 + 2T(x-y) \geq 0.$$

But $|T| = |m - (x-y)| \leq |m| + |x-y| \leq n - 1 + |x-y|$, so

$$2T(x-y) \geq -2|T|\,|x-y| \geq -2(n-1+|x-y|)|x-y| = -2(x-y)^2 - 2(n-1)|x-y|.$$

It follows that

$$\begin{aligned} x^2 + y^2 + 2n(x+y) + (x-y)^2 + 2T(x-y) \geq{}& x^2 + y^2 - (x-y)^2 + 2n(x+y) \\ & - 2(n-1)|x-y| \geq 0. \end{aligned}$$

(We have $x, y \geq 0$, so $x^2 + y^2 \geq x^2 - 2xy + y^2 = (x-y)^2$ and $x + y \geq |x-y|$, which implies $2n(x+y) \geq 2(n-1)|x-y|$.)

Note that if $|\sum_{i=1}^{2n} a_i| \geq n$ then $\sum_{i=1}^{2n} a_i^2 + \left(\sum_{i=1}^{2n} a_i\right)^2 \geq \beta + n^2 + n^2 = \gamma$, so we are done. Hence we may assume that $|\sum_{i=1}^{2n} a_i| \leq n-1$.

WLOG we may assume that $a_1 > \cdots > a_{2n}$. Since $|\sum_{i=1}^{2n} a_i| \le n-1$, we cannot have $a_{2n} > -n$, since this would imply $\sum_{i=1}^{2n} a_i \ge (1-n) + (2-n) + \cdots + (n-2) + (n-1) + n = n$. And we cannot have $a_1 < n$, since this would imply $\sum_{i=1}^{2n} a_i \le (n-1)+(n-2)+\cdots+(2-n)+(1-n)+(-n) = -n$. Thus $a_1 \ge n$ and $a_{2n} \le -n$.

We now prove our statement by induction. If $n = 1$, then $\gamma = 2$. Since $a_1 \ge 1$ and $a_2 \le -1$, we have $a_1^2 + a_2^2 + (a_1 + a_2)^2 \ge a_1^2 + a_2^2 \ge 2 = \gamma$, so we are done.

We now assume that $n > 1$ and we prove the induction step $n - 1 \to n$. As seen above, we may assume that $a_1 \ge n$, $a_{2n} \le -n$ and, if $m = \sum_{i=1}^{2n} a_i$, then $|m| \le n-1$. We put $T = \sum_{i=2}^{2n-1} a_i$, $x = a_1 - n \ge 0$, and $y = -a_{2n} - n \ge 0$. Then $a_1 = n + x$, $a_{2n} = -n - y$, and $m = T + a_1 + a_{2n} = T + x - y$.

The relation we want to prove writes as $\sum_{i=1}^{2n} a_i^2 + m^2 \ge \gamma$, i.e.

$$\sum_{i=2}^{2n-1} a_i^2 + a_1^2 + a_{2n}^2 + m^2 \ge \gamma = \beta + 2n^2.$$

But the induction hypothesis applied to $a_2, \dots, a_{2n-1}$ gives $\sum_{i=2}^{2n-1} a_i^2 + T^2 \ge \beta$. Hence it suffices to prove that $a_1^2 + a_{2n}^2 + m^2 - T^2 \ge 2n^2$, i.e. $(n+x)^2 + (-n-y)^2 + (T + x - y)^2 - T^2 \ge 2n^2$, which, after reductions, becomes

$$x^2 + y^2 + 2n(x+y) + (x-y)^2 + 2T(x-y) \ge 0.$$

But $|T| = |m - (x-y)| \le |m| + |x-y| \le n - 1 + |x-y|$, so

$$2T(x-y) \ge -2|T|\,|x-y| \ge -2(n-1+|x-y|)|x-y| = -2(x-y)^2 - 2(n-1)|x-y|.$$

It follows that

$$\begin{aligned} x^2 + y^2 + 2n(x+y) + (x-y)^2 + 2T(x-y) \ge & x^2 + y^2 - (x-y)^2 + 2n(x+y) \\ & - 2(n-1)|x-y| \ge 0. \end{aligned}$$

(We have $x, y \ge 0$, so $x^2 + y^2 \ge x^2 - 2xy + y^2 = (x-y)^2$ and $x + y \ge |x-y|$, which implies $2n(x+y) \ge 2(n-1)|x-y|$.)

Note that if $|\sum_{i=1}^{2n} a_i| \ge n$ then $\sum_{i=1}^{2n} a_i^2 + \left(\sum_{i=1}^{2n} a_i\right)^2 \ge \beta + n^2 + n^2 = \gamma$, so we are done. Hence we may assume that $|\sum_{i=1}^{2n} a_i| \le n-1$.

WLOG we may assume that $a_1 > \cdots > a_{2n}$. Since $|\sum_{i=1}^{2n} a_i| \le n-1$, we cannot have $a_{2n} > -n$, since this would imply $\sum_{i=1}^{2n} a_i \ge (1-n) + (2-n) + \cdots + (n-2) + (n-1) + n = n$. And we cannot have $a_1 < n$, since this would imply $\sum_{i=1}^{2n} a_i \le (n-1)+(n-2)+\cdots+(2-n)+(1-n)+(-n) = -n$. Thus $a_1 \ge n$ and $a_{2n} \le -n$.

We now prove our statement by induction. If $n = 1$, then $\gamma = 2$. Since $a_1 \ge 1$ and $a_2 \le -1$, we have $a_1^2 + a_2^2 + (a_1 + a_2)^2 \ge a_1^2 + a_2^2 \ge 2 = \gamma$, so we are done.

We now assume that $n > 1$ and we prove the induction step $n-1 \to n$. As seen above, we may assume that $a_1 \geq n$, $a_{2n} \leq -n$ and, if $m = \sum_{i=1}^{2n} a_i$, then $|m| \leq n-1$. We put $T = \sum_{i=2}^{2n-1} a_i$, $x = a_1 - n \geq 0$, and $y = -a_{2n} - n \geq 0$. Then $a_1 = n + x$, $a_{2n} = -n - y$, and $m = T + a_1 + a_{2n} = T + x - y$.

The relation we want to prove writes as $\sum_{i=1}^{2n} a_i^2 + m^2 \geq \gamma$, i.e.

$$\sum_{i=2}^{2n-1} a_i^2 + a_1^2 + a_{2n}^2 + m^2 \geq \gamma = \beta + 2n^2.$$

But the induction hypothesis applied to $a_2, \dots, a_{2n-1}$ gives $\sum_{i=2}^{2n-1} a_i^2 + T^2 \geq \beta$. Hence it suffices to prove that it holds

$$a_1^2 + a_{2n}^2 + m^2 - T^2 \geq 2n^2, \quad \text{i.e.} \quad (n+x)^2 + (-n-y)^2 + (T+x-y)^2 - T^2 \geq 2n^2,$$

which, after reductions, becomes

$$x^2 + y^2 + 2n(x+y) + (x-y)^2 + 2T(x-y) \geq 0.$$

But $|T| = |m - (x-y)| \leq |m| + |x-y| \leq n - 1 + |x-y|$, so

$$2T(x-y) \geq -2|T|\,|x-y| \geq -2(n-1+|x-y|)|x-y| = -2(x-y)^2 - 2(n-1)|x-y|.$$

It follows that

$$\begin{aligned} x^2 + y^2 + 2n(x+y) + (x-y)^2 + 2T(x-y) \geq{}& x^2 + y^2 - (x-y)^2 + 2n(x+y) \\ &- 2(n-1)|x-y| \geq 0. \end{aligned}$$

(We have $x, y \geq 0$, so $x^2 + y^2 \geq x^2 - 2xy + y^2 = (x-y)^2$ and $x + y \geq |x-y|$, which implies $2n(x+y) \geq 2(n-1)|x-y|$.)

We prove that the equality holds iff $(a_1, \dots, a_{2n})$ is a $2n$-arrangement of the set $S = \{0, \pm 1, \dots, \pm n\}$. The number of such arrangements is $\dfrac{(2n+1)!}{1!} = (2n+1)!$.

For the "if" part we note that $\sum_{x \in S} x = 0$ and $\sum_{x \in S} x^2 = 0^2 + 2(1^2 + \cdots + n^2) = \gamma$. If $(a_1, \dots, a_{2n})$ is an arrangement of S, then $\{a_1, \dots, a_{2n}\} = S \setminus \{m\}$ for some $m \in S$. It follows that $\sum_{i=1}^{2n} a_i = \sum_{x \in S \setminus \{m\}} x = \sum_{x \in S} x - m = -m$ and so

$$\sum_{i=1}^{2n} a_i^2 + \left(\sum_{i=1}^{2n} a_i \right)^2 = \sum_{x \in S \setminus \{m\}} x^2 + (-m)^2 = \sum_{x \in S} x^2 = \gamma.$$

For the "only if" part, we may assume that $n \geq 2$, since the case $n = 1$ is trivial. Suppose that $\sum_{i=1}^{2n} a_i^2 + \left(\sum_{i=1}^{2n} a_i \right)^2 = \gamma = \beta + 2n^2$.

If $\left| \sum_{i=1}^{2n} a_i \right| \geq n$, then $\beta + 2n^2 = \sum_{i=1}^{2n} a_i^2 + \left(\sum_{i=1}^{2n} a_i \right)^2 \geq \sum_{i=1}^{2n} a_i^2 + n^2$, so $\sum_{i=1}^{2n} a_i^2 \leq \beta + n^2$. But, as seen from the proof, $\sum_{i=1}^{2n} a_i^2 \geq \beta + n^2$, so we must have equality, which happens iff $\{a_1, \dots, a_n\} = \{0, \pm 1, \dots, \pm(n-1)\} \cup \{n\}$ or $\{0, \pm 1, \dots, \pm(n-1)\} \cup \{-n\}$. In both cases, $\{a_1, \dots, a_n\} \subseteq S$.

If $\left|\sum_{i=1}^{2n} a_i\right| \leq n-1$, then, WLOG, we may assume that $a_1 > \cdots > a_{2n}$. As seen from the proof of the induction step, in this case $a_1 = n + x$ and $a_{2n} = -n - y$, with $x, y \geq 0$ and, in order to have equality, we must have $2n(x + y) = 2(n-1)|x - y|$. Since $|x - y| \leq x + y$, this implies $2n(x + y) \leq 2(n-1)(x + y)$, whence $x + y \leq 0$. It follows that $x = y = 0$, that is, $a_1 = n$ and $a_{2n} = -n$. Since $n = a_1 > \cdots > a_{2n} = -n$, we have $\{a_1, \ldots, a_{2n}\} \subseteq S$, and we are done. ☐

539. Let $n \geq 1$ be an integer and let $X = \{1, \ldots, n\}$. We denote by F_X the set of all functions $f : X \to X$ and by S_X the symmetric group on X, i.e., the set of all permutations on X. If $f, g \in F_X$, we say that f and g are conjugate and we write $f \sim g$ if there is $\sigma \in S_X$ such that $g = \sigma f \sigma^{-1}$.

Let M_X be the set of all $f \in F_X$ such that for every $\emptyset \neq Y \subseteq X$ with $f(Y) \subseteq Y$ we have $f(Y) = f(X)$.

(i) Prove that if $f \in M_X$ and $g \sim f$, then $g \in M_X$.

(ii) Prove that $|M_X/\sim| = \frac{1}{n}\sum_{d|n} \phi(d)\, 2^{n/d} - 1$.

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Solution by the author. (i) Let $\sigma \in S_X$ such that $g = \sigma f \sigma^{-1}$. If $\emptyset \neq Y \subseteq X$ such that $g(Y) \subseteq Y$, then $\sigma f \sigma^{-1}(Y) \subseteq Y$ and when we apply σ^{-1} to both sides we get $f\sigma^{-1}(Y) \subseteq \sigma^{-1}(Y)$. Since $f \in M_X$, we have $f\sigma^{-1}(Y) = f(X) = f\sigma^{-1}(X)$ (the latter equality holds because σ^{-1} is a bijection, so $X = \sigma^{-1}(X)$). We apply σ to both sides and we get $\sigma f \sigma^{-1}(Y) = \sigma f \sigma^{-1}(X)$, i.e. $g(Y) = Y$. Hence $g \in M_X$.

(ii) We prove that $f \in M_X$ iff $f_{|f(X)} : f(X) \to f(X)$ is a cyclic permutation. First assume that $f \in M_X$. Note that if $Y = f(X)$ then $Y \neq \emptyset$ and $f(Y) \subseteq f(X) = Y$. Since $f \in M_X$, we have $f(Y) = f(X) = Y$. Hence $f_{|Y} : Y \to Y$, i.e. $f_{|f(X)} : f(X) \to f(X)$ is a surjective function. Since $f(X)$ is finite, we have $f_{|f(X)} \in S_{f(X)}$. If $(x_1, \ldots, x_k)$ is a cycle of the permutation $f_{|f(X)}$ and $Y = \{x_1, \ldots, x_k\}$, then $f(Y) = Y$, so, by hypothesis, $f(X) = f(Y) = Y = \{x_1, \ldots, x_k\}$. Hence $f_{|f(X)}$ coincides with the cycle $(x_1, \ldots, x_k)$.

Conversely, assume that $f(X) = \{x_1, \ldots, x_k\}$ and $f_{|f(X)}$ is the cyclic permutation $(x_1, \ldots, x_k)$, i.e. $f(x_i) = x_{i+1}$ for $1 \leq i \leq k-1$ and $f(x_k) = x_1$. Let $\emptyset \neq Y \subseteq X$ be such that $f(Y) \subseteq Y$. Let $y \in Y$ be arbitrary. Then $f(y) \in f(Y) \subseteq Y$, so $f^{(2)}(y) \in f(Y) \subseteq Y$ and so on. Hence $f^{(l)}(y) \in f(Y)$ for every $l \geq 1$. Since $f(y) \in f(X)$, we have $f(y) = x_i$ for some $1 \leq i \leq k$. Then the sequence $f(y), f^{(2)}(y), \ldots, f^{(k)}(y)$, which is contained in $f(Y)$, is $x_i, \ldots, x_k, x_1, \ldots, x_{i-1}$. Hence $f(X) = \{x_1, \ldots, x_k\} \subseteq f(Y)$. The reverse inclusion is trivial, so $f(Y) = f(X)$. Hence $f \in M_X$.

We have $M_X = \bigcup_{k=1}^{n} M_{X,k}$, where $M_{X,k} = \{f \in M_X : |f(X)| = k\}$.

Suppose now that $f \in M_X$, $|f(X)| = k$ and $f_{|f(X)}$ is the cycle $(x_1, \dots, x_k)$. For convenience, the indices in $x_1, \dots, x_k$ will be assumed to be from $\mathbb{Z}_k$. For every x_i we denote $\alpha_i = |f^{-1}(x_i)| - 1 = |\{x \in X \setminus f(X) : f(x) = x_i\}|$. (We have $|\{x \in f(X) : f(x) = x_i\}| = |\{x_{i-1}\}| = 1$.) We have $\sum_{i=1}^{k} \alpha_i = \sum_{x \in f(X)}(|f^{-1}(x)| - 1) = \sum_{x \in f(X)} |f^{-1}(x)| - |f(X)| = |X| - |f(X)| = n - k$. So to the $(k+1)$-uple $(f, x_1, \dots, x_k)$ we may associate the element $(\alpha_1, \dots, \alpha_k) \in A_{n,k}$, where $A_{n,k} = \{(\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k : \alpha_1 + \cdots + \alpha_k = n - k\}$. (Here we use the notations $\mathbb{N} := \mathbb{Z}_{\geq 0}$ and $\mathbb{N}^* := \mathbb{Z}_{\geq 1}$.)

But the cycle $f_{|f(X)}$ is not uniquely written as $(x_1, \dots, x_k)$. Instead, for every $h \in \mathbb{Z}_k$ it can be written as $(x_{h+1}, \dots, x_k, x_1, \dots, x_h)$. To the $(k+1)$-uple $(f, x_{h+1}, \dots, x_k, x_1, \dots, x_h)$ we associate the element $(\alpha_{h+1}, \dots, \alpha_k, \alpha_1, \dots, \alpha_h) \in A_{n,k}$. On $A_{n,k}$ we introduce the equivalence relation $\approx$, with $\alpha \approx \beta$ if $\beta_i = \alpha_{i+h}$ $\forall i \in \mathbb{Z}_k$, i.e. if $(\beta_1, \dots, \beta_k) = (\alpha_{h+1}, \dots, \alpha_k, \alpha_1, \dots, \alpha_h)$ for some $h \in \mathbb{Z}_k$. For every $\alpha \in A_{n,k}$ we denote by $\bar{\alpha}$ its equivalence class in $B_{n,k} = A_{n,k}/\approx$. Then we have a map $\Psi_k : M_{X,k} \to B_{n,k}$, where if $f_{|f(X)} = (x_1, \dots, x_k)$ and $\alpha_i = |f^{-1}(x_i)| - 1$, then $\Psi_k(f) = \bar{\alpha}$. The definition of $\Psi_k(f)$ is independent on how the cyclic permutation $f_{|f(X)}$ is written as $(x_1, \dots, x_k)$, as the class $\bar{\alpha}$ is invariant to the cyclic permutations of the entries $\alpha_1, \dots, \alpha_k$ of α.

Since $M_X = \bigcup_{k=1}^{n} M_{X,k}$, we have a map $\Psi : M_X \to \bigcup_{k=1}^{n} B_{n,k}$ given by $\Psi_{|M_{X,k}} = \Psi_k$. This map is surjective as every Ψ_k is surjective. Indeed, if $\bar{\alpha} \in B_{n,k}$, with $\alpha = (\alpha_1, \dots, \alpha_k) \in A_{n,k}$, then $k + \alpha_1 + \cdots + \alpha_k = n$, so we have a partition $X = \{1, \dots, k\} \cup X_1 \cup \cdots \cup X_k$, with $|X_i| = \alpha_i$ $\forall i$. Then we define $f : X \to X$ by $f_{|\{1,\dots,k\}} = (1, \dots, k)$ and $f_{|X_i} \equiv i$. We have $f(X) = \{1, \dots, k\}$ and $f_{|f(X)}$ is the cyclic permutation $(1, \dots, k)$. Hence $f \in M_{X,k}$. We have $f^{-1}(1) = \{m\} \cup X_1$ and $f^{-1}(i) = \{i-1\} \cup X_i$ for $2 \leq i \leq k$. Hence for every $1 \leq i \leq k$ we have $|f^{-1}(i)| - 1 = |X_i| = \alpha_i$. Thus $\Psi(f) = \Psi_k(f) = \bar{\alpha}$. So Ψ is surjective.

We claim that if $f, g \in M_X$ then $f \sim g$ iff $\Psi(f) = \Psi(g)$, and so Ψ induces a bijection between $M_X/\sim$ and $\bigcup_{k=1}^{n} B_{n,k}$, which implies that $a_n := |M_X/\sim| = \sum_{k=1}^{n} a_{n,k}$, where $a_{n,k} = |B_{n,k}|$.

First assume that $f \sim g$, so $g = \sigma f \sigma^{-1}$ for some $\sigma \in S_X$. Let $f(X) = \{x_1, \dots, x_k\}$ (with indices in $\mathbb{Z}_k$), such that $f_{|f(X)}$ is the cycle $(x_1, \dots, x_k)$. We have $g\sigma = \sigma f$. Since $\sigma(X) = X$, we get $g(X) = g\sigma(X) = \sigma f(X) = \sigma(\{x_1, \dots, x_k\}) = \{y_1, \dots, y_k\}$, with $y_i = \sigma(x_i)$. For each i we have $g(y_i) = g\sigma(x_i) = \sigma f(x_i) = \sigma(x_{i+1}) = y_{i+1}$, so $g_{|g(X)}$ is the cycle $(y_1, \dots, y_k)$. Let $i \in \mathbb{Z}_k$. For every $y \in X$ we have $y = \sigma(x)$ for some unique $x \in X$. Then $g(y) = y_i$ writes as $\sigma f(x) = g\sigma(x) = \sigma(x_i)$, which is equivalent to $f(x) = x_i$. Hence $y = \sigma(x) \in g^{-1}(y_i)$ iff $x \in f^{-1}(x_i)$. It follows that $g^{-1}(y_i) = \sigma(f^{-1}(x_i))$, which implies that $|g^{-1}(y_i)| = |f^{-1}(x_i)|$. Consequently, $f, g \in M_{X,k}$ and $\Psi(f) = \Psi(g) = \bar{\alpha}$, where $\alpha = (\alpha_1, \dots, \alpha_k)$ is given by $\alpha_i = |f^{-1}(x_i)| = |g^{-1}(y_i)|$.

Conversely, assume that $\Psi(f)=\Psi(g)$. If $\Psi(f)=\Psi(g)\in B_{n,k}$, then $f,g\in M_{X,k}$. Let $f(X)=\{x_1,\dots,x_k\}$ and $g(X)=\{y_1,\dots,y_k\}$ be such that $f_{|f(X)}$ and $g_{|g(X)}$ are the cycles $(x_1,\dots,x_k)$ and $(y_1,\dots,y_k)$, respectively. We have $\Psi(f)=\bar{\alpha}$ and $\Psi(g)=\bar{\beta}$, where $\alpha_i=|f^{-1}(x_i)|-1$ and $\beta_i=|g^{-1}(y_i)|-1$. Since $\bar{\alpha}=\bar{\beta}$, we have $\beta_i=\alpha_{i+h}$ for some $h\in\mathbb{Z}_k$. If we denote $z_i=y_{i-h}$, then the cycle $(y_1,\dots,y_k)$ also writes as $(z_1,\dots,z_k)$. Also note that $|g^{-1}(z_i)|-1=|g^{-1}(y_{i-h})|-1=\beta_{i-h}=\alpha_i$. We have the partitions $X=\{x_1,\dots,x_k\}\cup X_1\cup\cdots\cup X_k$ and $X=\{z_1,\dots,z_k\}\cup Z_1\cup\cdots\cup Z_k$, where $X_i=f^{-1}(x_i)\setminus\{x_{i-1}\}$ and $Z_i=g^{-1}(z_i)\setminus\{z_{i-1}\}$. For every $i\in\mathbb{Z}_k$ we have $|Z_i|=|g^{-1}(z_i)|-1=\alpha_i=|f^{-1}(x_i)|-1=|X_i|$, so there is a bijection $\sigma_i:X_i\to Z_i$. We also have the bijection $\sigma_0:\{x_1,\dots,x_k\}\to\{z_1,\dots,z_k\}$ given by $x_i\mapsto z_i$. Then we define $\sigma\in S_X$ by $\sigma_{|\{x_1,\dots,x_k\}}=\sigma_0$ and $\sigma_{|X_i}=\sigma_i$ $\forall i\in\mathbb{Z}_k$. For each $i\in\mathbb{Z}_k$ we have $g\sigma(x_i)=g(z_i)=z_{i+1}=\sigma(x_{i+1})=\sigma f(x_i)$ and if $x\in X_i$, then $\sigma(x)\in Z_i$. Since $X_i\subset f^{-1}(x_i)$ and $Z_i\subset g^{-1}(z_i)$, this implies that $g\sigma(x)=z_i$ and $f(x)=x_i$, so $g\sigma(x)=z_i=\sigma(x_i)=\sigma f(x)$. In conclusion, $g\sigma=\sigma f$, that is, $g=\sigma f\sigma^{-1}$, and so $f\sim g$.

We now evaluate $a_n=\sum_{k=1}^n a_{n,k}$, with $a_{n,k}=|B_{n,k}|=|A_{n,k}/\approx|$.

First note that $|A_{n,k}|=\binom{n-k+k-1}{k-1}=\binom{n-1}{k-1}$. (Here we use a well known result, which states that the cardinal of $\{(n_1,\dots,n_k)\in\mathbb{N}^k:\sum_{i=1}^k n_i=n\}$ is the coefficient of X^n in the series $(1+X+X^2+\cdots)^k=(1-X)^{-k}=\sum_{n\geq 0}\binom{-k}{n}(-X)^n$, i.e. it is $(-1)^n\binom{-k}{n}=\binom{n+k-1}{n}=\binom{n+k-1}{k-1}$.)

We also use the following elementary result.

Lemma 1. If $\sim$ is an equivalence relation on a set S and for every $x\in S$, its class in $S/\sim$ is denoted by $\hat{x}$, then

$$|S/\sim|=\sum_{x\in S}\frac{1}{|\hat{x}|}.$$

Proof. We have $S=\bigsqcup_{\xi\in S/\sim}\xi$, so

$$\sum_{x\in S}\frac{1}{|\hat{x}|}=\sum_{\xi\in S/\sim}\sum_{x\in\xi}\frac{1}{|\hat{x}|}=\sum_{\xi\in S/\sim}\sum_{x\in\xi}\frac{1}{|\xi|}=\sum_{\xi\in S/\sim}1=|S/\sim|. \qquad \square$$

In our case $a_{n,k}=|A_{n,k}/\approx|$ writes as $a_{n,k}=\sum_{\alpha\in A_{n,k}}\dfrac{1}{|\bar{\alpha}|}$.

Let $\alpha=(\alpha_1,\dots,\alpha_k)\in A_{n,k}$. (Again, here the indices are from $\mathbb{Z}_k$.) By definition, $\bar{\alpha}=\{\alpha[0],\alpha[1],\alpha[2],\dots\}$, where $\alpha[h]$ is α shifted by h, i.e. $\alpha[h]_i=\alpha_{i+h}$. Now for every $h,h'\in\mathbb{Z}$ we have $\alpha[h]=\alpha[h']$ iff $\alpha_{i+h}=\alpha_{i+h'}$ $\forall i$, i.e. iff the map $i\mapsto\alpha_i$ has period $h-h'$. Thus the maps $h\mapsto\alpha[h]$ and $i\mapsto\alpha_i$ have the same periodicity. It follows that $|\bar{\alpha}|=T$, where T is the smallest period of α, i.e. of the map $i\mapsto\alpha_i$.

We have $\alpha_{i+k}=\alpha_i$, so k is a period of α. Therefore T, the smallest period of α, is a divisor of k. We write $T=k/d$ for some d with $d\mid k$. Because the periodicity, the sequence $\alpha_1,\dots,\alpha_k$ is made of d copies of $\alpha_1,\dots,\alpha_{k/d}$. It follows that $n-k=\sum_{i=1}^k\alpha_i=d\sum_{i=1}^{k/d}\alpha_i$. Hence $d\mid n-k$, which, together with $d\mid k$, implies that $d\mid(n,k)$. Hence $A_{n,k}=\bigcup_{d\mid(n,k)}A_{n,k,d}$, where $A_{n,k,d}$ is the set of all $\alpha\in A_{n,k}$ for which the smallest period is k/d. It follows that

$$\sum_{d\mid(n,k)}|A_{n,k,d}|=|A_{n,k}|=\binom{n-1}{k-1}.$$

Also for every $\alpha\in A_{n,k,d}$ we have $|\bar{\alpha}|=k/d$, so $\dfrac{1}{|\bar{\alpha}|}=\dfrac{d}{k}$. Hence

$$a_{n,k}=\sum_{\alpha\in A_{n,k}}\frac{1}{|\bar{\alpha}|}=\sum_{d\mid(n,k)}\sum_{\alpha\in A_{n,k,d}}\frac{1}{|\bar{\alpha}|}=\sum_{d\mid(n,k)}\sum_{\alpha\in A_{n,k,d}}\frac{d}{k}=\sum_{d\mid(n,k)}\frac{d}{k}|A_{n,k,d}|.$$

We denote $C_{n,k}=A_{n,k,1}$, i.e. $C_{n,k}$ is the set of all $\alpha\in A_{n,k}$ that have no periods smaller than k. We also put $c_{n,k}=|C_{n,k}|$.

If $d\mid(n,k)$ and $\alpha\in A_{n,k,d}$, then the sequence $\alpha_1,\dots,\alpha_k$ is made of d copies of $\alpha_1,\dots,\alpha_{k/d}$. If we introduce $\alpha'=(\alpha_1,\dots,\alpha_{n/k})\in\mathbb{Z}^{k/d}$, then $\alpha\in(\mathbb{Z}^{k/d})^d=\mathbb{Z}^k$ writes as $\alpha=\alpha'^d$, which is the concatenation of d copies of α'. We have $n-k=\sum_{i=1}^k\alpha_i=d\sum_{i=1}^{k/d}\alpha_i$, so $\sum_{i=1}^{k/d}\alpha_i=n/d-k/d$. Thus $\alpha'\in A_{n/d,k/d}$. Also α and α' have the same periodicity, so the smallest period of α' is k/d. Hence $\alpha'\in A_{n/d,k/d,1}=C_{n/d,k/d}$. Conversely, if $\alpha'\in C_{n/d,k/d}$ and $\alpha=\alpha'^d$, then $\sum_{i=1}^{k/d}\alpha_i=n/d-k/d$, so $\sum_{i=1}^k\alpha_i=d\sum_{i=1}^{k/d}\alpha_i=n-k$, and the smallest period of α is the same as that of α', i.e., n/k. Thus $\alpha\in A_{n,k,d}$. So we have a bijection $C_{n/d,k/d}\to A_{n,k,d}$, given by $\alpha'\mapsto\alpha'^d$. It follows that $|A_{n,k,d}|=|C_{n/d,k/d}|=c_{n/d,k/d}$.

Then the two relations above may be written as

$$\sum_{d\mid(n,k)}c_{n/d,k/d}=\binom{n-1}{k-1}\quad\text{and}\quad\sum_{d\mid(n,k)}\frac{d}{k}c_{n/d,k/d}=a_{n,k}.$$

Lemma 2. If $m\geq l\geq 1$, $(m,l)=1$, and $s\geq 1$, then

$$c_{ms,ls}=\sum_{t\mid s}\mu(t)\binom{ms/t-1}{ls/t-1}.$$

Proof. We have $(ms,ls)=s$, whence

$$\sum_{t\mid s}c_{ms/t,ls/t}=\binom{ms-1}{ls-1}.$$

Hence, if we define $f,F:\mathbb{N}^*\to\mathbb{Z}$ by $f(s)=c_{ms,ls}$ and $F(s)=\binom{ms-1}{ls-1}$, then $F(s)=\sum_{t\mid s}f(s/t)=\sum_{t\mid s}f(t)$. By the Möbius inversion formula, we get $f(s)=\sum_{t\mid s}\mu(t)f(s/t)$, which is precisely what our lemma states. $\square$

If $n \geq k \geq 1$ and $d \mid (n,k)$, we apply Lemma 2 to $m = \frac{n}{(n,k)}$, $l = \frac{k}{(n,k)}$, and $s = \frac{(n,k)}{d}$. We get

$$c_{n/d,k/d} = \sum_{t \mid (n,k)/d} \mu(t) \binom{n/dt - 1}{k/dt - 1}.$$

It follows that

$$a_{n,k} = \sum_{d \mid (n,k)} \frac{d}{k} c_{n/d,k/d} = \sum_{d \mid (n,k)} \sum_{t \mid (n,k)/d} \frac{d}{k} \mu(t) \binom{n/dt - 1}{k/dt - 1}.$$

From

$$\{(d,t) : \, d \mid (n,k), \, t \mid (n,k)/d\} = \{(d,t) : dt \mid (n,k)\} = \{(e/t,t) : t \mid e \mid (n,k)\}$$

we obtain

$$a_{n,k} = \sum_{e \mid (n,k)} \sum_{t \mid e} \frac{\mu(t)}{t} \cdot \frac{e}{k} \binom{n/e - 1}{k/e - 1} = \sum_{e \mid (n,k)} \frac{\phi(e)}{k} \binom{n/e - 1}{k/e - 1}.$$

(Here we used the formula $\sum_{t \mid e} \frac{\mu(t)}{t} = \frac{\phi(e)}{e}$.)

It follows that

$$a_n = \sum_{k=1}^{n} a_{n,k} = \sum_{k=1}^{n} \sum_{d \mid (n,k)} \frac{\phi(d)}{k} \binom{n/d - 1}{k/d - 1}.$$

But $\{(k,d) : 1 \leq k \leq n, \, d \mid (n,k)\} = \{(di,d) : \, d \mid n, \, 1 \leq i \leq n/d\}$, so

$$a_n = \sum_{d \mid n} \sum_{i=1}^{n/d} \frac{\phi(d)}{di} \binom{n/d - 1}{i - 1} = \sum_{d \mid n} \frac{\phi(d)}{d} f(n/d),$$

where

$$f(n) = \sum_{i=1}^{n} \frac{1}{i} \binom{n-1}{i-1} = \sum_{i=0}^{n-1} \frac{1}{i+1} \binom{n-1}{i}.$$

We have $f(n) = g_n(1)$, where $g_n(x) = \sum_{i=0}^{n-1} \frac{x^{i+1}}{i+1} \binom{n-1}{i}$. Note that from $g_n'(x) = \sum_{i=0}^{n-1} \binom{n-1}{i} x^i = (1+x)^{n-1}$ and $g_n(0) = 0$ it follows that $g_n(x) = \frac{1}{n}((1+x)^n - 1)$, so $f(n) = \frac{1}{n}(2^n - 1)$. We get

$$a_n = \sum_{d \mid n} \frac{\phi(d)}{d} \cdot \frac{1}{n/d} (2^{n/d} - 1) = \frac{1}{n} \sum_{d \mid n} \phi(d)(2^{n/d} - 1) = \frac{1}{n} \sum_{d \mid n} \phi(d) 2^{n/d} - 1.$$

(We have $\frac{1}{n} \sum_{d \mid n} \phi(d) = \frac{1}{n} \cdot n = 1$.) □

Remark. This problem was inspired by an easier problem, which Gigel Militaru, from the Faculty of Mathematics and Informatics, University of Bucharest, proposed to his students. In his problem X was an arbitrary nonempty set and M_X was the set of all functions $f \in F_X$ such that the only subset $Y \neq \emptyset$ of X such that $f(Y) \subseteq Y$ is $Y = X$. It turns out that if X is infinite then $M_X = \emptyset$. And if $|X| = n < \infty$, then M_X consists of the cyclic permutations of length n, which are all conjugated to each other. So the answer for this problem is $|M_X/\sim| = 0$ if X is infinite and $|M_X/\sim| = 1$ if X is finite.

540. For any matrix M, denote $M^* = \overline{M}^t$ the transpose conjugate of M. Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be such that $A^*B = O_n$. Prove that

$$\operatorname{rank}(A^*A + B^*B) \leq \operatorname{rank}(AA^* + BB^*).$$

`Proposed by Mihai Opincariu, Brad, Romania, and Vasile Pop, Technical University of Cluj-Napoca, Romania.`

Solution by the authors. It is known that for any matrix M, the following equalities take place:

$$\operatorname{rank} M = \operatorname{rank} M^* = \operatorname{rank}(MM^*) = \operatorname{rank}(M^*M).$$

(Since M and M^*M have the same number of columns, to prove that they have the same rank it is enough to show that $\ker M = \ker M^*M$. The '$\subseteq$' inclusion is trivial. Conversely, if $M^*MX = 0$, then also $0 = X^*M^*MX = (MX)^*(MX) = |MX|^2$, so $MX = 0$. (If $Y := MX$ is the column vector $(b_1, \ldots, b_m)^t$, then $Y^*Y = |Y|^2 := |b_1|^2 + \cdots + |b_m|^2$, which is 0 iff $Y = 0$.) Similarly $\operatorname{rank} M^* = \operatorname{rank} MM^*$. And $\operatorname{rank} M = \operatorname{rank} M^*$ is trivial.)

Note that $A^*B = O_n$ implies $B^*A = (A^*B)^* = O_n^* = O_n$. It follows that

$$(A+B)^*(A+B) = (A^*+B^*)(A+B) = A^*A+A^*B+B^*A+B^*B = A^*A+B^*B,$$

which implies

$$\operatorname{rank}(A^*A + B^*B) = \operatorname{rank}\big((A+B)^*(A+B)\big) = \operatorname{rank}(A+B). \tag{1}$$

Next, let $M = \begin{pmatrix} A & B \end{pmatrix} \in \mathcal{M}_{n,2n}(\mathbb{C})$. Then $M^* = \begin{pmatrix} A^* \\ B^* \end{pmatrix} \in \mathcal{M}_{2n,n}(\mathbb{C})$ and we have

$$MM^* = AA^* + BB^* \text{ and } M^*M = \begin{pmatrix} A^*A & A^*B \\ B^*A & B^*B \end{pmatrix} = \begin{pmatrix} A^*A & O_n \\ O_n & B^*B \end{pmatrix}.$$

Since $\operatorname{rank}(MM^*) = \operatorname{rank}(M^*M)$, we obtain that

$$\operatorname{rank}(AA^* + BB^*) = \operatorname{rank}(A^*A) + \operatorname{rank}(B^*B) = \operatorname{rank} A + \operatorname{rank} B. \tag{2}$$

Since $\operatorname{rank}(A+B) \leq \operatorname{rank} A + \operatorname{rank} B$, the required inequality follows from (1) and (2).

Remark. Relation (2) can be obtained using an alternative approach. Since AA^* and BB^* are Hermitian, they are diagonalizable. Moreover, $(AA^*)(BB^*) = A(A^*B)B^* = O_n$ and $(BB^*)(AA^*) = B(B^*A)A^* = O_n$, so AA^* and BB^* commute, which implies that they are simultaneously diagonalizable. Therefore, there exists some basis with respect to which we have $AA^* = \operatorname{diag}[a_1, \dots, a_n]$ and $BB^* = \operatorname{diag}[b_1, \dots, b_n]$, hence $AA^* + BB^* = \operatorname{diag}[a_1 + b_1, \dots, a_n + b_n]$. Also, $(AA^*)(BB^*) = O_n$ leads to $a_i b_i = 0$, for all $i = 1, \dots, n$. From here, it is easy to check that the number of non-zero elements of $AA^* + BB^*$ is equal to the sum of the number of non-zero elements of AA^* and the number of non-zero elements of BB^*. This is enough to justify (2).

Solution by Moubinool Omarjee, Lycée Henri IV, Paris, France. We use the Frobenius inequality, $\operatorname{rank}(XYZ) + \operatorname{rank} Y \geq \operatorname{rank}(XY) + \operatorname{rank}(YZ)$, for $X = B^*$, $Y = AA^* + BB^*$, and $Z = A$. Since $A^*B = 0$, so also $B^*A = 0$, we have $XYZ = 0$, $XY = B^*BB^*$, and $YZ = AA^*A$. Then the Frobenius inequality writes as

$$0 + \operatorname{rank}(AA^* + BB^*) \geq \operatorname{rank}(B^*BB^*) + \operatorname{rank}(AA^*A).$$

But for every complex matrix X we have $\operatorname{rank} XX^*X = \operatorname{rank} X^*X$. It follows that

$$\operatorname{rank} AA^*A + \operatorname{rank} B^*BB^* = \operatorname{rank} A^*A + \operatorname{rank} B^*B \geq \operatorname{rank}(A^*A + B^*B),$$

which concludes the proof.

Editor's note. For the relation $\operatorname{rank} XX^*X = \operatorname{rank} X^*X$, note that X^*X is a Hermitian matrix, and so it is diagonalizable. This implies that X^*X and $(X^*X)^2 = X^*XX^*X$ have the same rank. Then from the inequalities $\operatorname{rank} X^*XX^*X \leq \operatorname{rank} XX^*X \leq \operatorname{rank} X^*X$ we get the claimed relation.

541. Calculate

$$\sum_{n=1}^{\infty} \frac{H_n H_{n+1}}{(2n+1)(2n+3)},$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the nth harmonic number.

`Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania.`

Solution by the authors. We prove that the series equals $\dfrac{\pi^2}{12}$.

We have

$$\begin{aligned}
\frac{H_nH_{n+1}}{(2n+1)(2n+3)} &= \frac{1}{2}\left(\frac{H_nH_{n+1}}{2n+1}-\frac{H_nH_{n+1}}{2n+3}\right)\\
&= \frac{1}{2}\left(\frac{H_n\left(H_n+\frac{1}{n+1}\right)}{2n+1}-\frac{\left(H_{n+1}-\frac{1}{n+1}\right)H_{n+1}}{2n+3}\right)\\
&= \frac{1}{2}\left(\frac{H_n^2}{2n+1}-\frac{H_{n+1}^2}{2n+3}\right)\\
&\quad+\frac{1}{2}\left(\frac{H_n}{(n+1)(2n+1)}+\frac{H_{n+1}}{(n+1)(2n+3)}\right)\\
&= \frac{1}{2}\left(\frac{H_n^2}{2n+1}-\frac{H_{n+1}^2}{2n+3}\right)\\
&\quad+\frac{1}{2}\left[H_n\left(\frac{2}{2n+1}-\frac{1}{n+1}\right)+H_{n+1}\left(\frac{1}{n+1}-\frac{2}{2n+3}\right)\right]\\
&= \frac{1}{2}\left(\frac{H_n^2}{2n+1}-\frac{H_{n+1}^2}{2n+3}\right)+\frac{H_n}{2n+1}-\frac{H_{n+1}}{2n+3}+\frac{H_{n+1}-H_n}{2(n+1)}\\
&= \frac{1}{2}\left(\frac{H_n^2}{2n+1}-\frac{H_{n+1}^2}{2n+3}\right)+\left(\frac{H_n}{2n+1}-\frac{H_{n+1}}{2n+3}\right)+\frac{1}{2(n+1)^2}.
\end{aligned}$$

We note that, with the exception of the last term, our sum telescopes. Hence we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{H_nH_{n+1}}{(2n+1)(2n+3)}\\
&= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{H_n^2}{2n+1}-\frac{H_{n+1}^2}{2n+3}\right)+\sum_{n=1}^{\infty}\left(\frac{H_n}{2n+1}-\frac{H_{n+1}}{2n+3}\right)+\sum_{n=1}^{\infty}\frac{1}{2(n+1)^2}\\
&= \frac{1}{2}\cdot\frac{H_1^2}{3}+\frac{H_1}{3}+\frac{1}{2}\left(\zeta(2)-1\right)=\frac{1}{6}+\frac{1}{3}+\frac{\zeta(2)}{2}-\frac{1}{2}=\frac{\zeta(2)}{2}=\frac{\pi^2}{12}.
\end{aligned}$$

This concludes the proof. □

Solution by Nandan Sai Dasireddy, Hyderabad, Telangana, India. By convention, we put $H_0 = 0$. For $n \geq 0$ we define A_n by the formula

$$A_n = \frac{H_nH_{n-1}}{2n+1}.$$

We have

$$\begin{aligned}A_n - A_{n+1} &= H_n\left(\frac{H_{n-1}}{2n+1} - \frac{H_{n+1}}{2n+3}\right)\\ &= H_n\left(\frac{H_{n+1} - \left(\frac{1}{n} + \frac{1}{n+1}\right)}{2n+1} - \frac{H_{n+1}}{2n+3}\right)\\ &= \frac{H_nH_{n+1}}{2n+1} - \frac{H_n\left(\frac{1}{n} + \frac{1}{n+1}\right)}{2n+1} - \frac{H_nH_{n+1}}{2n+3}\\ &= \frac{2H_nH_{n+1}}{(2n+1)(2n+3)} - \frac{H_n}{n} + \frac{H_n}{n+1}\\ &= \frac{2H_nH_{n+1}}{(2n+1)(2n+3)} - \frac{H_n}{n} + \frac{H_{n+1}}{n+1} - \frac{1}{(n+1)^2}.\end{aligned}$$

We sum from 1 to infinity. Since we have telescoping sums on both sides, we get

$$A_1 = 2\sum_{n=1}^{\infty}\frac{H_nH_{n+1}}{(2n+1)(2n+3)} - \frac{H_1}{1} - \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}.$$

But $A_1 = H_1H_0/3 = 0$ and $H_1/1 = 1$. Hence

$$\sum_{n=1}^{\infty}\frac{H_nH_{n+1}}{(2n+1)(2n+3)} = \frac{1}{2}\left(1 + \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{12}.$$

(Here we used the well-known Euler sum $\sum_{i=1}^{\infty} 1/n^2 = \zeta(2) = \pi^2/6$.)

542. Let $n \geq 2$ be an integer and let $f : \mathbb{R}^n \to \mathbb{R}$, given by $f(x_1, \ldots, x_n) = 0$ if $(x_1, \ldots, x_n) = (0, \ldots, 0)$ and

$$f(x_1, \ldots, x_n) = \frac{\sqrt[5]{x_1^4 \cdots x_n^4}}{\sqrt[3]{x_1^2 \cdots x_n^2} + (x_2 - x_1)^2 + (x_3 - x_1)^2 + \cdots + (x_n - x_1)^2}$$

otherwise.

Prove that:

(i) f is continuous at $(0, \ldots, 0)$.

(ii) f is Fréchet differentiable at $(0, \ldots, 0)$ if and only if $n \geq 8$.

`Proposed by Dumitru Popa, University of Constanța, Romania.`

Solution by the author. (i) We use the inequality

$$\sqrt[3]{x_1^2 \cdots x_n^2} + (x_2 - x_1)^2 + (x_3 - x_1)^2 + \cdots + (x_n - x_1)^2 \geq \sqrt[3]{x_1^2 \cdots x_n^2}$$

valid for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. It implies that if all $x_1, \ldots, x_n$ are $\neq 0$, then $0 \leq f(x_1, \ldots, x_n) \leq \frac{\sqrt[5]{x_1^4 \cdots x_n^4}}{\sqrt[3]{x_1^2 \cdots x_n^2}} = \sqrt[15]{x_1^2 \cdots x_n^2}$. If $x_i = 0$ for some i, then

$f(x_1,\dots,x_n)=0=\sqrt[15]{x_1^2\cdots x_n^2}$. Hence for all $(x_1,\dots,x_n)\in\mathbb{R}^n$ we have the double inequality $0\le f(x_1,\dots,x_n)\le\sqrt[15]{x_1^2\cdots x_n^2}$. By the squeeze theorem,

$$\lim_{(x_1,\dots,x_n)\to(0,\dots,0)} f(x_1,\dots,x_n)=0=f(0,\dots,0).$$

(ii) For every $1\le i\le n$ we have $f(0,\dots,0,x_i,,0,\dots,0)=0\quad\forall x_i\in\mathbb{R}$, so $\frac{\partial f}{\partial x_i}(0,\dots,0)=0$. Hence f is Fréchet differentiable at $(0,\dots,0)$ if and only if its differential at $(0,\dots,0)$ is zero, which is equivalent to $\lim\limits_{(x_1,\dots,x_n)\to(0,\dots,0)} g(x_1,\dots,x_n)=0$, where $g:\mathbb{R}^n\setminus\{(0,\dots,0)\}\to\mathbb{R}$ is given by

$$\begin{aligned} g(x_1,\dots,x_n)&=\frac{f(x_1,\dots,x_n)-f(0,\dots,0)-\sum_{k=1}^n\frac{\partial f}{\partial x_k}(0,\dots,0)x_k}{\sqrt{x_1^2+\cdots+x_n^2}}\\ &=\frac{f(x_1,\dots,x_n)}{\sqrt{x_1^2+\cdots+x_n^2}}\end{aligned}.$$

If $\lim\limits_{(x_1,\dots,x_n)\to(0,\dots,0)} g(x_1,\dots,x_n)=0$, then $\lim\limits_{k\to\infty} g\left(\frac{1}{k},\dots,\frac{1}{k}\right)=0$, i.e. $\lim\limits_{k\to\infty}\frac{1}{k^{\frac{2n}{15}-1}}=0$, which is equivalent to $n>\frac{15}{2}$. Since n is an integer, this means $n\ge 8$.

Conversely, let us suppose that $n\ge 8$. Let $(x_1,\dots,x_n)\in\mathbb{R}^n\setminus\{(0,\dots,0)\}$. If all $x_1,\dots,x_n$ are $\neq 0$, then, as seen in the proof of (i), $0\le f(x_1,\dots,x_n)\le\sqrt[15]{x_1^2\cdots x_n^2}$, so

$$0\le g(x_1,\dots,x_n)\le\frac{\sqrt[15]{x_1^2\cdots x_n^2}}{\sqrt{x_1^2+\cdots+x_n^2}}\le\frac{\sqrt[15]{x_1^2\cdots x_n^2}}{n\sqrt[2n]{x_1^2\cdots x_n^2}}=\frac{1}{n}(x_1^2\cdots x_n^2)^{\frac{2n-15}{30n}}.$$

Here we applied the AM-GM inequality to $x_1^2,\dots,x_n^2$. If $x_i=0$ for some i, then $g(x_1,\dots,x_n)=0=\frac{1}{n}(x_1^2\cdots x_n^2)^{\frac{2n-15}{30n}}$. Hence $0\le g(x_1,\dots,x_n)\le\frac{1}{n}\left(x_1^2\cdots x_n^2\right)^{\frac{2n-15}{30n}}$ for all $(x_1,\dots,x_n)\in\mathbb{R}^n\setminus\{(0,\dots,0)\}$. But $n\ge 8$, so $\dfrac{2n-15}{30n}>0$. It follows that $\lim\limits_{(x_1,\dots,x_n)\to(0,\dots,0)}\left(x_1^2\cdots x_n^2\right)^{\frac{2n-15}{30n}}=0$. By the squeeze theorem, we get $\lim\limits_{(x_1,\dots,x_n)\to(0,\dots,0)} g(x_1,\dots,x_n)=0$. □