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Series involving products of odd harmonic numbers

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Abstract. In this paper we calculate nonlinear series involving products of odd harmonic numbers O_n and O_{n+1}, where $O_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$, $n \geq 1$. Other series with harmonic numbers H_n and O_n are also calculated.

Keywords: De Doelder's series, generating functions, odd harmonic numbers, nth harmonic number, nonlinear harmonic series, Riemann zeta function.

MSC: 40A05, 40C10.

1. Introduction and the main results

Let $n \geq 1$ be an integer, let O_n be the *odd harmonic number* defined by

$$O_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1},$$

and let $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ be the nth harmonic number. In this paper we calculate nonlinear series involving products of odd harmonic numbers O_n and O_{n+1}, and series with products of O_n and H_n. This work is motivated by the results given in [3] and by [4, problems 3.105–3.108].

The main results of this paper are the following theorems.

Theorem 1. ***A mosaic of series with odd harmonic numbers O_n and O_{n+1}***

The following identities hold:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{O_n O_{n+1}}{n(n+1)} = \frac{3}{2}\zeta(2);$

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(b) $\displaystyle\sum_{n=1}^{\infty} \frac{O_n O_{n+1}}{n(n+1)^2} = \frac{7}{2}\zeta(2) - \frac{7}{4}\zeta(3) - \frac{45}{16}\zeta(4)$;

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{O_n O_{n+1}}{n^2(n+1)} = -\frac{5}{2}\zeta(2) + \frac{7}{4}\zeta(3) + \frac{45}{16}\zeta(4)$;

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{O_n O_{n+1}}{n^2(n+1)^2} = -6\zeta(2) + \frac{7}{2}\zeta(3) + \frac{45}{8}\zeta(4)$;

(e) $\displaystyle\sum_{n=1}^{\infty} (2n+1)\frac{O_n O_{n+1}}{n^2(n+1)^2} = \zeta(2)$;

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{O_n O_{n+1}}{n^3(n+1)^3} = 24\zeta(2) - \frac{35}{2}\zeta(3) - \frac{135}{8}\zeta(4)$.

Theorem 2. ***Series with harmonic numbers*** O_n***,*** H_n ***and*** H_{n+1}
The following identities hold:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{O_n H_{n+1}}{n(n+1)} = 6\ln 2 - 2\ln^2 2$;

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{O_n H_n}{n(n+1)} = -2\ln^2 2 + \zeta(2) + \frac{7}{4}\zeta(3)$.

The proofs of Theorems 1 and 2 are based on the fact that the general terms of the series telescope. Before we prove the theorems we collect some results we need in our analysis.

Lemma 3. ***Series with the odd harmonic number*** O_n
The following equalities hold:

(a) *a* $\zeta(2)$ *series*

$$\sum_{n=1}^{\infty} \frac{O_n}{n(2n-1)} = \zeta(2).$$

(b) *another* $\zeta(2)$ *series*

$$\sum_{n=1}^{\infty} \frac{O_n}{n(2n+1)} = \frac{\zeta(2)}{2}.$$

(c) *an Euler series with* O_n

$$\sum_{n=1}^{\infty} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3).$$

(d) *De Doelder's quadratic series*

$$\sum_{n=1}^{\infty} \frac{O_n^2}{n^2} = \frac{45}{16}\zeta(4).$$

Proof. (a) The following generating function for the sequence $\left(\frac{O_n}{n}\right)_{n\geq 1}$ is known (see [1, p. 1197], [4, problem 3.65 (a), p. 94])

$$\frac{1}{4}\ln^2\left(\frac{1+x}{1-x}\right) = \sum_{n=1}^{\infty} \frac{O_n}{n} x^{2n}, \quad -1 < x < 1. \tag{1}$$

We have, based on (1), that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{O_n}{n(2n-1)} &= \sum_{n=1}^{\infty} \frac{O_n}{n} \int_0^1 x^{2n-2}\mathrm{d}x \\
&= \int_0^1 \frac{1}{x^2}\left(\sum_{n=1}^{\infty} \frac{O_n}{n} x^{2n}\right)\mathrm{d}x \\
&\overset{(1)}{=} \frac{1}{4}\int_0^1 \frac{\ln^2\left(\frac{1-x}{1+x}\right)}{x^2}\mathrm{d}x \\
&\overset{\frac{1-x}{1+x}=t}{=} \frac{1}{2}\int_0^1 \frac{\ln^2 t}{(1-t)^2}\mathrm{d}t \\
&= \frac{1}{2}\ln^2 t \cdot \frac{t}{1-t}\bigg|_0^1 - \int_0^1 \frac{\ln t}{1-t}\mathrm{d}t \\
&= -\int_0^1 \frac{\ln t}{1-t}\mathrm{d}t \\
&= -\int_0^1 \ln t \left(\sum_{i=0}^{\infty} t^i\right)\mathrm{d}t \\
&= -\sum_{i=0}^{\infty} \int_0^1 t^i \ln t \,\mathrm{d}t \\
&= \sum_{i=0}^{\infty} \frac{1}{(i+1)^2} \\
&= \zeta(2),
\end{aligned}$$

and part(a) of the lemma is proved.

(b) Part (b) of the lemma, which can be proved similarly to the technique given in part (a), is left as an exercise to the interested reader. The proofs of parts (c) and (d) can be found in [1, p. 1198].

We mention that the series formula in part (d) of Lemma 3, which is analogous with the famous Sandham-Yeung series [4, p. 245], is an identity due to de Doelder [2, entry (22), p. 138], which is corrected by Borwein in [1]. □

Now we are ready to prove Theorem 1.

Proof. (a) The proof of this *remarkable formula* is based on the fact that the series *telescopes*. We have

$$\begin{aligned}
\frac{O_nO_{n+1}}{n(n+1)} &= \frac{O_nO_{n+1}}{n} - \frac{O_nO_{n+1}}{n+1} \\
&= \frac{O_n\left(O_n + \frac{1}{2n+1}\right)}{n} - \frac{\left(O_{n+1} - \frac{1}{2n+1}\right)O_{n+1}}{n+1} \\
&= \frac{O_n^2}{n} - \frac{O_{n+1}^2}{n+1} + \frac{O_n}{n(2n+1)} + \frac{O_{n+1}}{(2n+1)(n+1)} \\
&= \frac{O_n^2}{n} - \frac{O_{n+1}^2}{n+1} + O_n\left(\frac{1}{n} - \frac{2}{2n+1}\right) + O_{n+1}\left(\frac{2}{2n+1} - \frac{1}{n+1}\right) \\
&= \frac{O_n^2}{n} - \frac{O_{n+1}^2}{n+1} + \frac{O_n}{n} - \frac{O_{n+1}}{n+1} + 2\frac{O_{n+1} - O_n}{2n+1} \\
&= \frac{O_n^2}{n} - \frac{O_{n+1}^2}{n+1} + \frac{O_n}{n} - \frac{O_{n+1}}{n+1} + \frac{2}{(2n+1)^2}.
\end{aligned} \tag{2}$$

It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{O_nO_{n+1}}{n(n+1)} &= \sum_{n=1}^{\infty}\left(\frac{O_n^2}{n} - \frac{O_{n+1}^2}{n+1}\right) + \sum_{n=1}^{\infty}\left(\frac{O_n}{n} - \frac{O_{n+1}}{n+1}\right) + \sum_{n=1}^{\infty}\frac{2}{(2n+1)^2} \\
&= 1 + 1 + \sum_{n=1}^{\infty}\frac{2}{(2n+1)^2} \\
&= 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \\
&= \frac{3}{2}\zeta(2) \\
&= \frac{\pi^2}{4}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\frac{O_nO_{n+1}}{n(n+1)^2} &= \frac{O_nO_{n+1}}{n(n+1)} - \frac{O_nO_{n+1}}{(n+1)^2} \\
&= \frac{O_nO_{n+1}}{n(n+1)} - \frac{\left(O_{n+1} - \frac{1}{2n+1}\right)O_{n+1}}{(n+1)^2} \\
&= \frac{O_nO_{n+1}}{n(n+1)} - \frac{O_{n+1}^2}{(n+1)^2} + \frac{O_{n+1}}{(2n+1)(n+1)^2} \\
&= \frac{O_nO_{n+1}}{n(n+1)} - \frac{O_{n+1}^2}{(n+1)^2} + \frac{O_{n+1}}{n+1}\left(\frac{2}{2n+1} - \frac{1}{n+1}\right) \\
&= \frac{O_nO_{n+1}}{n(n+1)} - \frac{O_{n+1}^2}{(n+1)^2} + \frac{2O_{n+1}}{(n+1)(2n+1)} - \frac{O_{n+1}}{(n+1)^2}.
\end{aligned}$$

It follows, based on part (a) of the theorem, and parts (a), (c) and (d) of Lemma 3, that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{O_nO_{n+1}}{n(n+1)^2} &= \sum_{n=1}^{\infty}\frac{O_nO_{n+1}}{n(n+1)} - \sum_{n=1}^{\infty}\frac{O_{n+1}^2}{(n+1)^2} \\
&\quad + 2\sum_{n=1}^{\infty}\frac{O_{n+1}}{(n+1)(2n+1)} - \sum_{n=1}^{\infty}\frac{O_{n+1}}{(n+1)^2} \\
&= \frac{3}{2}\zeta(2) - \left(\frac{45}{16}\zeta(4) - 1\right) + 2(\zeta(2) - 1) - \left(\frac{7}{4}\zeta(3) - 1\right) \\
&= \frac{7}{2}\zeta(2) - \frac{7}{4}\zeta(3) - \frac{45}{16}\zeta(4).
\end{aligned}$$

(c) We have

$$\begin{aligned}
\frac{O_nO_{n+1}}{n^2(n+1)} &= \frac{O_nO_{n+1}}{n^2} - \frac{O_nO_{n+1}}{n(n+1)} \\
&= \frac{O_n\left(O_n + \frac{1}{2n+1}\right)}{n^2} - \frac{O_nO_{n+1}}{n(n+1)} \\
&= \frac{O_n^2}{n^2} + \frac{O_n}{(2n+1)n^2} - \frac{O_nO_{n+1}}{n(n+1)} \\
&= \frac{O_n^2}{n^2} + \frac{O_n}{n}\left(\frac{1}{n} - \frac{2}{2n+1}\right) - \frac{O_nO_{n+1}}{n(n+1)} \\
&= \frac{O_n^2}{n^2} + \frac{O_n}{n^2} - \frac{2O_n}{n(2n+1)} - \frac{O_nO_{n+1}}{n(n+1)}.
\end{aligned}$$

It follows, based on part (a) of the theorem and parts (b), (c) and (d) of Lemma 3, that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{O_nO_{n+1}}{n^2(n+1)} &= \sum_{n=1}^{\infty}\frac{O_n^2}{n^2}+\sum_{n=1}^{\infty}\frac{O_n}{n^2}-2\sum_{n=1}^{\infty}\frac{O_n}{n(2n+1)}-\sum_{n=1}^{\infty}\frac{O_nO_{n+1}}{n(n+1)}\\ &= \frac{45}{16}\zeta(4)+\frac{7}{4}\zeta(3)-\zeta(2)-\frac{3}{2}\zeta(2)\\ &= -\frac{5}{2}\zeta(2)+\frac{7}{4}\zeta(3)+\frac{45}{16}\zeta(4).\end{aligned}$$

(d) Since

$$\frac{O_nO_{n+1}}{n^2(n+1)^2}=\frac{O_nO_{n+1}}{n^2(n+1)}-\frac{O_nO_{n+1}}{n(n+1)^2},$$

the result follows by subtracting the series in parts (c) and (b) of Theorem 1.

(e) Since

$$(2n+1)\frac{O_nO_{n+1}}{n^2(n+1)^2}=\frac{O_nO_{n+1}}{n^2(n+1)}+\frac{O_nO_{n+1}}{n(n+1)^2},$$

the result follows by adding the series in parts (c) and (b) of Theorem 1.

(f) We have

$$\begin{aligned}\frac{O_nO_{n+1}}{n^3(n+1)^3} &= O_nO_{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right)^3\\ &= O_nO_{n+1}\left[\frac{1}{n^3}-3\frac{1}{n(n+1)}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{1}{(n+1)^3}\right]\\ &= \frac{O_nO_{n+1}}{n^3}-3\frac{O_nO_{n+1}}{n^2(n+1)}+3\frac{O_nO_{n+1}}{n(n+1)^2}-\frac{O_nO_{n+1}}{(n+1)^3}\\ &= \frac{O_n\left(O_n+\frac{1}{2n+1}\right)}{n^3}-3\frac{O_nO_{n+1}}{n^2(n+1)}+3\frac{O_nO_{n+1}}{n(n+1)^2}\\ &\quad -\frac{\left(O_{n+1}-\frac{1}{2n+1}\right)O_{n+1}}{(n+1)^3}\\ &= \frac{O_n^2}{n^3}-\frac{O_{n+1}^2}{(n+1)^3}+\frac{O_n}{n^3(2n+1)}+\frac{O_{n+1}}{(2n+1)(n+1)^3}\\ &\quad -3\frac{O_nO_{n+1}}{n^2(n+1)}+3\frac{O_nO_{n+1}}{n(n+1)^2}.\end{aligned} \tag{3}$$

Since

$$\frac{1}{n^3(2n+1)}=\frac{1}{n^3}-\frac{2}{n^2}+\frac{4}{n}-\frac{8}{2n+1},$$

we have that

$$\frac{O_n}{n^3(2n+1)} = \frac{O_n}{n^3} - \frac{2O_n}{n^2} + \frac{4O_n}{n} - \frac{8O_n}{2n+1}.$$

Similarly,

$$\frac{1}{(2n+1)(n+1)^3} = \frac{8}{2n+1} - \frac{4}{n+1} - \frac{2}{(n+1)^2} - \frac{1}{(n+1)^3}$$

and we have

$$\frac{O_{n+1}}{(2n+1)(n+1)^3} = \frac{8O_{n+1}}{2n+1} - \frac{4O_{n+1}}{n+1} - \frac{2O_{n+1}}{(n+1)^2} - \frac{O_{n+1}}{(n+1)^3}.$$

It follows that

$$\begin{aligned} &\frac{O_n}{n^3(2n+1)} + \frac{O_{n+1}}{(2n+1)(n+1)^3} \\ &= \frac{O_n}{n^3} - \frac{O_{n+1}}{(n+1)^3} + 4\left(\frac{O_n}{n} - \frac{O_{n+1}}{n+1}\right) - 2\frac{O_n}{n^2} - 2\frac{O_{n+1}}{(n+1)^2} + \frac{8}{(2n+1)^2}. \end{aligned} \tag{4}$$

We obtain, based on (3) and (4), that

$$\begin{aligned} \frac{O_nO_{n+1}}{n^3(n+1)^3} &= \frac{O_n^2}{n^3} - \frac{O_{n+1}^2}{(n+1)^3} + \frac{O_n}{n^3} - \frac{O_{n+1}}{(n+1)^3} + 4\left(\frac{O_n}{n} - \frac{O_{n+1}}{n+1}\right) \\ &\quad - 2\frac{O_n}{n^2} - 2\frac{O_{n+1}}{(n+1)^2} + \frac{8}{(2n+1)^2} - 3\frac{O_nO_{n+1}}{n^2(n+1)} + 3\frac{O_nO_{n+1}}{n(n+1)^2}. \end{aligned}$$

It follows, based on parts (b) and (c) of Theorem 1, and part (c) of Lemma 3, that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{O_nO_{n+1}}{n^3(n+1)^3} &= \sum_{n=1}^{\infty}\left(\frac{O_n^2}{n^3} - \frac{O_{n+1}^2}{(n+1)^3}\right) + \sum_{n=1}^{\infty}\left(\frac{O_n}{n^3} - \frac{O_{n+1}}{(n+1)^3}\right) \\ &\quad + 4\sum_{n=1}^{\infty}\left(\frac{O_n}{n} - \frac{O_{n+1}}{n+1}\right) - 2\sum_{n=1}^{\infty}\frac{O_n}{n^2} - 2\sum_{n=1}^{\infty}\frac{O_{n+1}}{(n+1)^2} \\ &\quad + \sum_{n=1}^{\infty}\frac{8}{(2n+1)^2} - 3\sum_{n=1}^{\infty}\frac{O_nO_{n+1}}{n^2(n+1)} + 3\sum_{n=1}^{\infty}\frac{O_nO_{n+1}}{n(n+1)^2} \\ &= 1 + 1 + 4 - \frac{7}{2}\zeta(3) - 2\left(\frac{7}{4}\zeta(3) - 1\right) + 8\left(\frac{3}{4}\zeta(2) - 1\right) \\ &\quad - 3\left(-\frac{5}{2}\zeta(2) + \frac{7}{4}\zeta(3) + \frac{45}{16}\zeta(4)\right) \\ &\quad + 3\left(\frac{7}{2}\zeta(2) - \frac{7}{4}\zeta(3) - \frac{45}{16}\zeta(4)\right) \\ &= 24\zeta(2) - \frac{35}{2}\zeta(3) - \frac{135}{8}\zeta(4), \end{aligned}$$

and Theorem 1 is proved. □

Now we give the proof of Theorem 2.

Proof. (a) We have

$$
\begin{aligned}
\frac{O_nH_{n+1}}{n(n+1)} &= \frac{O_nH_{n+1}}{n} - \frac{O_nH_{n+1}}{n+1} \\
&= \frac{O_n\left(H_n + \frac{1}{n+1}\right)}{n} - \frac{\left(O_{n+1} - \frac{1}{2n+1}\right)H_{n+1}}{n+1} \\
&= \frac{O_nH_n}{n} - \frac{O_{n+1}H_{n+1}}{n+1} + \frac{O_n}{n(n+1)} + \frac{H_{n+1}}{(n+1)(2n+1)} \\
&= \frac{O_nH_n}{n} - \frac{O_{n+1}H_{n+1}}{n+1} + \frac{O_n}{n} - \frac{O_n}{n+1} + \frac{H_{n+1}}{(n+1)(2n+1)} \\
&= \frac{O_nH_n}{n} - \frac{O_{n+1}H_{n+1}}{n+1} + \frac{O_n}{n} - \frac{O_{n+1}}{n+1} + \frac{1}{(2n+1)(n+1)} \\
&\quad + \frac{H_{n+1}}{(n+1)(2n+1)}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{O_nH_{n+1}}{n(n+1)} &= \sum_{n=1}^{\infty}\left(\frac{O_nH_n}{n} - \frac{O_{n+1}H_{n+1}}{n+1}\right) + \sum_{n=1}^{\infty}\left(\frac{O_n}{n} - \frac{O_{n+1}}{n+1}\right) \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{(2n+1)(n+1)} + \sum_{n=1}^{\infty} \frac{H_{n+1}}{(n+1)(2n+1)} \\
&= 1 + 1 + \sum_{i=2}^{\infty} \frac{1}{i(2i-1)} + \sum_{i=2}^{\infty} \frac{H_i}{i(2i-1)} \\
&= \sum_{i=1}^{\infty} \frac{1}{i(2i-1)} + \sum_{i=1}^{\infty} \frac{H_i}{i(2i-1)}.
\end{aligned}
\tag{5}
$$

A calculation shows that $\sum\limits_{i=1}^{\infty} \frac{1}{i(2i-1)} = \ln 4$. Now we calculate the harmonic series $\sum\limits_{i=1}^{\infty} \frac{H_i}{i(2i-1)}$.

We need the following power series formula (see [4, entry (11.9), p. 403])

$$
\sum_{n=1}^{\infty} \frac{H_n}{n} x^n = \mathrm{Li}_2(x) + \frac{1}{2}\ln^2(1-x), \quad x \in [-1, 1), \tag{6}
$$

where Li_2 denotes the Dilogarithm function defined, for $|z| \leq 1$, by

$$
\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\ln(1-t)}{t}\mathrm{d}t.
$$

The function $\mathrm{Li}_2(x)+\frac{1}{2}\ln^2(1-x)$, which appears in the right hand side of (6), is known as the generating function for the harmonic sequence $\left(\frac{H_n}{n}\right)_{n\geq 1}$. Letting $x=t^2$ in (6), we obtain that

$$\sum_{n=1}^{\infty}\frac{H_n}{n}t^{2n}=\mathrm{Li}_2(t^2)+\frac{1}{2}\ln^2(1-t^2),\quad t\in(-1,1).$$

Dividing by t^2 both sides of the previous equality and integrating from 0 to 1 we have that

$$\sum_{n=1}^{\infty}\frac{H_n}{n(2n-1)}=\int_0^1\frac{\mathrm{Li}_2(t^2)}{t^2}\mathrm{d}t+\frac{1}{2}\int_0^1\frac{\ln^2(1-t^2)}{t^2}\mathrm{d}t. \tag{7}$$

We calculate the first integral in (7).

$$\begin{aligned}\int_0^1\frac{\mathrm{Li}_2(t^2)}{t^2}\mathrm{d}t&=\int_0^1\frac{1}{t^2}\left(\sum_{n=1}^{\infty}\frac{t^{2n}}{n^2}\right)\mathrm{d}t\\&=\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 t^{2n-2}\mathrm{d}t\\&=\sum_{n=1}^{\infty}\frac{1}{n^2(2n-1)}\\&=2\sum_{n=1}^{\infty}\frac{1}{n(2n-1)}-\sum_{n=1}^{\infty}\frac{1}{n^2}\\&=2\ln 4-\zeta(2).\end{aligned}$$

We calculate the second integral in (7). We integrate by parts, with $f(t)=\ln^2(1-t^2)$, $f'(t)=-\frac{4t}{1-t^2}\ln(1-t^2)$, $g'(t)=\frac{1}{t^2}$, $g(t)=\frac{t-1}{t}$, and we have that

$$\begin{aligned}\int_0^1\frac{\ln^2(1-t^2)}{t^2}\mathrm{d}t&=\frac{t-1}{t}\ln^2(1-t^2)\Big|_0^1-4\int_0^1\frac{\ln(1-t^2)}{1+t}\mathrm{d}t\\&=-4\int_0^1\frac{\ln(1-t^2)}{1+t}\mathrm{d}t\\&=-4\int_0^1\frac{\ln(1-t)}{1+t}\mathrm{d}t-4\int_0^1\frac{\ln(1+t)}{1+t}\mathrm{d}t\\&=-4\left(\frac{\ln^2 2}{2}-\frac{\pi^2}{12}\right)-2\ln^2 2\\&=\frac{\pi^2}{3}-4\ln^2 2.\end{aligned}$$

We used that $\int_0^1 \frac{\ln(1-t)}{1+t}\mathrm{d}t = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}$. We prove this formula by direct computation. We have

$$\begin{aligned}
\int_0^1 \frac{\ln(1-t)}{1+t}\mathrm{d}t &\overset{1-t=x}{=} \int_0^1 \frac{\ln x}{2-x}\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \frac{\ln x}{1-\frac{x}{2}}\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \ln x \left(\sum_{n=0}^{\infty} \frac{x^n}{2^n}\right)\mathrm{d}x \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2^n}\int_0^1 x^n \ln x \,\mathrm{d}x \\
&= -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+1)^2} \\
&= -\mathrm{Li}_2\left(\frac{1}{2}\right).
\end{aligned}$$

Now the result follows since $\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}$. Putting all these together we get the formula

$$\sum_{i=1}^{\infty} \frac{H_i}{i(2i-1)} = 4\ln 2 - 2\ln^2 2. \tag{8}$$

Combining (5) and (8) we have that part (a) of Theorem 2 is proved.

(b) Similarly as in the proof of part (a) one can show that

$$\frac{O_n H_n}{n(n+1)} = \frac{O_n H_n}{n} - \frac{O_{n+1}H_{n+1}}{n+1} + \frac{O_{n+1}}{(n+1)^2} + \frac{H_{n+1}}{(2n+1)(n+1)} - \frac{1}{(n+1)^2(2n+1)}.$$

This implies, based on part (c) of Lemma 3 and equality (8), that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{O_n H_n}{n(n+1)} &= \sum_{n=1}^{\infty}\left(\frac{O_n H_n}{n} - \frac{O_{n+1}H_{n+1}}{n+1}\right) + \sum_{n=1}^{\infty} \frac{O_{n+1}}{(n+1)^2} \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{H_{n+1}}{(2n+1)(n+1)} - \frac{1}{(n+1)^2(2n+1)}\right) \\
&= 1 + \left(\frac{7}{4}\zeta(3) - 1\right) + \sum_{i=1}^{\infty}\left(\frac{H_i}{i(2i-1)} - \frac{1}{i^2(2i-1)}\right) \\
&= \frac{7}{4}\zeta(3) + 4\ln 2 - 2\ln^2 2 - 4\ln 2 + \zeta(2) \\
&= \frac{7}{4}\zeta(3) - 2\ln^2 2 + \zeta(2).
\end{aligned}$$

An alternative proof of part (b) is based on observing that

$$\frac{O_nH_{n+1}}{n(n+1)} - \frac{O_nH_n}{n(n+1)} = \frac{O_n}{n(n+1)^2}$$
$$= \frac{O_n}{n} - \frac{O_{n+1}}{n+1} - \frac{O_{n+1}}{(n+1)^2} + \frac{1}{(2n+1)(n+1)} + \frac{1}{(2n+1)(n+1)^2}$$

and then by using part (a) of the theorem. □

We mention the alternating version of the series in part (a) of Theorem 1

$$\sum_{n=1}^{\infty}(-1)^n\frac{O_nO_{n+1}}{n(n+1)} = -\frac{3}{4}\zeta(2) - \frac{7}{8}\zeta(3) + 2\mathrm{G},$$

where $\mathrm{G} = \sum\limits_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$ denotes the Catalan constant.

The above formula can be proved based on equality (2) and the series formulae

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{O_n}{n} = -\frac{3}{8}\zeta(2);$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{O_n^2}{n} = -\frac{7}{16}\zeta(3).$

The formula in part (a) can be proved, this may be challenging!, by using that $O_n = \int_0^1 \frac{1-x^{2n}}{1-x^2}\mathrm{d}x$, or it follows directly from the first formula in section 4.1 in [2, p. 136] with $x = 1$, and the alternating series in part (b) is [2, entry (21), p. 137].

We leave, as a *challenge*, to the interested reader the calculation of the alternating version of the series in Theorems 1 and 2.

We also mention the *open problem* of calculating the quadratic and the cubic series involving the odd harmonic number O_n

$$\sum_{n=1}^{\infty}\frac{O_n^2}{n^3},\quad \sum_{n=1}^{\infty}\frac{O_n^3}{n^2}\quad \text{and}\quad \sum_{n=1}^{\infty}\frac{O_n^3}{n^3}.$$

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Evaluating divergent integrals

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Abstract. We show that many divergent integrals can be assigned reasonable values in a consistent manner.

Keywords: Improper integrals, divergent integrals, generalized functions, Fourier transforms.

MSC: 26A42, 30G99, 42A38, 44A10.

1. Introduction

Sometimes we can evaluate divergent integrals. More precisely, we can assign reasonable values to certain divergent integrals.

One well-known improper integral is

$$\int_0^\infty \frac{\sin x}{x} \mathrm{d}x = \frac{\pi}{2}.$$

This is a challenging integral, because the convergence is not directly visible. We can make it even more challenging if we remove x from the denominator. What happens is that we obtain the formula

$$\int_0^\infty \sin x \mathrm{d}x = 1. \tag{1}$$

The integral is clearly divergent, but the value 1 can be justified in a certain sense. We will discuss this very soon.

Also, we can use the Laplace transform formula

$$\int_0^\infty e^{-zx} \sin(xy) \mathrm{d}x = \frac{y}{y^2+z^2} \quad (y>0,\ z>0) \tag{2}$$

and let $z \to 0$. This gives symbolically

$$\int_0^\infty \sin(xy) \mathrm{d}x = \frac{1}{y} \tag{3}$$

and with $y=1$ we get (1).

The integral in (3) is clearly divergent, as the antiderivative of $\sin x$ does not have a limit at infinity. Moreover, passing to the limit when $z \to 0$ may not be legitimate.

Let us try another well-known integral from the popular table of Gradshteyn and Ryzhik [4, (3.761.4)]

$$\int_0^\infty \frac{\sin(xy)}{x^p} \mathrm{d}x = y^{p-1}\Gamma(1-p)\cos\frac{\pi p}{2} \quad (0<y,\ 0<p<1).$$

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Letting $p \to 0$ we recover formula (3). Here is yet another similar manipulation. Starting from the well-known and very important integral [4, (3.911.1)]

$$\int_0^\infty \frac{\sin(xy)}{e^{ax}+1}\mathrm{d}x = \frac{1}{2y} - \frac{\pi}{2a\sinh\frac{\pi y}{a}} \qquad (a, y > 0)$$

with $a \to 0$ we confirm (3) as $\lim\limits_{a\to 0^+} \frac{\pi y}{a\sinh(\pi y/a)} = 0$.

Is this a coincidence? Obviously, there is some hidden truth in (3)!

One way to explain the evaluation (3) is to look at the function

$$f(z) = \frac{y}{y^2+z^2}$$

where $y > 0$ is fixed. This function is analytic for $\operatorname{Re} z > 0$ and has the integral representation (2). The function is also analytic on the open disk $|z| < y$ and we may assume that $f(z)$ represents the integral in (2) on this disk. In this sense the equation$f(0) = 1/y$ assigns value $1/y$ to the integral in (2). Moving in this direction we can take also $y = 1$ and $z = -i/2$ to get

$$\int_0^\infty e^{i\frac{x}{2}} \sin x \,\mathrm{d}x = \frac{4}{3} \tag{4}$$

and we will see in Section 3 that this evaluation makes good sense.

Even more curious is the (symbolic) evaluation

$$\int_0^\infty x\sin(xy)\mathrm{d}x = 0 \tag{5}$$

resulting from (2) after differentiating with respect to z and then letting $z \to 0$.

We have a similar situation with some divergent series. For example, the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s},$$

defined for $\operatorname{Re} s > 1$ has analytic extension on the entire complex plane with a simple pole at $s = 1$ and we can write symbolically

$$1 + 1 + 1 + \cdots = \zeta(0) = -\frac{1}{2},$$

a shocking equation that appears often in mathematical puzzles. The situation with divergent series is so interesting that Hardy wrote a book on this topic [5].

Accepting the symbolic evaluation (3) we can make further steps and manipulate it. For example, a bold differentiation with respect to y in (3) gives the interesting symbolic equation

$$\int_0^\infty x\cos(xy)\mathrm{d}x = -\frac{1}{y^2}. \tag{6}$$

We can confirm (6) also from another direction by differentiating the Laplace transform of the cosine function

$$\int_0^\infty e^{-zx}\cos(xy)\mathrm{d}x = \frac{z}{y^2+z^2} \tag{7}$$

with respect to z. This gives

$$\int_0^\infty e^{-zx}x\cos(xy)\mathrm{d}x = \frac{z^2-y^2}{(y^2+z^2)^2}$$

and then letting $z \to 0$ leads to (6).

Next we will explain these equations in a consistent mathematical manner.

2. Revealing the mystery

In one sentence: We are performing Fourier transforms of generalized functions. Now some definitions and details are needed to clarify the general picture.

A generalized function (also distribution) is a continuous linear functional on an appropriate space of basic functions. For example, we can take the topological linear space S of all infinitely differentiable functions $\varphi(t)$ on $\mathbb{R}$ satisfying

$$M_{p,k}(\varphi) = \sup\{(1+|t|^p)\,|\varphi^{(k)}(t)|,\ t\in\mathbb{R}\} < \infty$$

for all non-negative integers p, k (the Schwartz space). The topology on S is defined by the seminorms $M_{p,k}$. The space of continuous linear functionals on S (the distributions) is denoted by S'. Appropriate functions $g(t)$ on $\mathbb{R}$ define distributions in S' by

$$\varphi \mapsto (g,\varphi) \equiv \int_{-\infty}^\infty g(t)\,\varphi(t)\mathrm{d}t.$$

For convenience we will identify this distribution with g.

Another approach to generalized functions is this: suppose $f(t)$ is a bounded function on the real line and $g(t)$ is an almost everywhere continuous function on $\mathbb{R}$ with moderate growth (at most polynomial). Consider sequences of functions $g_n \in L^1(\mathbb{R})$, $g_n \to g$ pointwise, such that

$$\lim_{n\to\infty}\int_{-\infty}^\infty g_n(t)f(t)\mathrm{d}t = (g,f)$$

exists and is the same for any such sequence. This way $g(t)$ defines a generalized function which we again denote by $g(t)$. For details see [1], [3], [6].

A popular distribution is the usual step function

$$u(t) = \begin{cases} 0 & t<0 \\ 1 & t\ge 0 \end{cases}$$

together with its shifts $u_a(t) = u(t-a)$. Another well-known distribution is the Dirac delta function $\delta(t)$ with shifts $\delta_a(t) = \delta(t-a)$. Recall that $\delta(t-a) = 0$ for $t \neq a$ and symbolically $\delta(0) = \infty$. The action of $\delta_a(t)$ is defined by

$$(\delta_a, f) = \int_{-\infty}^{\infty} \delta_a(t) f(t) \mathrm{d}t = \int_{-\infty}^{\infty} \delta(t-a) f(t) \mathrm{d}t = f(a).$$

Now we introduce the Fourier transform for functions $f \in L^1(\mathbb{R})$

$$\begin{aligned} F\{f\}(y) &= \int_{-\infty}^{\infty} e^{ixy} f(x) \mathrm{d}x = \int_{-\infty}^{\infty} \left(f(x)\cos(xy) + i f(x)\sin(xy)\right) \mathrm{d}x \\ &= \int_{-\infty}^{\infty} f(x)\cos(xy)\mathrm{d}x + i \int_{-\infty}^{\infty} f(x)\sin(xy)\mathrm{d}x. \end{aligned} \tag{8}$$

From (8) we can write

$$\begin{aligned} F\{u(x)f(x)\}(y) &= \int_0^{\infty} f(x)\cos(xy)\mathrm{d}x + i\int_0^{\infty} f(x)\sin(xy)\mathrm{d}x \\ &= F_c\{f\}(y) + iF_s\{f\}(y), \end{aligned} \tag{9}$$

where the two integrals are the Fourier cosine and sine transforms of $f(x)$.

Clearly, $S \subset L^1(\mathbb{R})$. We also have $F\{\varphi\} \in S$ for $\varphi \in S$. The Fourier transform extends to distributions in the following way. Let $g \in S'$ be a distribution on the set of basic functions S and let $\varphi \in S$. Then $F\{g\}$ is the distribution acting by $(F\{g\}, \varphi) = (g, F\{\varphi\})$.

The book [2] contains a table of Fourier transforms of distributions. Entry 195 in [2] says that for $y > 0$

$$F\{u(x)\}(y) = \int_0^{\infty} \cos(xy)\mathrm{d}x + i\int_0^{\infty} \sin(xy)\mathrm{d}x = \pi\delta(y) + \frac{i}{y},$$

and comparing real and imaginary parts we find (in the sense of distributions)

$$\begin{aligned} F_c\{u(x)\}(y) &= \int_0^{\infty} \cos(xy)\mathrm{d}x = \pi\delta(y), \\ F_s\{u\}(y) &= \int_0^{\infty} \sin(xy)\mathrm{d}x = \frac{1}{y}, \end{aligned} \tag{10}$$

where the second integral is exactly equation (3). The first equation gives for $y > 0$

$$\int_0^{\infty} \cos(xy)\mathrm{d}x = 0.$$

Entry 203 in [2] states that

$$F\{xu(x)\}(y) = \int_0^{\infty} x\cos(xy)\mathrm{d}x + i\int_0^{\infty} x\sin(xy)\mathrm{d}x = -\frac{1}{y^2} - i\pi\delta'(y)$$

and from here

$$\int_0^\infty x\cos(xy)\mathrm{d}x = -\frac{1}{y^2}, \quad \int_0^\infty x\sin(xy)\mathrm{d}x = -\pi\,\delta'(y).$$

The first equation is (6) and the second reads as (5), since $\delta'(y) = 0$ for $y > 0$. We use the definition

$$\delta'(y) = \lim_{h\to 0}\frac{\delta(y+h) - \delta(y)}{h}$$

and $\delta'(y)$ is known as the "unit doublet".

Although the theory of distributions is good and consistent, equations like (1) may cause cognitive dissonance. If we really have the number 1 on the right hand side in (1), then we need to have something that has value 1 on the left hand side. However, the integral in (1) is really divergent. We may assume this integral is just a symbol which is 1 by definition. Like $0! = 1$.

We also see that equations (3), (4), (5) have a better "raison d'être" compared to the symbolic divergent series for $\zeta(0)$.

3. More strange integrals

Entry 466 in [2] reads

$$\int_0^\infty \sin(ax)\,e^{ixy}\mathrm{d}x = \frac{a}{a^2 - y^2} - i\frac{\pi}{2}\left[\delta(y+a) - \delta(y-a)\right]. \tag{11}$$

Separating real and imaginary parts gives

$$\int_0^\infty \sin(ax)\cos(xy)\mathrm{d}x = \frac{a}{a^2 - y^2}, \tag{12}$$

$$\int_0^\infty \sin(ax)\sin(xy)\mathrm{d}x = -\frac{\pi}{2}\left[\delta(y+a) - \delta(y-a)\right]. \tag{13}$$

Setting $a = 1, y = 1/2$ in (11) we get the evaluation (4) obtained before in a different way.

Now we turn to [2, (204)]. For every integer $n \geq 0$

$$\int_0^\infty x^n e^{ixy}\mathrm{d}x = i^{n+1}\frac{n!}{y^{n+1}} + (-i)^n\pi\,\delta^{(n)}(y).$$

Yes, we have such things like $\delta^{(n)}$!. The cases $n = 0, 1$ were considered at the end of Section 2. It is convenient to separate the cases when n is even or odd. Euler's formula

$$e^{ixy} = \cos(xy) + i\sin(xy)$$

yields, after separating real and imaginary parts,

$$\begin{aligned}\int_0^\infty x^{2n}\cos(xy)\mathrm{d}x &= (-1)^n\pi\,\delta^{(2n)}(y),\\ \int_0^\infty x^{2n}\sin(xy)\mathrm{d}x &= (-1)^n\frac{(2n)!}{y^{2n+1}},\end{aligned} \tag{14}$$

$$\begin{aligned}\int_0^\infty x^{2n+1}\cos(xy)\mathrm{d}x &= (-1)^{n+1}\frac{(2n+1)!}{y^{2n+2}},\\ \int_0^\infty x^{2n+1}\sin(xy)\mathrm{d}x &= (-1)^{n+1}\pi\delta^{(2n+1)}(y),\end{aligned} \tag{15}$$

where $\delta^{(n)}(y) = 0$ for $y \neq 0$.

It is worth mentioning also some interesting formulas from [2] representing Laplace transforms of generalized functions. The Laplace transform is defined by

$$L\{f(x)\}(y) = \int_0^\infty f(x)\, e^{-xy}\mathrm{d}x,\ y > 0,$$

for functions f of at most exponential growth at infinity. According to entry 742 in [2] we have for every integer $n \geq 0$

$$\int_0^\infty \frac{1}{x^{n+1}}\, e^{-xy}\mathrm{d}x = (-1)^{n+1}\frac{y^n}{n!}\left[\ln y - \psi(n+1)\right], \tag{16}$$

where $\psi(z) = \frac{d}{dz}\ln\Gamma(z)$ is the digamma function. The integral in (16) is "very" divergent, but the formula is valuable. For $n = 0$ it becomes

$$\int_0^\infty \frac{e^{-xy}}{x}\,\mathrm{d}x = -\ln y - \gamma, \tag{17}$$

with $\gamma = -\psi(1)$ being Euler's constant. Differentiating (17) with respect to y gives the obviously correct result

$$\int_0^\infty e^{-xy}\mathrm{d}x = \frac{1}{y}.$$

Another curious entry in [2] is 787

$$\int_0^\infty \frac{\cos(ax)}{x}\, e^{-xy}\mathrm{d}x = -\frac{1}{2}\ln(y^2+a^2) - \gamma, \tag{18}$$

and the case $a = 0$ confirms (17). Differentiating (18) with respect to a leads to the well-known Laplace transform of the sine function

$$\int_0^\infty \sin(ax)\, e^{-xy}\mathrm{d}x = \frac{a}{a^2+y^2}.$$

At the same time the Laplace transform formula

$$\int_0^\infty \frac{\sin(ax)}{x}\, e^{-xy}\mathrm{d}x = \arctan\frac{a}{y}$$

compared to (18) is quite legitimate with an obediently convergent integral.

One divergent (but nice) integral similar to (18) appears in entry 771 of [2]

$$\int_0^\infty \frac{\ln x}{x}\, e^{-xy}\mathrm{d}x = \frac{\pi^2}{12} + \frac{1}{2}(\gamma + \ln y)^2. \tag{19}$$

This equation can be viewed as the result of "integration" with respect to y in the legitimate well-known formula [4, (4.331.1)]

$$\int_0^\infty \ln x\, e^{-xy} \mathrm{d}x = -\frac{1}{y}\,(\gamma + \ln y) \tag{20}$$

as differentiation with respect to y in (19) gives (20).

At the end we want to mention that not all divergent integrals can be evaluated by means of distributions or by somehow involving them in useful formulas. Most likely the integral

$$\int_0^\infty e^{x^5} \mathrm{d}x$$

cannot be evaluated like those above.

Important divergent integrals appearing in physics can be approached by the method of regularization. We will not discuss this method here. Instead, we direct the interested readers to the publication [7].

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16th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2022

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Abstract. The 16th South Eastern European Mathematical Olympiad for University Students (SEEMOUS 2022) took place between May 27 and June 1, 2022, in Palić, Serbia. We present the competition problems and their solutions, as given by the corresponding authors, members of the jury or contestants.

Keywords: Sylvester's inequality, functional equation, numerical series, spectral norm.

MSC: Primary 15A03; Secondary 15A24, 39B22, 40A05.

The Mathematical Society of South-Eastern Europe (MASSEE) has launched in 2022 the 16th South Eastern European Mathematical Competition for University Students with International Participation (SEEMOUS 2022), which is addressed to students in the first or second year of undergraduate studies, from universities in countries that are members of MASSEE or from invited countries that are not affiliated to MASSEE.

This year's competition was hosted by the Faculty of Mathematics of the University of Belgrade, Serbia, between May 27 and June 1, 2022, at the Student Resort in Palić. The number of students that participated in the contest was 61, representing 16 universities from Bulgaria, Greece, North Macedonia, Romania and Serbia. The jury awarded 7 gold medals, 12 silver medals and 21 bronze medals. The winner of the competition was Răzvan-Gabriel Petec from Babeş-Bolyai University, Cluj-Napoca, Romania.

We present the four problems from the contest and their solutions as given by the corresponding authors, members of the jury or contestants. The interested reader may also find alternative (or similar) solutions, by following the discussions on the *AoPS Online Community* website, see [1–4].

Problem 1. Let $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $AB^2A = AB$. Prove that:

(a) $(AB)^2 = AB$;
(b) $(AB - BA)^3 = O_n$.

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Authors' solution. Since $AB(BA - I_n) = O_n$, the Sylvester inequality for AB and $BA - I_n$ leads to

$$\operatorname{rank}(AB) + \operatorname{rank}(BA - I_n) \leq n + \operatorname{rank}\left(AB(BA - I_n)\right) = n. \qquad (1)$$

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It is true in general (see Remark 1) that

$$\operatorname{rank}(AB - I_n) = \operatorname{rank}(BA - I_n), \tag{2}$$

so (1) becomes

$$\operatorname{rank}(AB - I_n) + \operatorname{rank}(AB) \le n. \tag{3}$$

Moreover, $\ker(AB - I_n) \subseteq \operatorname{Im}(AB)$, leading to equality in the Sylvester inequality for $AB - I_n$ and AB:

$$\operatorname{rank}(AB - I_n) + \operatorname{rank}(AB) = n + \operatorname{rank}\Big((AB)^2 - AB\Big). \tag{4}$$

By combining (3) and (4), the conclusion (a) follows.

Next, using the identity from the hypothesis and (a), we obtain

$$\begin{aligned}(AB - BA)^2 &= (AB)^2 + (BA)^2 - AB^2A - BA^2B \\ &= (BA)^2 - BA^2B = -BA(AB - BA), \\ (AB - BA)^3 &= -BA(AB - BA)^2 = (BA)^2(AB - BA), \\ (AB - BA)^4 &= (BA)^2(AB - BA)^2 = -(BA)^3(AB - BA) \\ &= -B(AB)^2A(AB - BA) = -B(AB)A(AB - BA) \\ &= -(BA)^2(AB - BA) = -(AB - BA)^3,\end{aligned}$$

hence

$$(AB - BA)^4 = -(AB - BA)^3. \tag{5}$$

Let λ be any eigenvalue of $AB - BA$. Then $\lambda^4 = -\lambda^3$, by (5), so $\lambda \in \{0, -1\}$. Since $\operatorname{Tr}(AB - BA) = 0$, all the eigenvalues of $AB - BA$ must be 0, hence $(AB - BA)^n = O_n$, which by (5) leads to (b).

Alternative solution. *The following solution of* (a) *was given by Alexandru Buzea, from University Politehnica of Bucharest, and by Konstantinos Tsirkas, from National and Kapodistrian University of Athens (contestants).*

First, (3) is obtained as in the authors' solution. Next, denote by k_0 and k_1 the number of Jordan blocks of the matrix AB that correspond to the eigenvalues 0 and 1, respectively, and by J denote the Jordan canonical form of the matrix AB. Then $\operatorname{rank}(AB) = n - k_0$ and $AB = PJP^{-1}$, where $P \in \mathcal{M}_n(\mathbb{C})$ is some invertible matrix. It follows that $AB - I_n = P(J - I_n)P^{-1}$, hence $AB - I_n$ has k_1 Jordan blocks corresponding to the eigenvalue 0, so $\operatorname{rank}(AB - I_n) = n - k_1$.

Considering (3), we obtain that $n - k_0 + n - k_1 \le n$, so $k_0 + k_1 \ge n$. Consequently, the matrix AB is diagonalizable and has no eigenvalues outside $\{0, 1\}$, hence $J^2 = J$, which leads to $(AB)^2 = PJ^2P^{-1} = PJP^{-1} = AB$.

Alternative solution. *This follows the ideas from the solution of Minas Margaritis (contestant), from National and Kapodistrian University of Athens.*

Let $X = I_n - A^T B^T$ and $Y = I_n - B^T A^T$. It is well-known that $\operatorname{rank} X = \operatorname{rank} Y$ (see (2)). Since $AB(BA - I_n) = O_n$, it follows that $\left(A^T B^T - I_n\right) B^T A^T = O_n$, hence $X(I_n - Y) = O_n$, so $XY = X$, which leads to $\ker Y \subseteq \ker X$. But $\dim \ker X = n - \operatorname{rank} X = n - \operatorname{rank} Y = \dim \ker Y$, which means that $\ker X = \ker Y$. Then $X(I_n - Y) = O_n$ implies that $Y(I_n - Y) = O_n$, hence $Y^2 = Y$. Using the definition of Y, this expands immediately to $(AB)^2 = AB$.

In order to prove (b), we first obtain $(AB - BA)^3 = (BA)^2(AB - BA)$, by a similar computation as in the authors' solution. Also, by (a),

$$(BA)^3 = B(AB)^2 A = BABA = (BA)^2, \tag{6}$$

which leads to $(BA)^2(AB - BA) = (BA)^2(AB - I_n)$, hence (b) is verified if and only if $(BA)^2(AB - I_n) = O_n$, which can be written as $(B^T A^T - I_n)(A^T B^T)^2 = O_n$, or $Y(I_n - X)^2 = O_n$. But since X and Y have the same kernel, it suffices to prove that $X(I_n - X)^2 = O_n$. However, $X(I_n - X)^2 = O_n$ rewrites as $\left(I_n - A^T B^T\right)\left(A^T B^T\right)^2 = O_n$, which is equivalent to $(BA)^2(I_n - BA) = O_n$, that follows from (6).

Remark 1. In order to prove the identity (2), which is a general property for all $A, B \in \mathcal{M}_n(\mathbb{C})$, we may use $\operatorname{Ker}(I_n - AB) \subseteq \operatorname{Im} A$ and $\operatorname{Ker} A \subseteq \operatorname{Im}(I_n - BA)$, which means that the equality holds in the corresponding Sylvester inequalities, i.e.,

$$\operatorname{rank}(I_n - AB) + \operatorname{rank} A = n + \operatorname{rank}(A - ABA),$$
$$\operatorname{rank} A + \operatorname{rank}(I_n - BA) = n + \operatorname{rank}(A - ABA).$$

Problem 2. Let $a, b, c \in \mathbb{R}$ be such that

$$a + b + c = a^2 + b^2 + c^2 = 1, \quad a^3 + b^3 + c^3 \neq 1.$$

We say that a function f is a *Palić function* if $f : \mathbb{R} \to \mathbb{R}$, f is continuous, and satisfies

$$f(x) + f(y) + f(z) = f(ax + by + cz) + f(bx + cy + az) + f(cx + ay + bz) \quad (P)$$

for all $x, y, z \in \mathbb{R}$.

Prove that any Palić function is infinitely many times differentiable and find all the Palić functions.

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Author's solution. Note that the given conditions imply that a, b, c are nonzero and $ab + bc + ca = 0$. Let f be a Palić function. Putting $z = 0$ in (P), we obtain

$$f(x) + f(y) + f(0) = f(ax + by) + f(bx + cy) + f(cx + ay) \tag{1}$$

for all $x, y \in \mathbb{R}$. Since f is continuous, it follows that $F(x) = \displaystyle\int_0^x f(t)\,\mathrm{d}t$ is a primitive of f. By integrating in (1) with respect to y over $[0, 1]$, it follows that

$$f(x)+\int_0^1 f(y)\,\mathrm{d}y+f(0)=\tfrac{F(ax+b)-F(ax)}{b}+\tfrac{F(bx+c)-F(bx)}{c}+\tfrac{F(cx+a)-F(cx)}{a}, \quad (2)$$

for all $x, y \in \mathbb{R}$. Since F is differentiable, it follows from (2) that f is also differentiable, hence F is twice differentiable. By repeating the argument (using (2)), we easily obtain that f is infinitely many times differentiable.

Next, we differentiate in (P) three times with respect to x to obtain

$$f'''(x)=a^3f'''(ax+by+cz)+b^3f'''(bx+cy+az)+c^3f'''(cx+ay+bz),$$

then let $y=z=x$, whence $f'''(x)=(a^3+b^3+c^3)f'''(x)$, for all $x \in \mathbb{R}$. Because $a^3+b^3+c^3 \neq 1$, it follows that $f'''(x)=0$, so any Palić function is of the type

$$f(x)=px^2+qx+r \quad (p,q,r \in \mathbb{R}). \quad (3)$$

Replacing the expression of f from (3) in (P) leads to

$$\begin{aligned} f(ax+by+cz)+&f(bx+cy+az)+f(cx+ay+bz)\\ &=p\left(a^2+b^2+c^2\right)\left(x^2+y^2+z^2\right)\\ &\quad+2p(ab+bc+ca)(xy+yz+xz)\\ &\quad+q(a+b+c)(x+y+z)+3r\\ &=p(x^2+y^2+z^2)+q(x+y+z)+3r=f(x)+f(y)+f(z), \end{aligned}$$

for all $x, y, z \in \mathbb{R}$, so any function f of the form (3) is a Palić function.

Alternative solution. *This is based on the solutions given by Dimitris Emmanouil (contestant) and Kyprianos–Iason Prodromidis (member of the jury), from National and Kapodistrian University of Athens.*

First, we prove the following result.

Proposition. Let $a, b, c \in \mathbb{R}$ be as in the hypothesis of the problem and let $g : \mathbb{R} \to \mathbb{R}$ be a function whose second derivative at 0 exists, such that the relations

$$g(0)=g'(0)=g''(0)=0 \quad \text{and} \quad g(x)=g(ax)+g(bx)+g(cx), \quad \text{for all } x \in \mathbb{R},$$

hold. Then $g(x)=0$, for all $x \in \mathbb{R}$.

Proof. Since the functions $g(-x)$, $-g(x)$, $-g(-x)$ also satisfy the same conditions as g, we can assume for the sake of contradiction that $g(x_0)=\lambda>0$, for some $x_0>0$. Also, the given conditions imply that a, b, c are nonzero, so $m=\max\{|a|,|b|,|c|\}<1$.

We construct inductively a sequence $(x_n)_{n\geq 0}$ with the following properties: $g(x_n) \geq \lambda\left(\dfrac{x_n}{x_0}\right)^2$ and $0<|x_{n+1}| \leq m|x_n|$, for all $n \geq 0$.

Indeed, suppose $g(x_k) \geq \lambda \left(\frac{x_k}{x_0}\right)^2$ for some $k \geq 0$ (obviously, true for $k = 0$). Then

$$g(ax_k) + g(bx_k) + g(cx_k) = g(x_k) \geq \lambda \left(\frac{x_k}{x_0}\right)^2 = \lambda(a^2 + b^2 + c^2) \left(\frac{x_k}{x_0}\right)^2,$$

which leads to

$$\left(g(ax_k) - \lambda \left(\frac{ax_k}{x_0}\right)^2\right) + \left(g(bx_k) - \lambda \left(\frac{bx_k}{x_0}\right)^2\right) + \left(g(cx_k) - \lambda \left(\frac{cx_k}{x_0}\right)^2\right) \geq 0,$$

so at least one of these parentheses is non-negative. It is enough to let $x_{k+1} = ax_k$, bx_k, or cx_k, according to the term that is non-negative, to obtain the next term in the sequence. Also, it follows that $x_{k+1} \leq mx_k$, which concludes the argument. In particular, it follows that $\lim_{n\to\infty} x_n = 0$.

Yet, $g(0) = g'(0) = g''(0) = 0$ implies that $\lim_{x\to 0} \frac{g(x)}{x^2} = 0$, hence $\lim_{n\to\infty} \frac{g(x_n)}{x_n^2} = 0$, which is in contradiction to $\frac{g(x_n)}{x_n^2} \geq \frac{\lambda}{x_0^2}$ for all $n \geq 0$. $\square$

Back to the initial problem, after proving that all Palić functions are infinitely many times differentiable, it is easy to check that if f is a Palić function, then so is $g(x) = f(x) - f(0) - f'(0)x - \frac{f''(0)}{2}x^2$ which satisfies $g(0) = g'(0) = g''(0) = 0$. By letting $y = z = 0$ in (P) for g leads to $g(x) = g(ax) + g(bx) + g(cx)$, for all $x \in \mathbb{R}$. By the Proposition above, it follows that $g = 0$, hence f is a polynomial of degree at most 2.

Problem 3. Let $\alpha \in \mathbb{C} \setminus \{0\}$ and $A \in \mathcal{M}_n(\mathbb{C})$, $A \neq O_n$, be such that

$$A^2 + (A^*)^2 = \alpha AA^*,$$

where $A^* = \left(\overline{A}\right)^T$ denotes the transpose conjugate of A.

Prove that $\alpha \in \mathbb{R}$, $|\alpha| \leq 2$, and $AA^* = A^*A$.

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Authors' solution. Let $A = (a_{ij})_{1\leq i,j\leq n}$. Applying the trace operator in the given identity, it follows that $\sum_{i,j=1}^{n} a_{ij} \cdot a_{ji} + \sum_{i,j=1}^{n} \overline{a_{ji}} \cdot \overline{a_{ij}} = \alpha \cdot \sum_{i,j=1}^{n} a_{ij} \cdot \overline{a_{ij}}$, hence

$$2\,\mathrm{Re}\left(\sum_{i,j=1}^{n} a_{ij} \cdot a_{ji}\right) = \alpha \sum_{i,j=1}^{n} |a_{ij}|^2, \tag{1}$$

which leads to $\alpha \in \mathbb{R}$. Since $2\,|\mathrm{Re}(xy)| \le 2\,|x|\cdot|y| \le |x|^2+|y|^2$, for all $x, y \in \mathbb{C}$, it follows by (1) that

$$|\alpha|\sum_{i,j=1}^{n}|a_{ij}|^2 = 2\left|\mathrm{Re}\left(\sum_{i,j=1}^{n} a_{ij}\cdot a_{ji}\right)\right| \le \sum_{i,j=1}^{n}|a_{ij}|^2+\sum_{i,j=1}^{n}|a_{ji}|^2 = 2\sum_{i,j=1}^{n}|a_{ij}|^2.$$

Since $A \ne O_n$, $\sum_{i,j=1}^{n}|a_{ij}|^2 \ne 0$ and we get $|\alpha| \le 2$.

Let $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$ be the solutions of $z^2-\alpha z+1=0$, so $\varepsilon_1+\varepsilon_2=\alpha$ and $\varepsilon_1\varepsilon_2=1$. Let $X = A-\varepsilon_1 A^*$ and $Y = A-\varepsilon_2 A^*$. Then $XY = A^2 + \varepsilon_1\varepsilon_2\,(A^*)^2 - \varepsilon_1 A^*A - \varepsilon_2 AA^* = \alpha AA^* - \varepsilon_1 A^*A - \varepsilon_2 AA^* = \varepsilon_1(AA^*-A^*A)$ and, similarly, $YX = \varepsilon_2(AA^*-A^*A)$.
Then $XY = \dfrac{\varepsilon_1}{\varepsilon_2}YX = \varepsilon_1^2 YX$, so $(XY)^2 = \varepsilon_1^4(YX)^2$. Since $\mathrm{Tr}\left((XY)^2\right) = \mathrm{Tr}\left((YX)^2\right)$, it follows that $(\varepsilon_1^4-1)\,\mathrm{Tr}\left((XY)^2\right)=0$, so we distinguish the following cases:

- $\varepsilon_1 \in \{-i, i\}$; then $\alpha = 0$ (which contradicts the hypothesis);
- $\varepsilon_1 \in \{-1, 1\}$; then $\alpha \in \{-2, 2\}$ and the equality from the hypothesis becomes $(A \pm A^*)^2 = \pm(A^*A - AA^*)$; the equality of the traces gives $\mathrm{Tr}\left((A\pm A^*)^2\right) = 0$, which leads to $A \pm A^* = O_n$, and the conclusion follows;
- $\mathrm{Tr}\left((XY)^2\right) = 0$; then $\mathrm{Tr}\left((AA^*-A^*A)^2\right) = 0$, so $AA^*-A^*A = O_n$.

Alternative solutions. *There were several solutions proposed by the contestants for the first part of the problem. One such solution was given by Alexandru Buzea, from University Politehnica of Bucharest.*

Taking the conjugate transpose of both sides of the equality from the statement, we obtain that $(A^*)^2 + A^2 = \overline{\alpha}AA^*$, hence $(\alpha-\overline{\alpha})\,AA^* = O_n$. Since $A \ne O_n$, we obtain that $\alpha - \overline{\alpha} = 0$, so $\alpha \in \mathbb{R}$.
Next, consider the matrix norm induced by the ℓ^2–norm,

$$\|B\|_2 = \sup_{\|\mathbf{x}\|_2=1}\|B\mathbf{x}\|_2, \quad \text{for all } B \in \mathcal{M}_n(\mathbb{C}).$$

By taking the norm in both sides of the equality from the statement, it follows that $\left\|A^2+(A^*)^2\right\|_2 = |\alpha|\,\|AA^*\|_2$, therefore $|\alpha| = \dfrac{\left\|A^2+(A^*)^2\right\|_2}{\|AA^*\|_2}$.
Considering the subadditivity of the norm and that $\|AA^*\|_2 = \|A\|_2^2 = \|A^*\|_2^2$ (see [1], p. 283), we obtain $|\alpha| \le 2$.

Aggelos Gatos and Konstantinos Tsirkas, from National and Kapodistrian University of Athens, used a similar argument.

Consider any $\mathbf{x} \in \mathbb{C}^n$ such that $A^*\mathbf{x} \ne \mathbf{0}$. Such $\mathbf{x}$ exists, because $A \ne O_n$. Let $\langle\cdot,\cdot\rangle$ be the standard inner product on $\mathbb{C}^n$ and let $\|\cdot\|$ be the induced norm. Then

$$\langle\left(A^2+(A^*)^2\right)\mathbf{x},\mathbf{x}\rangle = \langle A^2\mathbf{x},\mathbf{x}\rangle + \langle (A^*)^2\mathbf{x},\mathbf{x}\rangle = \langle A\mathbf{x},A^*\mathbf{x}\rangle + \langle A^*\mathbf{x},A\mathbf{x}\rangle$$
$$= 2\,\mathrm{Re}\langle A\mathbf{x},A^*\mathbf{x}\rangle.$$

Moreover, $\langle \alpha AA^*\mathbf{x},\mathbf{x}\rangle = \alpha\langle A^*\mathbf{x},A^*\mathbf{x}\rangle = \alpha\|A^*\mathbf{x}\|^2$. These relations, together with the hypothesis of the problem, imply that

$$\alpha = \frac{2\,\mathrm{Re}\langle A\mathbf{x},A^*\mathbf{x}\rangle}{\|A^*\mathbf{x}\|^2}\in\mathbb{R}.$$

Now, let $\mathbf{x}$ be a unit vector such that $\|A^*\mathbf{x}\| = \|A^*\|_2 = \|A\|_2$, where $\|\cdot\|_2$ is the matrix norm induced by the ℓ^2–norm. The Cauchy–Schwarz inequality tells us that

$$|\alpha| \le 2\frac{|\langle A\mathbf{x},A^*\mathbf{x}\rangle|}{\|A^*\mathbf{x}\|^2} \le 2\frac{\|A\mathbf{x}\|\cdot\|A^*\mathbf{x}\|}{\|A^*\mathbf{x}\|^2} = 2\frac{\|A\mathbf{x}\|}{\|A^*\mathbf{x}\|} \le 2\frac{\|A\|_2\cdot\|\mathbf{x}\|}{\|A^*\|_2} = 2.$$

Minas Margaritis and Kyprianos–Iason Prodromidis (member of the jury), from National and Kapodistrian University of Athens, used a different approach.

Observe that $(A+A^*)^2 = (\alpha+1)\,AA^* + A^*A$. Since $A+A^*$ is Hermitian, the eigenvalues of its square are non-negative, which means that $\mathrm{Tr}\left((A+A^*)^2\right) \ge 0$, so $(\alpha+1)\,\mathrm{Tr}\,(AA^*) + \mathrm{Tr}\,(A^*A) \ge 0$, which leads to $(\alpha+2)\,\mathrm{Tr}\,(AA^*) \ge 0$. Since $\mathrm{Tr}\,(AA^*) > 0$, we conclude that $\alpha \ge -2$. By using $A - A^*$ in place of $A + A^*$, it follows that $\alpha \le 2$ in an almost identical way.

Problem 4. Let $\mathcal{F}$ be the family of all nonempty finite subsets of $\mathbb{N}\cup\{0\}$, where $\mathbb{N}$ denotes the set of positive integers. Find all positive real numbers a for which the series $\displaystyle\sum_{A\in\mathcal{F}}\frac{1}{\sum\limits_{k\in A}a^k}$ is convergent.

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Author's solution. Let $a = 2$. Since every $n\in\mathbb{N}$ can be uniquely represented in base 2, i.e., as a sum of distinct powers of 2

$$n = 2^{k_1}+2^{k_2}+\cdots+2^{k_p},\quad 0\le k_1<k_2<\cdots<k_p \text{ integers},$$

the map $\varphi:\mathbb{N}\to\mathcal{F}$, with $\varphi(n)=\{k_1,k_2,\dots,k_p\}$, is well defined and bijective, with the inverse $\varphi^{-1}(A) = \displaystyle\sum_{k\in A}2^k$, $A\in\mathcal{F}$.

Because all the terms of the series in question are positive, they can be rearranged at our convenience. Since φ is bijective, we can write

$$\sum_{A\in\mathcal{F}}\frac{1}{\sum\limits_{k\in A}2^k} = \sum_{n=1}^{\infty}\frac{1}{\sum\limits_{k\in\varphi(n)}2^k} = \sum_{n=1}^{\infty}\frac{1}{n},$$

which is divergent.

If $a < 2$, then $\dfrac{1}{\sum\limits_{k\in A} a^k} \geq \dfrac{1}{\sum\limits_{k\in A} 2^k}$, for all $A \in \mathcal{F}$, so the series dominates the divergent harmonic series, which means it is also divergent.

Now, let $a > 2$. For every $n \in \mathbb{N} \cup \{0\}$, denote by $\mathcal{F}_n$ the family of all nonempty finite subsets of $\mathbb{N} \cup \{0\}$ whose greatest element is n. Clearly, there are 2^n sets in $\mathcal{F}_n$ and $\{\mathcal{F}_n : n \in \mathbb{N} \cup \{0\}\}$ is a partition of $\mathcal{F}$. By regrouping the terms of the series, we obtain

$$\sum_{A\in\mathcal{F}} \frac{1}{\sum\limits_{k\in A} a^k} = \sum_{n=0}^{\infty} \sum_{A\in\mathcal{F}_n} \frac{1}{\sum\limits_{k\in A} a^k} \leq \sum_{n=0}^{\infty} \sum_{A\in\mathcal{F}_n} \frac{1}{a^n} = \sum_{n=0}^{\infty} \left(\frac{2}{a}\right)^n,$$

which means that the series is dominated by a convergent geometric series, hence it is convergent.

Concluding, the series is convergent if and only if $a > 2$.

Alternative solutions. *Other solutions, proposed by members of the jury or by contestants, use the same idea of grouping the terms of the series according to the largest member of the subsets also when obtaining the divergence (for $a \leq 2$), while the convergence (for $a > 2$) follows just like in the author's solution.*

For $a \leq 2$, Mircea Rus (member of the jury), from Technical University of Cluj-Napoca, used $\sum\limits_{k\in A} a^k \leq \sum\limits_{k\in A} 2^n \leq (n+1)2^n$, for all $A \in \mathcal{F}_n$ and $n \in \mathbb{N} \cup \{0\}$, to obtain

$$\sum_{A\in\mathcal{F}} \frac{1}{\sum\limits_{k\in A} a^k} = \sum_{n=0}^{\infty} \sum_{A\in\mathcal{F}_n} \frac{1}{\sum\limits_{k\in A} a^k} \geq \sum_{n=0}^{\infty} \sum_{A\in\mathcal{F}_n} \frac{1}{(n+1)2^n} = \sum_{n=0}^{\infty} \frac{1}{n+1},$$

which is enough to prove that the series is divergent. A different domination

$$\sum_{k\in A} a^k \leq 1 + a + a^2 + \cdots + a^n, \quad \text{for all } A \in \mathcal{F}_n \text{ and } n \in \mathbb{N} \cup \{0\},$$

was given by Răzvan-Gabriel Petec (contestant), from Babeş-Bolyai University, Cluj-Napoca, who used it to obtain

$$\sum_{A\in\mathcal{F}} \frac{1}{\sum\limits_{k\in A} a^k} = \sum_{n=0}^{\infty} \sum_{A\in\mathcal{F}_n} \frac{1}{\sum\limits_{k\in A} a^k} \geq \sum_{n=0}^{\infty} \frac{2^n}{1 + a + a^2 + \cdots + a^n}.$$

It is then elementary to show that the series $\sum\limits_{n=0}^{\infty} \dfrac{2^n}{1 + a + a^2 + \cdots + a^n}$ is divergent for all $a \leq 2$.

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Traian Lalescu national mathematics contest for university students, 2022 edition

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Abstract. This note presents the problems from the 2022 edition of the "Traian Lalescu" National Mathematics Contest for University Students, hosted by the West University of Timişoara between May 12 and May 14.

Keywords: Rank of a matrix, Sylvester's inequality for ranks, eigenvalue of a matrix, Jordan form, trace of a matrix, finite Abelian group, group isomorphism, Wallis formula, numerical series, Cauchy condensation test, Cauchy-Schwarz inequality, Riemann integral, integration by parts, solid of revolution.

MSC: Primary 15A03; Secondary 15A04, 15A18, 15A20, 15A23, 20D40, 20D45, 26A42, 40A05.

This year's edition of the "Traian Lalescu" National Mathematics Contest for University Students took place between May 12 and May 14 and was organized by The Faculty of Mathematics and Computer Science of the West University of Timişoara, with the support of the "Traian Lalescu" Foundation and the Ministry of Education.

There were 79 students that participated in the competition, representing 13 universities from Braşov, Bucharest, Cluj-Napoca, Craiova, Iaşi, Sibiu and Timişoara. The contest was organized in five distinct sections:

- Section A for first- and second-year students that follow a specialization in Mathematics;
- Section B for first-year students that follow some specialization in Electrical Engineering, or in Computer Science;
- Section C for first-year students from technical faculties with a specialization outside the field of Electrical Engineering;

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- Section D for second-year students from technical faculties with specializations in Electrical Engineering;
- Section E for second-year students from technical faculties with a specialization outside the field of Electrical Engineering.

In total, 17 students were awarded prizes or honorable mentions. The interested reader may find additional details at the competition's website: `https://sites.google.com/e-uvt.ro/concursultraianlalescu`.

We present the problems for Sections A and B of the contest and their solutions as given by the corresponding authors.

SECTION A

Problem 1. Find the integer values n, $n \geq 2$, for which there exist $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $A^2 = A$ and $AB - BA$ is invertible.

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Authors' solution. We prove that the required numbers n are the positive even numbers.

Indeed, for $n = 2k$, $k \geq 1$, we may consider the block matrices $A = \begin{pmatrix} I_k & O_k \\ O_k & O_k \end{pmatrix}$ and $B = \begin{pmatrix} O_k & I_k \\ -I_k & O_k \end{pmatrix}$, which verify $A^2 = A$ and $AB - BA = \begin{pmatrix} O_k & I_k \\ I_k & O_k \end{pmatrix}$, which is invertible.

It remains to show that if $n = 2k + 1$, $k \geq 1$, and $A \in \mathcal{M}_n(\mathbb{C})$ such that $A^2 = A$, then there is no matrix $B \in \mathcal{M}_n(\mathbb{C})$ such that $AB - BA$ is invertible. This statement can be proven by several methods.

First method: Assume, by contradiction, that there exists $B \in \mathcal{M}_n(\mathbb{C})$ such that $C := AB - BA$ is invertible. Since $A^2 = A$, it follows that $AC + CA = A^2B - ABA + ABA - BA^2 = AB - BA = C$, therefore $C^{-1}AC + A = I_n$. Since $\operatorname{Tr}(C^{-1}AC) = \operatorname{Tr} A$, we obtain that $\operatorname{Tr} A = \dfrac{n}{2}$. It is a known result that the trace of an idempotent matrix equals its rank, so we may conclude that $\operatorname{rank} A = \dfrac{n}{2}$, which fails to be an integer, hence the contradiction with our assumption.

Second method: By the Sylvester inequality for ranks, it follows that $\operatorname{rank} A + \operatorname{rank}(A - I_n) \leq \operatorname{rank}\left(A^2 - A\right) + n = 2k+1$, so either $\operatorname{rank} A \leq k$, or $\operatorname{rank}(A - I_n) \leq k$. Let $B \in \mathcal{M}_n(\mathbb{C})$. If $\operatorname{rank} A \leq k$, then

$$\operatorname{rank}(AB - BA) \leq \operatorname{rank}(AB) + \operatorname{rank}(BA) \leq 2 \operatorname{rank} A \leq 2k < n.$$

If $\operatorname{rank}(A - I_n) \le k$, then

$$\begin{aligned}\operatorname{rank}(AB - BA) &= \operatorname{rank}\left((A - I_n)B - B(A - I_n)\right)\\ &\le \operatorname{rank}\left((A - I_n)B\right) + \operatorname{rank}\left(B(A - I_n)\right)\\ &\le 2\operatorname{rank}(A - I_n) \le 2k < n.\end{aligned}$$

In both cases, $AB - BA$ is not invertible.

Third method: Since $A^2 = A$, there exists a base in which A can be represented as the block matrix $\begin{pmatrix} I_p & O_{p,n-p} \\ O_{n-p,p} & O_{n-p} \end{pmatrix}$, with $p \ge 0$. With respect to the same base, we can also write B as $\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$, where $X \in \mathcal{M}_p(\mathbb{C})$, $Y \in \mathcal{M}_{p,n-p}(\mathbb{C})$, $Z \in \mathcal{M}_{n-p,p}(\mathbb{C})$ and $T \in \mathcal{M}_{n-p}(\mathbb{C})$. Then

$$AB - BA = \begin{pmatrix} O_p & Y \\ -Z & O_{n-p} \end{pmatrix},$$

so

$$\operatorname{rank}(AB - BA) = \operatorname{rank} Y + \operatorname{rank} Z \le 2\min\{p, n-p\} \le 2k < n,$$

hence $AB - BA$ is not invertible.

Problem 2. Let H and G be finite Abelian groups, with $\operatorname{ord} H = m$ and $\operatorname{ord} G = n$, and let $f : H \to G$ be a surjective group homomorphism with $\operatorname{ord}(\operatorname{Ker} f) = p$ and $(p, n) = 1$. Prove that:

a) The mapping $u : \operatorname{Ker} f \to \operatorname{Ker} f$, defined by $u(x) = x^n$ for all $x \in \operatorname{Ker} f$, is an automorphism of the group $\operatorname{Ker} f$.

b) The mapping $\varphi : H \to (\operatorname{Ker} f) \times G$, defined by $\varphi(x) = \left(u^{-1}(x^n), f(x)\right)$ for all $x \in \operatorname{Ker} f$, is a group isomorphism.

* * *

Solution. Let e_H and e_G denote the identity elements of H and G, respectively.

a) Since f is a group homomorphism, it follows that $\operatorname{Ker} f$ is a subgroup of H. Also, if $x \in \operatorname{Ker} f$, then $f(x^n) = (f(x))^n = e_G^n = e_G$, therefore $x^n \in \operatorname{Ker} f$, which means that u is well-defined.

Since H is Abelian, it follows that $u(xy) = (xy)^n = x^n y^n = u(x) \cdot u(y)$, for all $x, y \in \operatorname{Ker} f$, hence u is an endomorphism of $\operatorname{Ker} f$.

Next, let $x \in \operatorname{Ker} u$, i.e., $x^n = e_H$. It follows that $\operatorname{ord} x \mid n$. Also, by the Lagrange theorem, we have that $\operatorname{ord} x \mid p$. Since p and n are relatively prime, this leads to $\operatorname{ord} x = 1$, so $x = e_H$. In conclusion, $\operatorname{Ker} u = \{e_H\}$, which means that u is injective, hence bijective (since $\operatorname{Ker} f$ is finite).

b) Since $\operatorname{ord} G = n$, it follows that $f(x^n) = (f(x))^n = e_G$, which means that $x^n \in \operatorname{Ker} f$ for all $x \in H$, hence the mapping φ is well-defined. It is straightforward to check the group homomorphism property for φ. Also, since

$$\begin{aligned}\operatorname{ord} H &= \operatorname{ord}(\operatorname{Ker} f)\cdot\operatorname{ord}(H/\operatorname{Ker} f) = p\cdot\operatorname{ord}(\operatorname{Im} f) = p\cdot\operatorname{ord} G\\ &= \operatorname{ord}((\operatorname{Ker} f)\times G),\end{aligned}$$

it remains only to show that φ is injective, i.e., $\operatorname{Ker}\varphi = \{e_H\}$. Indeed, if $x\in\operatorname{Ker}\varphi$, then $u^{-1}(x^n) = e_H$ and $f(x) = e_G$, meaning that $x^n = e_H$ and $x\in\operatorname{Ker} f$, which leads to $x\in\operatorname{Ker} u = \{e_H\}$.

Remark. The problem is known and can be found in [1] (Problem 31, p. 15). The original statement (which asks only to prove that H and $G\times\operatorname{Ker} f$ are isomorphic) was modified, in order to guide the contestants, by providing them with an actual isomorphism.

Problem 3. Prove that there exists a maximum value of $\alpha\in\mathbb{R}$ and find that value, for which there is a sequence $(a_n)_{n\geq 1}$ of positive real numbers such that $\displaystyle\sum_{n=1}^{\infty} a_n^2$ is convergent and $\displaystyle\sum_{n=1}^{\infty}\frac{a_n}{n^\alpha}$ is divergent.

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Authors' solution. We prove that the maximum value of α exists and it is $\dfrac{1}{2}$.

Let $\alpha\in\mathbb{R}$ such that a sequence $(a_n)_{n\geq 1}$ exists with the required properties. By the Cauchy-Schwarz inequality, $\displaystyle\left(\sum_{k=1}^{n}\frac{a_k}{k^\alpha}\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n}\frac{1}{k^{2\alpha}}\right)$, for all $n\geq 1$, therefore $\displaystyle\left(\sum_{n=1}^{\infty}\frac{a_n}{n^\alpha}\right)^2 \leq \left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^{2\alpha}}\right)$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{a_n}{n^\alpha}$ is divergent and $\displaystyle\sum_{n=1}^{\infty} a_n^2$ is convergent, it follows that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{2\alpha}}$ is divergent, hence $\alpha\leq\dfrac{1}{2}$.

For $\alpha = \dfrac{1}{2}$, it is enough to give an example of a sequence $(a_n)_{n\geq 1}$ with the required properties. Let $a_1 = 0$ and $a_n = \dfrac{1}{\sqrt{n}\ln n}$, for $n\geq 2$. Then $\displaystyle\sum_{n=1}^{\infty} a_n^2 = \sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{a_n}{\sqrt{n}} = \sum_{n=2}^{\infty}\frac{1}{n\ln n}$. Since $\displaystyle\left(\frac{1}{n(\ln n)^2}\right)_{n\geq 2}$ and $\displaystyle\left(\frac{1}{n\ln n}\right)_{n\geq 2}$ are decreasing, the nature of both series can be studied using the *condensation test* of Cauchy. Using the symbol $\sim$ between two series to denote they have the same nature, it follows that

$$\sum_{n=1}^{\infty} a_n^2 \sim \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n \left(\ln 2^n\right)^2} = \frac{1}{(\ln 2)^2} \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{which is convergent,}$$

and

$$\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} \sim \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n \ln 2^n} = \frac{1}{\ln 2} \sum_{n=1}^{\infty} \frac{1}{n}, \quad \text{which is divergent.}$$

Problem 4. Let $\mathcal{F}$ be the set of differentiable functions $f : [0,1] \to \mathbb{R}$, with f' continuous over $[0,1]$ and which verify the equalities

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 x f(x)\,\mathrm{d}x = 1.$$

Find $\min \left\{ \int_0^1 \left(f'(x)\right)^2 \,\mathrm{d}x \;\middle|\; f \in \mathcal{F} \right\}$.

* * *

Solution. Using integration by parts, we obtain that

$$1 = \int_0^1 f(x)\,\mathrm{d}x = x f(x)\big|_0^1 - \int_0^1 x f'(x)\,\mathrm{d}x = f(1) - \int_0^1 x f'(x)\,\mathrm{d}x$$

and

$$1 = \int_0^1 x f(x)\,\mathrm{d}x = \frac{1}{2} x^2 f(x)\big|_0^1 - \frac{1}{2}\int_0^1 x^2 f'(x)\,\mathrm{d}x = \frac{f(1)}{2} - \frac{1}{2}\int_0^1 x^2 f'(x)\,\mathrm{d}x.$$

By canceling out $f(1)$ in the two equalities, we get that $\int_0^1 (x-x^2) f'(x)\,\mathrm{d}x = 1$. The Cauchy-Schwarz inequality then leads to

$$\begin{aligned} 1 = \left(\int_0^1 (x-x^2) f'(x)\,\mathrm{d}x\right)^2 &\le \int_0^1 (x-x^2)^2\,\mathrm{d}x \;\cdot \int_0^1 \left(f'(x)\right)^2\,\mathrm{d}x \\ &= \frac{1}{30}\int_0^1 \left(f'(x)\right)^2\,\mathrm{d}x, \end{aligned}$$

with equality if and only if $f'(x) = \alpha(x - x^2)$ for some $\alpha \in \mathbb{R}$, since f' is continuous. This happens when $f(x) = \alpha\left(\frac{x^2}{2} - \frac{x^3}{3}\right) + \beta$, where the constants α and β are such that $f \in \mathcal{F}$. A simple computation leads to the linear system

$$\begin{cases} \dfrac{1}{12}\alpha + \beta = 1 \\[2ex] \dfrac{7}{120}\alpha + \dfrac{1}{2}\beta = 1 \end{cases},$$

with the solution $\alpha = 30$, $\beta = -\dfrac{3}{2}$.

Concluding, $\min\left\{\displaystyle\int_0^1 \left(f'(x)\right)^2 \,\mathrm{d}x \;\middle|\; f \in \mathcal{F}\right\} = 30$, which is achieved for the function $f(x) = -10x^3 + 15x^2 - \dfrac{3}{2}$.

Remark. The problem is known and can be found under a slightly different formulation in, e.g., [2] (p. 77, Problem 9.17).

Section B

Problem 1. Prove that there exists a maximum value of $\alpha \in \mathbb{R}$ and find that value, for which there is a sequence $(a_n)_{n\geq 1}$ of positive real numbers such that $\displaystyle\sum_{n=1}^{\infty} a_n^2$ is convergent and $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^\alpha}$ is divergent.

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Solution. This problem was also given in Section A (Problem 3), to which we refer the reader.

Problem 2. Let $n, k \in \mathbb{N}^*$, $a \in \mathbb{C}^*$ and $A \in \mathcal{M}_n(\mathbb{C})$ such that $A^{k+1} = a \cdot A$. Prove that $\operatorname{Tr}(A^k) = a \cdot \operatorname{rank} A$.

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Authors' solutions.
First solution. First, we claim that $\operatorname{rank}(A^k) = \operatorname{rank} A$.

Indeed, since $A^{k+1} = a \cdot A$ and $a \neq 0$, it follows that $\operatorname{rank}(A^{k+1}) = \operatorname{rank} A$. But $\operatorname{rank}(A^{k+1}) = \operatorname{rank}(A^k \cdot A) \leq \operatorname{rank}(A^k)$, hence $\operatorname{rank} A \leq \operatorname{rank}(A^k)$. At the same time, $\operatorname{rank}(A^k) = \operatorname{rank}(A \cdot A^{k-1}) \leq \operatorname{rank} A$, so our claim is verified.

By multiplying in $A^{k+1} = a \cdot A$ with $\dfrac{1}{a^2} \cdot A^{k-1}$, we obtain that $\dfrac{1}{a^2} \cdot A^{2k} = \dfrac{1}{a} \cdot A^k$, which means that the matrix $\dfrac{1}{a} \cdot A^k$ is idempotent. It is a known result that the trace and the rank of an idempotent matrix are equal, which assures that $\operatorname{rank}\left(\dfrac{1}{a} \cdot A^k\right) = \operatorname{Tr}\left(\dfrac{1}{a} \cdot A^k\right)$. From here and using that $\operatorname{rank}(A^k) = \operatorname{rank} A$, the conclusion follows immediately.

Second solution. If $\operatorname{rank} A = 0$, then $A = O_n$ and the conclusion is trivial, so we assume that $\operatorname{rank} A = p \geq 1$.

Using the rank factorization, there exist $B \in \mathcal{M}_{n,p}(\mathbb{C})$ and $C \in \mathcal{M}_{p,n}(\mathbb{C})$ such that $\operatorname{rank} B = \operatorname{rank} C = p$ and $A = BC$. Then $A^{k+1} = B(CB)^k C$, so $\operatorname{rank}(A^{k+1}) \leq \operatorname{rank}\left((CB)^k\right) \leq \operatorname{rank}(CB)$, therefore $p = \operatorname{rank}(a \cdot A) =$

$\operatorname{rank}(A^{k+1}) \le \operatorname{rank}(CB) \le p$, leading to $\operatorname{rank}(CB) = p$, i.e., $CB \in \mathcal{M}_p(\mathbb{C})$ is invertible.

Rewriting the equality from the hypothesis, it becomes $(BC)^{k+1} = a \cdot BC$. Multiplying with C to the left and with B to the right, it follows that $(CB)^{k+2} = a \cdot (CB)^2$, hence $(CB)^k = a \cdot I_p$, so $\operatorname{Tr}\left((CB)^k\right) = ap$.

On the other hand, since in general $\operatorname{Tr}(XY) = \operatorname{Tr}(YX)$ when both XY and YX make sense, it follows that

$$\begin{aligned}\operatorname{Tr}\left((CB)^k\right) &= \operatorname{Tr}\left(C(BC)^{k-1}B\right) = \operatorname{Tr}\left((BC)^{k-1}BC\right)\\ &= \operatorname{Tr}\left((BC)^k\right) = \operatorname{Tr}(A^k),\end{aligned}$$

hence $\operatorname{Tr}(A^k) = ap = a \cdot \operatorname{rank} A$.

Third solution. Since the eigenvalues λ of A satisfy $\lambda^{k+1} = a \cdot \lambda$, it follows that $\lambda = 0$ or $\lambda^k = a$. Denote by $\lambda_1, \lambda_2, \ldots, \lambda_p$ the non-zero eigenvalues of A and by J_A the Jordan canonical form of A.

Using that $\operatorname{rank} A \ge \operatorname{rank} A^2 \ge \cdots \ge \operatorname{rank} A^k \ge \operatorname{rank} A^{k+1} = \operatorname{rank} aA = \operatorname{rank} A$, we obtain $\operatorname{rank} A = \operatorname{rank} A^2 = \cdots = \operatorname{rank} A^k = \operatorname{rank} A^{k+1}$. Then $\operatorname{rank} J_A = \operatorname{rank} J_A^2$, which means that all the Jordan cells in J_A corresponding to the eigenvalue 0 have dimension 1. Therefore, $\operatorname{rank} A = \operatorname{rank} J_A = p$ and

$$\operatorname{Tr}(A^k) = \lambda_1^k + \lambda_2^k + \cdots + \lambda_p^k = p \cdot a = a \cdot \operatorname{rank} A.$$

Problem 3. Let $A, B \in \mathcal{M}_n(\mathbb{C})$. Prove that the following statements are equivalent:

(1) $AB = A$ and $\operatorname{rank}(I_n - A) + \operatorname{rank} B = n$;
(2) $BA = B$ and $\operatorname{rank}(I_n - B) + \operatorname{rank} A = n$.

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Authors' solutions. Due to the symmetry of the problem, it is enough to prove one of the implications. We will assume (1) and prove (2).

First solution. Since $AB = A$, it follows that $\operatorname{Ker} B \subseteq \operatorname{Ker} A$. Also, it is generally true that $\operatorname{Ker} A \subseteq \operatorname{Im}(I_n - A)$, meaning that $\operatorname{Ker} B \subseteq \operatorname{Im}(I_n - A)$.

On the other hand, the following characterization takes place for all $X, Y \in \mathcal{M}_n(\mathbb{C})$:

$$\operatorname{Ker} X \subseteq \operatorname{Im} Y \text{ if and only if } \operatorname{rank} X + \operatorname{rank} Y = n + \operatorname{rank}(XY)$$

(this is the equality case in the Sylvester inequality for ranks). Letting $X = B$ and $Y = I_n - A$, it follows that $n = \operatorname{rank} B + \operatorname{rank}(I_n - A) = n + \operatorname{rank}(B - BA)$, hence $BA = B$. This also leads to $\operatorname{Ker} A \subseteq \operatorname{Ker} B \subseteq \operatorname{Im}(I_n - B)$, which will assure that $\operatorname{rank} A + \operatorname{rank}(I_n - B) = n + \operatorname{rank}(A - AB) = n$.

Second solution. Let M be the block matrix $\begin{pmatrix} I_n - A & O_n \\ O_n & B \end{pmatrix}$. We will apply elementary transformations to M that preserve the rank, as follows:

$$\begin{aligned}\begin{pmatrix} I_n - A & O_n \\ O_n & B \end{pmatrix} &\xrightarrow{\cdot L_1 \text{ to the left}} \begin{pmatrix} I_n - A & AB \\ O_n & B \end{pmatrix} \\ &\xrightarrow{\cdot C_1 \text{ to the right}} \begin{pmatrix} I_n - A & I_n - A + AB \\ O_n & B \end{pmatrix} = \begin{pmatrix} I_n - A & I_n \\ O_n & B \end{pmatrix} \\ &\xrightarrow{\cdot C_2 \text{ to the right}} \begin{pmatrix} I_n & I_n \\ BA & B \end{pmatrix} \xrightarrow{\cdot C_3 \text{ to the right}} \begin{pmatrix} O_n & I_n \\ BA - B & B \end{pmatrix} \\ &\xrightarrow{\cdot L_2 \text{ to the left}} \begin{pmatrix} O_n & I_n \\ BA - B & O_n \end{pmatrix} = N\end{aligned}$$

with $L_1 = \begin{pmatrix} I_n & A \\ O_n & I_n \end{pmatrix}$, $L_2 = \begin{pmatrix} I_n & O_n \\ -B & I_n \end{pmatrix}$, $C_1 = \begin{pmatrix} I_n & I_n \\ O_n & I_n \end{pmatrix}$, $C_2 = \begin{pmatrix} I_n & O_n \\ A & I_n \end{pmatrix}$, and $C_3 = \begin{pmatrix} I_n & O_n \\ -I_n & I_n \end{pmatrix}$. Since, $N = L_2 \cdot L_1 \cdot M \cdot C_1 \cdot C_2 \cdot C_3$, it follows that $\operatorname{rank} N = \operatorname{rank} M$, i.e.,

$$\begin{aligned} n + \operatorname{rank}(BA - B) &= \operatorname{rank} \begin{pmatrix} O_n & I_n \\ BA - B & O_n \end{pmatrix} = \operatorname{rank} \begin{pmatrix} I_n - A & O_n \\ O_n & B \end{pmatrix} \\ &= \operatorname{rank}(I_n - A) + \operatorname{rank} B = n \end{aligned}$$

which leads to $BA = B$.

Next, by swapping A and B in the above transformations, we also obtain that

$$\begin{aligned} n = n + \operatorname{rank}(AB - A) &= \operatorname{rank} \begin{pmatrix} O_n & I_n \\ AB - A & O_n \end{pmatrix} = \operatorname{rank} \begin{pmatrix} I_n - B & O_n \\ O_n & A \end{pmatrix} \\ &= \operatorname{rank}(I_n - B) + \operatorname{rank} A, \end{aligned}$$

which leads to $\operatorname{rank}(I_n - B) + \operatorname{rank} A = n$.

Problem 4. Let $f : [0,1] \to \mathbb{R}$ be a function of class C^1 such that $f(0) = 0$, $f(1) = 1$ and $f' > 0$ over $(0,1)$. Let $a \in [0,1]$ such that the function $F(x) = \int\limits_a^x f(t)\,\mathrm{d}t$ $(x \in [0,1])$ satisfies

$$F(x) \cdot f'(x) = (f(x))^2 - 1, \quad \text{for all } x \in [0,1]. \tag{1}$$

a) Prove that $a = 1$ and that, for every $n \in \mathbb{N}^*$, the function $g_n(x) : [0,1] \to [0,1]$, $g_n(x) = (f(x))^n$, is bijective.

b) For every $n \in \mathbb{N}^*$, let $V_{x,n}$ and $V_{y,n}$ be the volumes of the solids generated by revolving the graph of the function g_n about the x- and y-axis, respectively, and define $a_n = \dfrac{V_{x,n}}{V_{y,n}}$.

Study the convergence of the series $\sum_{n=1}^{\infty} a_n^{\alpha}$, for any real parameter α.

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Author' solution. a) Obviously, $f(x) < 1$, for all $x \in (0,1)$, so $F(x) = \dfrac{(f(x))^2 - 1}{f'(x)} < 0$, for all $x \in (0,1)$. Assuming that $a < 1$, then $F(x) > 0$ for all $x \in (a,1)$, since f is continuous and positive over $(0,1]$, thus leading to a contradiction. Concluding, $a = 1$ and $F(1) = 0$.

Using that f is differentiable and strictly increasing, it follows that g_n has the same properties, which then easily leads to g_n being bijective, for every $n \in \mathbb{N}^*$.

b) It is well known that $V_{x,n} = \pi \int_0^1 (g_n(x))^2 \,\mathrm{d}x = \pi I_{2n}$, where $I_n = \int_0^1 (f(x))^n \,\mathrm{d}x$ for every $n \in \mathbb{N}$. Let us observe that, for every $n \geq 2$,

$$\begin{aligned}
I_n &= \int_0^1 (f(x))^n \,\mathrm{d}x = \int_0^1 (f(x))^{n-1} \cdot F'(x) \,\mathrm{d}x \\
&= (f(x))^{n-1} \cdot F(x) \Big|_0^1 - \int_0^1 (n-1)\,(f(x))^{n-2} \cdot f'(x) \cdot F(x) \,\mathrm{d}x \\
&= F(1) + (n-1) \int_0^1 (f(x))^{n-2} \cdot \left(1 - (f(x))^2\right) \mathrm{d}x \\
&= (n-1)I_{n-2} - (n-1)I_n,
\end{aligned}$$

which leads to $I_n = \dfrac{n-1}{n} I_{n-2}$, with $I_0 = 1$ and $I_1 = -F(0) > 0$. A simple argument gives

$$I_{2n} = \frac{(2n-1)!!}{(2n)!!}, \quad I_{2n+1} = -\frac{(2n)!!}{(2n+1)!!} \cdot F(0),$$

hence $V_{x,n} = \pi \dfrac{(2n-1)!!}{(2n)!!}$.

In turn, $V_{y,n} = \pi \int_0^1 \left(g_n^{-1}(y)\right)^2 \mathrm{d}y$. The change of variable $g_n^{-1}(y) = t$ leads to $y = g_n(t) = (f(t))^n$ and

$$\begin{aligned}
V_{y,n} &= \pi \int_0^1 t^2 \cdot g_n'(t) \,\mathrm{d}t = \pi \left(t^2 \cdot g_n(t) \Big|_0^1 - \int_0^1 2t \cdot g_n(t) \,\mathrm{d}t \right) \\
&= \pi \left(1 - \int_0^1 2t \cdot (f(t))^n \,\mathrm{d}t \right).
\end{aligned}$$

It is a classical result (either by the Wallis formula or by a squeezing argument) that $\lim_{n\to\infty} I_n = 0$. Then $\lim_{n\to\infty} V_{x,n} = 0$ and $\lim_{n\to\infty} V_{y,n} = \pi$, since $0 \le \int_0^1 2t \cdot (f(t))^n \, \mathrm{d}t \le 2I_n$. Using the Wallis formula, we obtain that

$$\lim_{n\to\infty} \sqrt{n} a_n = \lim_{n\to\infty} \sqrt{n} \cdot \frac{V_{x,n}}{V_{y,n}} = \lim_{n\to\infty} \sqrt{n} \cdot \frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}}.$$

Then, based on the comparison criteria, $\sum_{n=1}^{\infty} a_n^{\alpha}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha/2}}$ will have the same nature. Concluding, $\sum_{n=1}^{\infty} a_n^{\alpha}$ is convergent *if and only if* $\alpha > 2$.

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MATHEMATICAL NOTES

Predicting the result of e-learning testing with a survival function calculation algorithm

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Abstract. In this note we present an algorithm for calculating the survival function and the distribution function in the case of multivariate discrete distributions. This algorithm is implemented in C++. Then, an empirical application in predicting the result of e-learning testing based on an algorithm is presented. The latter algorithm simulates the marks from several tests on the same learning unit, calculates and displays the distribution with which one works and then calculates the survival function at the desired point.

Keywords: Survival analysis, multivariate distribution, algorithm, data structures.

MSC: 60E05, 68P05, 68W01.

1. Introduction and preliminaries

Survival functions are often used in areas of risk analysis. When we have a portfolio with several factors we have a distribution on $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$.

Tsiamyrtzis and Karlis [8] discussed an efficient algorithm that makes use of the recurrence relationships in the case of multivariate Poisson distribution. That algorithm reduces the computational effort and calculates easily the required probabilities. Shih [6] proposed a bivariate discrete survival distribution and calculated some conditional probabilities. Aitchison and Ho [1] studied the mathematical properties of the multivariate discrete Poisson-log normal distributions family. Karlis [4] proposed an algorithm for maximum likelihood estimation of the parameters of the multivariate Poisson distribution. Stein et al. [7] proposed a reparameterization of the Sichel distribution and gave an algorithm for computing the maximum likelihood estimates of the new parameters. Gomez-Deniz et al. [3] proposed and studied mathematical properties of a multivariate distribution with Poisson-Lindley marginals and with a flexible covariance structure.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X : \Omega \to \mathbb{R}^d$ be a random vector with $d \geq 2$. We consider $\mu_X(B) = P(X \in B)$ be its distribution on $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$,

$$F_X(t) = P(X \leq t) = P\left(X_1 \leq t_1, \dots, X_d \leq t_d\right)$$

its distribution function and

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$$F_X^*(t) = P(X \geq t,\ X \neq t).$$

Even if we know the multivariate distribution function, it would still be difficult to calculate $F_X^*(t)$.

In multivariate case Catana [2] proved that $F^* \neq 1 - F$.

We present few definitions that we will work with.

Definition 1.1. *We say that the d-dimensional random vector X is* discrete distributed *if $Supp\,(X)$ is an at most countable set of points from $\mathbb{R}^d$.*

If $Supp\,(X) = \{a_1, \ldots, a_n\}$ and $P\,(X = a_i) = p_i$ $(i \in \{1, 2, \ldots, n\})$, then we denote $X \sim \sum\limits_{i=1}^{n} p_i \delta_{a_i}$, where δ_x is the Dirac measure in $x \in \mathbb{R}^d$.

Definition 1.2. *(Shaked and Shanthikumar* [5]*) Let $x, y \in \mathbb{R}^d$. We say that x is* smaller *than y (and denote $x \leq y$) if $x_i \leq y_i\ \forall\ i \in \{1, 2, \ldots, d\}$.*

Definition 1.3. *Let X be a d-dimensional random vector. The function*

$$\overline{F}_X : \mathbb{R}^d \to [0, \infty)\,, \overline{F}_X\,(t) = P\,(X \geq t)$$

is called the survival function *of X.*

Remark 1.4. If the distribution μ_X is absolutely continuous with respect to the Lebesgue measure, then $\overline{F}_X = F_X^*$. If the distribution μ_X is discrete then $\overline{F}_X = F_X^* + p_X$, where p_X is the probability mass function.

In this note we present an algorithm that calculates the survival function as well as the distribution function of a discrete multivariate distribution implemented in C++ (regardless of its size). We show how we can implement as a data structure a point in $\mathbb{R}^d$, and how to compare two points in $\mathbb{R}^d$ (with a given rule) in C++. At the end, an empirical application in predicting the result of e-learning testing based on an algorithm was presented.

2. Survival function calculation algorithm for discrete multivariate distribution

In this section we present an algorithm for calculating the survival function implemented in C++. It reads the dimension (dim) in which it works, the points in $\mathbb{R}^{\dim}$, the range and the values of the probability mass function, and a point $t \in \mathbb{R}^{\dim}$. Then it calculates the survival function in t.

In this algorithm the basic idea is the following: we can not use matrix writing to represent a point because the size may differ from case to case and the matrices are initialized with fixed dimension. Then we use the vector class. Thus the points will be represented by vectors and the probability mass function support will be represented by a vector of vectors.

The algorithm compares each vector in v_i with t and if $v_i \geq t$, then sums in survival_probability (initialized with 0) the probability mass function in that point represented by v_i. The value of the mass probability function in v_i is retained after the coordinates in v_i.

The algorithm implemented in C++ is the following:

Algorithm 2.1.

```
#include <iostream>
#include <vector>

using namespace std;

int main()
{
    vector<vector<double>> v;
    vector<double> a,t;
    double p,x,survival_probability;
    unsigned int nr_points,dim,q;

    cout<<"dimension=";
    cin>>dim;
    cout<<"number of points=";
    cin>>nr_points;
    for(int i=0;i<nr_points;i++)
     {
        cout<<"point "<<i+1<<":"<<endl;
        cout<<"coordinates"<<endl;
        for(int j=0;j<dim;j++)
        {
            cin>>x;
            a.push_back(x);
         }
        cout<<"probability mass function"<<endl;
        cin>>p;
        cout<<endl;
        a.push_back(p);
        v.push_back(a);
        a.clear();
    }
    cout<<"coordinates of t"<<endl;
    for(int j=0;j<dim;j++)
     {
        cin>>x;
        t.push_back(x);
    }
    cout<<endl;
```

```
survival_probability=0;
for(int i=0;i<nr_points;i++)
{
    q=1;
    for(int j=0;j<dim;j++)
        if(v.at(i).at(j)<t.at(j))
            q=0;
    if(q==1)
      survival_probability=survival_probability+v.at(i).at(dim);
}
cout<<"The survival function at the point t is equal to
"<< survival_probability<<".";

return 0;
}
```

Example 2.2. If $X \sim 0.5\delta_{(2,1)} + 0.3\delta_{(4,2)} + 0.2\delta_{(1,3)}$, let us compute $\overline{F}_X(1,2)$, $\overline{F}_X(0.8,3)$ and $\overline{F}_X(4,4)$.

We run the program with this algorithm and obtain: $\overline{F}_X(1,2) = 0.5$, $\overline{F}_X(0.8,3) = 0.2$, $\overline{F}_X(4,4) = 0$.

Example 2.3. If $X \sim 0.4\delta_{(2,1,0,4,1)} + 0.2\delta_{(2,2,2,2,2)} + 0.3\delta_{(2,3,7,6,1)} + 0.1\delta_{(1,1,2,2,1)}$, then $\overline{F}_X(1,0,0,2,0) = 1$ and $\overline{F}_X(2,2,2,2,2) = 0.2$.

Remark 2.4. If we want to calculate the distribution function with this algorithm we replace the condition

$$\text{v.at(i).at(j)} < \text{t.at(j)}$$

with the condition

$$\text{v.at(i).at(j)} > \text{t.at(j)}$$

so that the algorithm will look for all points $v_i \leq t$, then sums in survival_probability (initialized with 0) the probability mass function in that point represented by v_i.

3. Empirical application in predicting the result of e-learning testing

Suppose a student has learned a unit of learning. At the end of it they work several tests with the same degree of difficulty. Suppose the test is grid and has several subjects consisting of a number of problems, each of which receiving the same marks. After several tests, we want to find out the probability that in the next test of this learning unit the student will cross certain scoring thresholds in each subject.

The following algorithm implemented in C++ simulates the marks taken in several tests (in number of nr_tests) per subject (they are nr_subjects). It randomly generates integers between 0 and max_value. Then the algorithm determines and displays the distribution with which one works. The probability mass function for each point is ((double) nr_aparitions) / ((double) nr_tests). Finally it calculates and displays the survival function in t.

Algorithm 3.1

```
#include <iostream>
#include <cstdlib>
#include <time.h>
#include <vector>
using namespace std;
int main()
{
  int nr_subjects,nr_tests,max_value,x,nr_aparitions,q;
  vector<vector<double>> v,distribution;
  vector<double> a,t;
  double r,survival_probability;

  cout<<"number of subjects in a test=";
  cin>>nr_subjects;
  cout<<"max_value=";
  cin>>max_value;
  cout<<"number of tests=";
  cin>>nr_tests;

  for(int i=0;i<nr_tests;i++)
  {
    for(int j=0;j<nr_subjects;j++)
    {
        x=rand()%max_value;
      a.push_back(x);
    }
    v.push_back(a);
    a.clear();
  }
   for(int i=0;i<nr_tests;i++)
   {
     for(int j=0;j<nr_subjects;j++)
       cout<<v.at(i).at(j)<<" ";
     cout<<endl;
   }
   cout<<endl;

  for(int i=0;i<nr_tests;i++)
```

```
{
   nr_aparitions=1;
   for(int j=i+1;j<nr_tests;j++)
   if(v.at(j).at(0)>=0)
      if(v.at(i)==v.at(j))
      {
         nr_aparitions++;
         v.at(j).at(0)=-1;
      }
   if(v.at(i).at(0)>=0)
   {
      v.at(i).push_back(((double)nr_aparitions)/((double)nr_tests));
      distribution.push_back(v.at(i));
   }
}
cout<<"The distribution is"<<endl;
for(int i=0;i<distribution.size();i++)
{
   for(int j=0;j<=nr_subjects;j++)
      cout<<distribution.at(i).at(j)<<" ";
   cout<<endl;
}
cout<<endl;


cout<<"coordinates of t"<<endl;
for(int j=0;j<nr_subjects;j++)
{
   cin>>x;
   t.push_back(x);
}
cout<<endl;


survival_probability=0;
for(int i=0;i<distribution.size();i++)
{
   q=1;
   for(int j=0;j<nr_subjects;j++)
      if(v.at(i).at(j)<t.at(j))
         q=0;
   if(q==1)
    survival_probability=survival_probability+v.at(i).at(nr_subjects);
}
```

```
    cout<<"The survival function at the point t is equal to
"<<survival_probability<<".";

    return 0;
  }
```

For instance, when number of subjects in a test = 2, max_value = 3, number of tests = 20 and t=$(1, 1)$, the output is

```
2 2
1 1
2 1
0 0
1 2
2 2
1 0
1 2
1 2
0 0
0 0
2 0
1 1
0 2
2 2
2 0
2 0
0 1
2 1
1 0
```

```
The distribution is
2 2 0.15
1 1 0.1
2 1 0.1
0 0 0.15
1 2 0.15
1 0 0.1
2 0 0.15
0 2 0.05
0 1 0.05
coordinates of t
1
1
```

The survival function at the point t is equal to 0.5.

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PROBLEMS

Authors should submit proposed problems to `gmaproblems@rms.unibuc.ro`. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of May 2023**.

PROPOSED PROBLEMS

529. Prove that

$$\prod_{n=2}^{\infty} e^{n^2+1/2}\left(1-\frac{1}{n^2}\right)^{n^4} = \pi \exp\left(\frac{-7}{4}+\frac{3\zeta(3)}{\pi^2}\right).$$

Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.

530. Let $a, b, c, d, e \in \mathbb{R}$ such that $a \geq b \geq c \geq d \geq e \geq 0$ and

$$ab + bc + cd + de + ea = 5.$$

Prove that

$$(3-a)^2 + (3-b)^2 + (3-c)^2 + (3-d)^2 + (3-e)^2 \geq 20.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania.

531. If $n \geq 2$ is an integer, it is known that if $k \nmid n$ $\forall k \leq \sqrt{n}$, then n is a prime.

Let $\theta \in [1/3, 1/2]$. Determine the probability for a number $n \geq 2$ with $k \nmid n$ $\forall k \leq n^{\theta}$ to be a prime. That is, determine the limit

$$\lim_{x\to\infty} \frac{\#\{p \leq x \,:\, p \text{ prime}\}}{\#\{n \leq x \,:\, k \nmid n \forall k \leq n^{\theta}\}}.$$

Proposed by Constantin-Nicolae Beli, IMAR, București, Romania.

532. Let $(S, \cdot)$ be a semigroup with the property that for every $x \in S$ there is a unique $x' \in S$ such that $(xx')^2 = xx'$.

Prove that S is a group.

Proposed by Gheorghe Andrei, Constanța, and Mihai Opincariu, Brad, Romania.

533. Prove that
$$\sum_{m=2}^{\infty}(-1)^m\left(\zeta(m)-\zeta(m+1)\right)\left(H_{\frac{m-1}{2}}-H_{\frac{m}{2}}\right)=\frac{\pi^2}{3}\left(1-\log 2\right)-2\gamma,$$
where $\zeta(k)=\sum_{n=1}^{\infty}1/n^k$ is the Riemann zeta function, $H_n=\int_0^1\frac{1-x^n}{1-x}\,\mathrm{d}x$, and $\gamma=\lim_{n\to\infty}\left(-\log n+\sum_{k=1}^{n}1/k\right)$ is the Euler–Mascheroni constant.

Proposed by Narendra Bhandari, Bajura, and Yogesh Joshi, Kailali, Nepal.

534. Let $n\geq 4$.

(a) Find the smallest positive constant k_n for which the inequality
$$\frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}+k_n(\sqrt{a_1}-\sqrt{a_n})^2\geq\frac{a_1+\cdots+a_n}{n}$$
holds for all $a_1,\ldots,a_n\in\mathbb{R}$ with $a_1\geq\cdots\geq a_n>0$.

(b) Find the largest positive constant c_n for which the inequality
$$\frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}+c_n(\sqrt{a_1}-\sqrt{a_n})^2\leq\frac{a_1+\cdots+a_n}{n}$$
holds for all $a_1,\ldots,a_n\in\mathbb{R}$ with $a_1\geq\cdots\geq a_n>0$.

Proposed by Leonard Giugiuc, Traian National College, Drobeta-Turnu Severin, and Vasile Cîrtoaje, Petroleum-Gas University of Ploieşti, Romania.

535. Let $A,B\in M_n(\mathbb{C})$ be two matrices of the same rank and let $k\in\mathbb{N}^*$. (Here by $\mathbb{N}^*$ we mean $\mathbb{Z}_{\geq 1}$.)

Then $A^{k+1}B^k=A$ if and only if $B^{k+1}A^k=B$.

Proposed by Vasile Pop, Cluj-Napoca, and Mihai Opincariu, Brad, Romania.

SOLUTIONS

513. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ which verify the identity

$$xf'(x) + kf(-x) = x^2 \quad \text{for all } x \in \mathbb{R},$$

where $k \geq 1$ is an integer.

`Proposed by Vasile Pop and Ovidiu Furdui, Technical University of Cluj-Napoca, Romania.`

Solution by Leonard Giugiuc, Drobeta-Turnu Severin, Romania. We replace x by $-x$ in the identity and we obtain

$$\begin{cases} xf'(x) + kf(-x) = x^2, \\ -xf'(-x) + kf(x) = x^2. \end{cases}$$

This implies, by subtraction, that

$$x(f'(x) + f'(-x)) + k(f(-x) - f(x)) = 0.$$

Let

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = f(x) - f(-x).$$

Note that $g(-x) = g(x)$ $\forall x \in \mathbb{R}$, i.e., g is odd. In particular, $g(0) = 0$.

The previous relation rewrites as

$$xg'(x) - kg(x) = 0, \quad \text{for all } x \in \mathbb{R}.$$

When we divide by x^{k+1} we get $x^{-k}g'(x) - kx^{-k-1}g(x) = 0$, i.e., $(x^{-k}g(x))' = 0$. This relation holds on the intervals $(-\infty, 0)$ and $(0, \infty)$, but not at 0. Hence there are the constants $C_1, C_2 \in \mathbb{R}$ such that $x^{-k}g(x) = C_1$ on $(0, \infty)$ and $x^{-k}g(x) = C_2$ on $(-\infty, 0)$. Thus $g(x) = C_1x^k$ if $x > 0$ and $g(x) = C_2x^k$ if $x < 0$. Then the relation $g(x) = -g(-x)$ $\forall x > 0$ writes as $C_1x^k = -C_2(-x)^k$ and it is equivalent to $C_2 = (-1)^{k+1}C_1$. Hence

$$g(x) = \begin{cases} C_1x^k & \text{if } x \geq 0, \\ (-1)^{k+1}C_1x^k & \text{if } x \leq 0. \end{cases}$$

Note that if $x < 0$, then $(-1)^{k+1}x^k = x(-x)^{k-1} = x|x|^{k-1}$. Hence we have a more compact formula for g, $g(x) = C_1x|x|^{k-1}$.

Since $f(x) - f(-x) = g(x) = C_1x|x|^{k-1}$, by replacing $f(-x) = f(x) - C_1x|x|^{k-1}$ in the initial identity, we get

$$xf'(x) + kf(x) = C_1kx|x|^{k-1} + x^2.$$

We multiply this equation by x^{k-1} and obtain

$$\left(x^k f(x)\right)' = C_1kx^k|x|^{k-1} + x^{k+1}, \quad \text{for all } x \in \mathbb{R}.$$

Note that on $(0, \infty)$ we have $x^k|x|^{k-1} = x^{2k-1}$, which has the primitive $\dfrac{x^{2k}}{2k}$, and on $(-\infty, 0)$ we have $x^k|x|^{k-1} = (-1)^{k-1}x^{2k-1}$, which has the primitive

$\dfrac{(-1)^{k-1}x^{2k}}{2k} = \dfrac{x^{k+1}(-x)^{k-1}}{2k} = \dfrac{x^{k+1}|x|^{k-1}}{2k}$. Hence $\dfrac{x^{k+1}|x|^{k-1}}{2k}$ is a primitive of $x^k|x|^{k-1}$ on $\mathbb{R}$. Therefore the relation above implies that

$$x^k f(x) = \frac{C_1 x^{k+1}|x|^{k-1}}{2} + \frac{x^{k+2}}{k+2} + C_3, \quad \text{for all } x \in \mathbb{R},$$

where $C_3 \in \mathbb{R}$ is a constant. By taking $x = 0$ we identify $C_3 = 0$. Letting $C := \dfrac{C_1}{2} \in \mathbb{R}$, we conclude that

$$f(x) = Cx|x|^{k-1} + \frac{x^2}{k+2}, \quad \text{for all } x \in \mathbb{R}.$$

Remark 1. The authors' solution had a mistake, which escaped us at the first reading. From $xg'(x) - kg(x) = 0$ they wrongly concluded that $g(x) = C_1 x^k \ \forall x \in \mathbb{R}$. This lead to a wrong solution, $f(x) = \dfrac{x^2}{k+2}$ if k is even and $f(x) = Cx^k + \dfrac{x^2}{k+2}$ for some $C \in \mathbb{R}$ if k is odd. We encountered the same error in two other solutions we received.

Remark 2. Instead of subtracting the formulas $xf'(x) + kf(-x) = x^2$ and $-xf'(-x) + kf(x) = x^2$, we may add them and we get

$$x(f'(x) - f'(-x)) + k(f(x) + f(-x)) = 2x^2, \quad \text{i.e.,} \quad xh'(x) + kh(x) = 2x^2,$$

where $h(x) = f(x) + f(-x)$. We multiply by x^{k-1} and we get $(x^k h(x))' = 2x^{k+1}$, so $x^k h(x) = \dfrac{2x^{k+2}}{k+2} + C_3$ for some constant $C_3 \in \mathbb{R}$. When we take $x = 0$ we get $C_3 = 0$. It follows that $f(x) + f(-x) = h(x) = \dfrac{2x^2}{k+2}$. Together with $f(x) - f(-x) = C_1 x|x|^{k-1}$, this implies the same relation $f(x) = Cx|x|^{k-1} + \dfrac{x^2}{k+2}$, where $C = \dfrac{C_1}{2}$.

514. Evaluate the integral

$$\int_0^1 \frac{\log^n x}{\sqrt{x(1-x)}}\,\mathrm{d}x,$$

where n is a positive integer.

`Proposed by Mircea Ivan, Technical University of Cluj-Napoca, Romania.`

Solution by the author. We use the following notations:

- $\Gamma(x) \quad := \displaystyle\int_0^\infty e^{-t}t^{x-1}\mathrm{d}t$, $x > 0$, the *gamma function*;

- $B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $a,b>0$, the *beta function*;
- $\psi(x) := \dfrac{\mathrm{d}}{\mathrm{d}x}\log(\Gamma(x)) = \dfrac{\Gamma'(x)}{\Gamma(x)}$, $x>0$, the *digamma function*;
- $\psi^{(m)}$ - the *polygamma function* of order m (the mth derivative of ψ);
- $\zeta(s,x) := \displaystyle\sum_{k=0}^{\infty}\frac{1}{(k+x)^s}$, $s>1$, $x>0$, the Hurwitz zeta function [2, (1.3.1)];
- $\zeta(s)$ - the Riemann function $\zeta(s,1)$;

We note that

$$\frac{\partial^n B}{\partial a^n}(a,b) = \int_0^1 \left(\frac{\partial}{\partial a}\right)\left(x^{a-1}(1-x)^{b-1}\right)\mathrm{d}x = \int_0^1 \log^n x x^{a-1}(1-x)^{b-1}\mathrm{d}x.$$

Hence, our integral is equal to $\frac{\partial^n B}{\partial a^n}(1/2,1/2)$.

To compute $\frac{\partial^n B}{\partial a^n}$, we write $B(a,b) = \exp(\log B(a,b))$ and we use the chain rule for higher derivatives (the Faà di Bruno formula):

$$\big(f(g)\big)^{(n)} = \sum_{1\cdot m_1+\cdots+n\cdot m_n=n} \frac{n!\cdot f^{(m_1+\cdots+m_n)}(g)}{m_1!\,\cdots\,m_n!}\prod_{i=1}^{n}\left(\frac{g^{(i)}}{i!}\right)^{m_i} \tag{1}$$

(the sum is over all n-tuples of nonnegative integers $(m_1,\ldots,m_n)$ satisfying the constraint $1\cdot m_1+\cdots+n\cdot m_n = n$; see, e.g., [5, p. 36] or [3, p. 137], [4, Eq. (1.1)] and the references therein).

In our case, we apply (1) to $f=\exp$ and $g=\log B(\cdot,b)$, i.e., to $g(a) = \log B(a,b)$.

Since $f^{(k)} = \exp$ for all $k\geq 1$, every factor $f^{(m_1+\cdots+m_n)}(g(a))$ is equal to $\exp(\log B(a,b))$. Therefore, $B(a,b)$ can be factored out in the sum from (1).

We also have $g(a) = \log B(a,b) = \log\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, so

$$g'(a) = (\log\Gamma)'(a) - (\log\Gamma)'(a+b) = \psi(a)-\psi(a+b).$$

More generally, for $\nu\geq 1$ we have

$$\left(\frac{\partial}{\partial a}\right)^{\nu}\log B(a,b) = g^{\nu}(a) = \psi^{(\nu-1)}(a) - \psi^{(\nu-1)}(a+b).$$

When $\nu\geq 2$ we have

$$\psi^{(\nu-1)}(x) = (-1)^{\nu}(\nu-1)!\,\zeta(\nu,x) \text{ (see, e.g., [1, 6.4.10]).}$$

In conclusion, $\frac{\partial^n B}{\partial a^n}(a,b) = (f(g))^{(n)}(a)$ writes as

$$
\begin{aligned}
&\sum \frac{n!B(a,b)}{\prod\limits_{\nu=1}^{n} m_\nu!} \prod_{\nu=1}^{n} \left(\frac{g^{(\nu)}(a)}{\nu!} \right)^{m_\nu} \\
= & \sum \frac{(-1)^n n!B(a,b)}{\prod\limits_{\nu=1}^{n} m_\nu!} \prod_{\nu=1}^{n} \left(\frac{(-1)^\nu g^{(\nu)}(a)}{\nu!} \right)^{m_\nu} \\
= & \sum \frac{(-1)^n n!\; B(a,b)}{\prod\limits_{\nu=1}^{n} m_\nu!} \prod_{\nu=1}^{n} \left(\frac{(-1)^\nu \left(\psi^{(\nu-1)}(a) - \psi^{(\nu-1)}(a+b)\right)}{\nu!} \right)^{m_\nu} \qquad (2) \\
= & (-1)^n n! B(a,b) \sum \frac{(\psi(a+b)-\psi(a))^{m_1}}{\prod\limits_{\nu=1}^{n} m_\nu!} \prod_{\nu=2}^{n} \left(\frac{\zeta(\nu,a)-\zeta(\nu,a+b)}{\nu} \right)^{m_\nu}
\end{aligned}
$$

where all the sums are over all n-tuples of nonnegative integers $(m_1, \ldots, m_n)$ satisfying the constraint $1 \cdot m_1 + \cdots + n \cdot m_n = n$.

When $a = b = 1/2$ we have $B(1/2, 1/2) = \pi$, $\psi(1) - \psi(1/2) = (-\gamma) - (-\gamma - 2\log 2) = \log 4$ and $\zeta(\nu, 1/2) - \zeta(\nu, 1) = (2^\nu - 2)\,\zeta(\nu)$ for $\nu \geq 2$. Hence, from (2) we get

$$
\begin{aligned}
&\int_0^1 \frac{\log^n x}{\sqrt{x(1-x)}}\, \mathrm{d}x = \frac{\partial^n B}{\partial a^n}(1/2, 1/2) \\
&= (-1)^n n! \pi \sum_{1\cdot m_1 + \cdots + n \cdot m_n = n} \frac{\log^{m_1} 4}{\prod\limits_{\nu=1}^{n} m_\nu!} \prod_{\nu=2}^{n} \left(\frac{2^\nu - 2}{\nu} \right)^{m_\nu} \zeta(\nu)^{m_\nu}.
\end{aligned}
$$

In particular, we have:

$$
\begin{aligned}
\int_0^1 \frac{\log^1 x}{\sqrt{x(1-x)}} \mathrm{d}x &= -\pi \log 4, \\
\int_0^1 \frac{\log^2 x}{\sqrt{x(1-x)}} \mathrm{d}x &= \pi \left(\log^2 4 + 2\zeta(2) \right), \\
\int_0^1 \frac{\log^3 x}{\sqrt{x(1-x)}} \mathrm{d}x &= -\pi \left(\log^3 4 + 6 \log 4 \zeta(2) + 12\zeta(3) \right).
\end{aligned}
$$

We note that the mathematical computer software Mathematica fails in calculating the integral.

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515. Let $S = \{(\alpha, \beta, \gamma) \in (0, \pi/2)^3 : \alpha + \beta + \gamma = \pi\}$. On S we define the real valued functions

$$a := a(\alpha, \beta, \gamma) = \sqrt{\sin^4 \beta + \sin^4 \gamma - 2\sin^2 \beta \sin^2 \gamma \cos 2\alpha},$$

$$b := b(\alpha, \beta, \gamma) = \sqrt{\sin^4 \alpha + \sin^4 \gamma - 2\sin^2 \alpha \sin^2 \gamma \cos 2\beta},$$

$$c := a(\alpha, \beta, \gamma) = \sqrt{\sin^4 \alpha + \sin^4 \beta - 2\sin^2 \alpha \sin^2 \beta \cos 2\gamma}.$$

Prove that the function $f : S \to S$ which sends (α, β, γ) to

$$\left(\arccos\left(\frac{b^2 + c^2 - a^2}{2bc}\right), \arccos\left(\frac{a^2 + c^2 - b^2}{2ac}\right), \arccos\left(\frac{a^2 + b^2 - c^2}{2ab}\right)\right)$$

is well defined.

Is f injective? Is it surjective?

Proposed by Leonard Giugiuc, Traian National College, Drobeta Turnu Severin, Romania, and Abdilkadîr Altıntaş, Emirdağ, Afyonkarahisar, Turkey.

Solution by the authors. Let M be a point in the plane. Since $2\alpha + 2\beta + 2\gamma = 2\pi$, there is a triangle ABC containing M inside, such that the angles BMC, CMA and AMB are 2α, 2β and 2γ, respectively, and $AM = \sin^2 \alpha$, $BM = \sin^2 \beta$, $CM = \sin^2 \gamma$. Let a, b, c be the side lengths of this triangle. The angles of ABC are $A = \arccos\left(\frac{b^2+c^2-a^2}{2bc}\right)$, $B = \arccos\left(\frac{a^2+c^2-b^2}{2ac}\right)$ and $C = \arccos\left(\frac{a^2+b^2-c^2}{2ab}\right)$. Obviously, A, B, C are functions of α, β, γ and we have $f(\alpha, \beta, \gamma) = (A, B, C)$.

Now $A, B, C \in (0, \pi)$ are well defined and $A + B + C = \pi$. To complete the proof of $f(\alpha, \beta, \gamma) = (A, B, C) \in S$, we must prove that $A, B, C < \frac{\pi}{2}$, i.e., that ABC is an acute triangle.

We prove that $b^2 + c^2 > a^2$. We have

$$\frac{b^2 + c^2 - a^2}{2} = \sin^4 \alpha - \sin^2 \alpha \sin^2 \beta \cos 2\gamma - \sin^2 \alpha \sin^2 \gamma \cos 2\beta + \sin^2 \beta \sin^2 \gamma \cos 2\alpha.$$

Since $\cos 2\alpha = 1 - 2\sin^2 \alpha$ and similarly for β, γ, the relation we want to prove, $\frac{b^2+c^2-a^2}{2} > 0$, writes as

$$\sin^4 \alpha + \sin^2 \beta \sin^2 \gamma + 2\sin^2 \alpha \sin^2 \beta \sin^2 \gamma - \sin^2 \alpha \sin^2 \beta - \sin^2 \alpha \sin^2 \gamma > 0.$$

By ignoring the positive term $\sin^2\beta\sin^2\gamma$ it is enough to prove that

$$\sin^4\alpha + 2\sin^2\alpha\sin^2\beta\sin^2\gamma > \sin^2\alpha\sin^2\beta + \sin^2\alpha\sin^2\gamma,$$

or, equivalently,

$$2\sin^2\beta\sin^2\gamma > \sin^2\beta + \sin^2\gamma - \sin^2\alpha.$$

Now in an arbitrary triangle we have $b^2+c^2-a^2 = 2bc\cos A$ and $a/\sin A = b/\sin B = c/\sin C$, so $\sin^2 B + \sin^2 C - \sin^2 A = 2\sin B\sin C\cos A$. In particular, since α,β,γ are the angles of a triangle, we have $\sin^2\beta+\sin^2\gamma-\sin^2\alpha = 2\sin\beta\sin\gamma\cos\alpha$. So our inequality writes as

$$2\sin^2\beta\sin^2\gamma > 2\sin\beta\sin\gamma\cos\alpha, \quad \text{i.e.,} \quad 2\sin\beta\sin\gamma > 2\cos\alpha.$$

But $2\sin\beta\sin\gamma = \cos(\beta-\gamma) - \cos(\beta+\gamma) = \cos(\beta-\gamma) + \cos\alpha$. (We have $\beta+\gamma = \pi-\alpha$.) So our inequality is equivalent to $\cos(\beta-\gamma) > \cos\alpha$. As $0 \le |\beta-\gamma| < \alpha < \pi/2$, we have $\cos(\beta-\gamma) = \cos|\beta-\gamma| > \cos\alpha$, as claimed.

Hence $b^2+c^2 > a^2$. Similarly, $a^2+c^2 > b^2$ and $a^2+b^2 > c^2$, so the triangle ABC is acute and so $(A,B,C) \in S$. Thus f is well defined.

We now prove that f is neither injective nor surjective. For both proofs, we consider triplets $(\alpha,\beta,\gamma) \in S$ such that $f(\alpha,\beta,\gamma) = (A,B,C)$, with $B = C = \dfrac{\pi-A}{2}$. For injectivity, we note that if $f(\alpha,\beta,\gamma) = (A,B,C)$, then $f(\alpha,\gamma,\beta) = (A,C,B)$. In particular, if we find $(\alpha,\gamma,\beta) \in S$ such that $\beta \ne \gamma$ but $B = C$, then $(\alpha,\beta,\gamma) \ne (\alpha,\gamma,\beta)$ and $f(\alpha,\beta,\gamma) = f(\alpha,\gamma,\beta)$, so f is not injective. We will also prove that there is some $\delta \in (0,\pi/2)$ such that for $A \in (\delta,\pi/2)$ we have $\left(\dfrac{\pi-A}{2}, \dfrac{\pi-A}{2}, A\right) \notin \operatorname{Im} f$ and so f is not surjective.

Note that

$$\begin{aligned} a^2 &= \sin^4\beta + \sin^4\gamma - 2\sin^2\beta\sin^2\gamma(1-2\sin^2\alpha) \\ &= 4\sin^2\alpha\sin^2\beta\sin^2\gamma + (\sin^2\beta - \sin^2\gamma)^2. \end{aligned}$$

Similarly, we have

$$\begin{aligned} b^2 &= 4\sin^2\alpha\sin^2\beta\sin^2\gamma + (\sin^2\alpha - \sin^2\gamma)^2, \\ c^2 &= 4\sin^2\alpha\sin^2\beta\sin^2\gamma + (\sin^2\alpha - \sin^2\beta)^2. \end{aligned}$$

We have $B = C$ iff $b = c$, i.e., iff $b^2 = c^2$, which is equivalent to $(\sin^2\alpha - \sin^2\gamma)^2 = (\sin^2\alpha - \sin^2\beta)^2$ and to $\sin^2\alpha - \sin^2\gamma = \pm(\sin^2\alpha - \sin^2\beta)$. So, we have two cases: $\sin^2\beta = \sin^2\gamma$ and $2\sin^2\alpha = \sin^2\beta + \sin^2\gamma$. We consider the two cases separately.

Case 1. $\sin^2\beta = \sin^2\gamma$. This is equivalent to $\beta = \gamma$, so $\alpha = \pi - 2\beta$. Since $\alpha \in (0,\pi/2)$, we have $\beta \in (\pi/4,\pi/2)$. Then $\sin\alpha = \sin 2\beta$, so that $a^2 = 4\sin^2 2\beta\sin^4\beta$ and $b^2 = c^2 = 4\sin^2 2\beta\sin^4\beta + (\sin 2\beta - \sin\beta)^2$. It follows that

$$A = \arccos\left(\frac{2\sin^2 2\beta\sin^4\beta + (\sin 2\beta - \sin\beta)^2}{4\sin^2 2\beta\sin^4\beta + (\sin 2\beta - \sin\beta)^2}\right) =: \phi(\beta),$$

with $\phi : (\pi/4, \pi/2) \to (0, \pi/2)$.

We have $\phi(\pi/4 + 0) = \arccos(3/5)$ and $\phi(\pi/2 - 0) = 0$. Since ϕ is continuous, we have either $\sup \phi(\beta) = \max \phi(\beta) = \phi(\beta_0)$ for some $\beta_0 \in (\pi/4, \pi/2)$ or $\sup \phi(\beta) \in \{\phi(\pi/4 + 0), \phi(\pi/2 - 0)\}$. From $\phi(\beta) < \pi/2$ for all $\beta \in (\pi/4, \pi/2)$ and $\max \{\phi(\pi/4 + 0), \phi(\pi/2 - 0)\} < \pi/2$ we deduce that $\delta_1 := \sup \phi(\beta) < \pi/2$.

Case 2. $2 \sin^2 \alpha = \sin^2 \beta + \sin^2 \gamma$. Now we have

$$\begin{aligned}\sin^2 \beta + \sin^2 \gamma &= \frac{1}{2}(2 - \cos 2\beta - 2 \cos 2\gamma) = 1 - \cos(\beta + \gamma) \cos(\beta - \gamma) \\ &= 1 + \cos \alpha \cos(\beta - \gamma)\end{aligned}$$

because $\cos(\beta + \gamma) = \cos(\pi - \alpha) = -\cos \alpha$.

Hence, our relation writes as $2-2\cos^2 \alpha = 2 \sin^2 \alpha = 1+\cos \alpha \cos(\beta-\gamma)$ and it is equivalent to $\cos(\beta - \gamma) = 2 \cos \alpha - \dfrac{1}{\cos \alpha}$.

In order that $(\alpha, \beta, \gamma) \in S$, we have restrictions on α, which result from conditions on $\beta - \gamma$. Without loss of generality, we may assume that $\beta \geq \gamma$, i.e., that $\beta - \gamma \geq 0$. We have the conditions $0 < \alpha < \pi/2$ and $0 \leq \gamma \leq \beta \leq \pi/2$. The condition $\beta < \pi/2$ is equivalent to $\pi > 2\beta = \beta + \pi - \alpha - \gamma$, i.e., to $\beta - \gamma < \alpha$. The condition $\gamma > 0$ is equivalent to $\pi - \alpha - \beta = \gamma > \gamma$, i.e., to $\beta - \gamma < \pi - \alpha$. Since $\alpha < \pi - \alpha$, the last condition is superfluous. So, the conditions on $\beta - \gamma$ are $0 < \beta - \gamma < \alpha$. Since $\alpha < \pi/2$, this is equivalent to $\cos \alpha < \cos(\beta - \gamma) \leq \cos 0 = 1$, i.e., to $\cos \alpha < \dfrac{1}{\cos \alpha} - 2 \cos \alpha \leq 1$. The left inequality is equivalent to $\cos \alpha < 1/\sqrt{3}$ and the right inequality is equivalent to $0 \leq 2 \cos^2 \alpha + \cos \alpha - 1 = (\cos \alpha + 1)(2 \cos \alpha - 1)$, i.e., to $\cos \alpha \geq 1/2$. So, the condition on α is $\alpha \in (\theta, \pi/3]$, where $\theta = \arccos(\sqrt{1/3})$.

Now $\beta - \gamma = g(\alpha)$, where $g : (\theta, \pi/3] \to \mathbb{R}$ is given by

$$g(\alpha) = \arccos \left(\frac{1}{\cos \alpha} - 2 \cos \alpha \right).$$

Note that, if $\alpha < \pi/3$, then $\cos \alpha > 1/2$, so the inequality $\dfrac{1}{\cos \alpha} - 2 \cos \alpha \leq 1$ becomes strict. Hence, $g(\alpha) > 0$ for $\alpha \in (\theta, \pi/3)$.

Since $\beta - \gamma = g(\alpha)$ and $\beta + \gamma = \pi - \alpha$, we have $\beta = \frac{1}{2}(\pi - \alpha + g(\alpha))$ and $\gamma = \frac{1}{2}(\pi - \alpha - g(\alpha))$. Similarly, when $\beta \leq \gamma$ we have $\beta = \frac{1}{2}(\pi - \alpha - g(\alpha))$ and $\gamma = \frac{1}{2}(\pi - \alpha + g(\alpha))$.

We define the function $\psi : (\theta, \pi/3] \to (0, \pi/2)$ by

$$\begin{aligned}\psi(\alpha) &= A \left(\alpha, \frac{1}{2}(\pi - \alpha + g(\alpha)), \frac{1}{2}(\pi - \alpha - g(\alpha)) \right) \\ &= A \left(\alpha, \frac{1}{2}(\pi - \alpha - g(\alpha)), \frac{1}{2}(\pi - \alpha + g(\alpha)) \right).\end{aligned}$$

We extend by continuity g and ψ at $\alpha = \theta$. We have $\cos\theta = \sqrt{1/3}$ and so $\sin\theta = \sqrt{2/3}$. Now $g(\theta) = \arccos\left(\frac{1}{\cos\theta} - 2\cos\theta\right) = \arccos(\sqrt{1/3}) = \theta$. It follows that $\beta = \frac{1}{/}2(\pi - \theta + g(\theta)) = \pi/2$ and $\gamma = \frac{1}{/}2(\pi - \theta - g(\theta)) = \pi/2 - \theta$. Note that for $(\alpha, \beta, \gamma) = (\theta, \pi/2, \pi/2 - \theta)$ we have $\sin\alpha = \sqrt{2/3}$, $\sin\beta = 1$, and $\sin\gamma = \cos\alpha = \sqrt{1/3}$. Then the formulas for a, b, c when $(\alpha, \beta, \gamma) = (\theta, \pi/2, \pi/2 - \theta)$ give $a = 4/3$ and $b = c = 1$. Consequently,

$$\psi(\theta) = A(\theta, \pi/2, \pi/2 - \theta) = \arccos\left(\frac{b^2 + c^2 - a^2}{2bc}\right) = \arccos(1/9).$$

Since ψ is continuous, we have either $\sup\psi(\alpha) = \max\psi(\alpha) = \psi(\alpha_0)$ for some $\alpha_0 \in (\theta, \pi/3]$ or $\sup\psi(\alpha) = \psi(\theta + 0)$. But $\psi(\alpha) < \pi/2\ \forall\alpha \in (\theta, \pi/3]$ and $\psi(\theta + 0) = \arccos(1/9) < \pi/2$, so that $\delta_2 := \sup\psi(\alpha) < \pi/2$.

We now prove that f is not surjective. Let $\delta = \max\{\delta_1, \delta_2\}$. Then $0 < \delta < \pi/2$. So, if $\delta < A < \pi/2$, then $\left(\dfrac{\pi - A}{2}, \dfrac{\pi - A}{2}, A\right) \in S$. Suppose that $\left(\dfrac{\pi - A}{2}, \dfrac{\pi - A}{2}, A\right) = f(\alpha, \beta, \gamma)$. Then (α, β, γ) is in one of the cases 1 and 2 above. In the first case we have $A = \phi(\beta) \le \delta_1 \le \delta$ and in the second case $A = \psi(\alpha) \le \delta_2 \le \delta$. As we always reached a contradiction, f is not surjective.

For injectivity, let $\alpha \in (\theta, \pi/3)$. Then $g(\alpha) > 0$, so, if $\beta = \frac{1}{2}(\pi - \alpha + g(\alpha))$ and $\gamma = \frac{1}{2}(\pi - \alpha - g(\alpha))$, then $\beta > \gamma$. It follows that $(\alpha, \beta, \gamma) \ne (\alpha, \gamma, \beta)$, but $f(\alpha, \beta, \gamma) = f(\alpha, \gamma, \beta) = \left(\dfrac{\pi - A}{2}, \dfrac{\pi - A}{2}, A\right)$, where $A = \psi(\alpha)$. Hence, f is not injective. $\square$

516. Let $n \ge 2$ be an integer. Let $v_1, \dots, v_{n-1}$ be some orthonormal vectors and let v be a unit vector in $\mathbb{R}^n$. We regard $v_1, \dots, v_{n-1}, v$ as column vectors, i.e., as $n \times 1$ matrices.

We consider the $n \times n$ matrix

$$A = v_1 \cdot v_1^T + \cdots + v_{n-1} \cdot v_{n-1}^T - v \cdot v^T.$$

If A is not invertible, prove that $A^2 = A$ and determine its rank.

`Proposed by Marian Panţiruc, Gheorghe Asachi Technical University of Iaşi, Romania.`

Solution by the author. We complete $v_1, \dots, v_{n-1}$ to an orthonormal basis $v_1, \dots, v_n$ of $\mathbb{R}^n$. We claim that if

$$B = v_1 v_1^T + \cdots + v_n v_n^T,$$

then $B = I_n$.

Recall that $x^T y = \langle x, y \rangle$, so the property that $v_1, \ldots, v_n$ is an orthonormal basis writes as $v_i^T v_j = \langle v_i, v_j \rangle = \delta_{i,j}$. Hence

$$Bv_i = \sum_{k=1}^{n} v_k v_k^T v_i = \sum_{k=1}^{n} v_k \delta_{k,i} = v_i,$$

i.e., $(B - I_n)v_i = 0$, so $v_i \in \ker(B - I_n)$ for $1 \leq i \leq n$. Since $v_1, \ldots, v_n$ generate $\mathbb{R}^n$, we have $\ker(B - I_n) = \mathbb{R}^n$, so that $B - I_n = 0$, i.e., $B = I_n$, as claimed.

Consequently, $v_1 v_1^T + \cdots + v_{n-1} v_{n-1}^T = B - v_n v_n^T = I_n - v_n v_n^T$. It follows that $A = I_n - v_n v_n^T - vv^T$. For convenience, we put $w = v_n$, so that $A = I_n - ww^T - vv^T$.

Assume now that A is not invertible. Then 0 is an eigenvalue and let $x \neq 0$ be an eigenvector for the eigenvalue 0. Since $v^T x = \langle x, v \rangle$ and $w^T x = \langle x, w \rangle$, we have

$$0 = Ax = x - ww^T x - vv^T x = x - \langle x, w \rangle \cdot w - \langle x, v \rangle \cdot v,$$

that is $x = \langle x, w \rangle \cdot w + \langle x, v \rangle \cdot v$. It follows that

$$\begin{aligned} \langle x, v \rangle &= \langle \langle x, w \rangle \cdot w + \langle x, v \rangle \cdot v, v \rangle \\ &= \langle x, w \rangle \cdot \langle w, v \rangle + \langle x, v \rangle \cdot \langle v, v \rangle \\ &= \langle x, w \rangle \cdot \langle w, v \rangle + \langle x, v \rangle . \end{aligned}$$

Hence, $\langle x, w \rangle \cdot \langle w, v \rangle = 0$. By a similar proof, $\langle x, v \rangle \cdot \langle w, v \rangle = 0$. If $\langle w, v \rangle \neq 0$, then one necessarily has $\langle x, w \rangle = \langle x, v \rangle = 0$, so that $x = \langle x, w \rangle \cdot w + \langle x, v \rangle \cdot v = 0$, which is a contradiction.

It results $\langle w, v \rangle = 0$. Recall that also $\langle w, w \rangle = \langle v, v \rangle = 1$. Hence, if $X = ww^T$ and $Y = vv^T$, then $X^2 = w(w^T w)w^T = w \cdot \langle w, w \rangle \cdot w^T = ww^T = X$, $XY = w(w^T v)v^T = w \cdot \langle w, v \rangle \cdot w^T = 0$ and, similarly, $Y^2 = Y$ and $YX = 0$. Therefore,

$$\begin{aligned} A^2 = (I_n - X - Y)^2 &= I_n + X^2 + Y^2 - 2X - 2Y + XY + YX \\ &= I_n + X + Y - 2X - 2Y + 0 + 0 = I_n - X - Y = A. \end{aligned}$$

In order to compute the rank of A, note that since v and w are non-zero and orthogonal, they are linearly independent.

We first determine $\dim \ker A$. As seen above, for every $x \in \ker A$ we have $x = \langle x, w \rangle \cdot w + \langle x, v \rangle \cdot v$, so x belongs to the space spanned by v and w. Conversely, we have

$$\begin{aligned} Av &= (I_n - ww^T - vv^T)v = v - w(w^T v) - v(v^T v) \\ &= v - w \langle w, v \rangle - v \langle v, v \rangle = v - 0 - v = 0, \end{aligned}$$

so $v \in \ker A$. Similarly, $w \in \ker A$, so $\ker A$ is the space spanned by v and w, of dimension 2. It follows that $\operatorname{rank} A = n - \dim \ker A = n - 2$. $\square$

517. Calculate the sum

$$S=\sum_{p,q,r=1}^{\infty}\frac{3p+r}{5^{p+q+r}r(p+q)(q+r)(r+p)}.$$

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania.

Solution by the author. We add the six formulas for S obtained by permutating the variables p, q and r. Since

$$\frac{3p+r}{5^{p+q+r}r(p+q)(q+r)(r+p)}=\frac{3p^2q+pqr}{5^{p+q+r}pqr(p+q)(q+r)(r+p)}$$

and similarly for the permutations of p, q and r, we get

$$\begin{aligned}6S&=\sum\frac{3(p^2q+q^2r+r^2p+pq^2+qr^2+rp^2)+6pqr}{5^{p+q+r}pqr(p+q)(q+r)(r+p)}\\&=\sum\frac{3(p+q)(q+r)(r+p)}{5^{p+q+r}pqr(p+q)(q+r)(r+p)}=\sum_{p+q+r=1}^{\infty}\frac{3}{5^{p+q+r}pqr}\\&=3\sum_{p-1}^{\infty}\frac{1}{p\cdot 5^p}\cdot\sum\frac{1}{q\cdot 5^q}\cdot\sum\frac{1}{r\cdot 5^r}=3\left(\sum_{n=1}^{\infty}\frac{1}{n\cdot 5^n}\right)^3.\end{aligned}$$

Define $F:(-1,1)\to\mathbb{R}$ by $F(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}$. Then, $F(0)=0$ and $F'(x)=\sum_{n=1}^{\infty}x^{n-1}=\frac{1}{1-x}$. It follows that $F(x)=-\ln(1-x)=\ln\frac{1}{1-x}$. In conclusion,

$$S=\frac{1}{2}F\left(\frac{1}{5}\right)^3=\frac{1}{2}\left(\ln\frac{5}{4}\right)^3.$$

□

518. Calculate the integral:

$$\int_0^{\infty}\frac{x^2\sqrt{x}\ln x}{x^4+x^2+1}\mathrm{d}x.$$

Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.

Solution by the author. Let us put

$$I=\int_0^{\infty}\frac{x^2\sqrt{x}\ln x}{x^4+x^2+1}\mathrm{d}x;\ A=\int_0^{1}\frac{x^2\sqrt{x}\ln x}{x^4+x^2+1}\mathrm{d}x;\ B=\int_1^{\infty}\frac{x^2\sqrt{x}\ln x}{x^4+x^2+1}\mathrm{d}x.$$

In integral A we make the substitution $x^2 = y$, which results in

$$\begin{aligned}
A &= \frac{1}{4}\int_0^1 \frac{(1-y)y^{\frac{3}{4}}\ln y}{1-y^3}\,\mathrm{d}y = \frac{1}{4}\left(\int_0^1 \frac{y^{\frac{3}{4}}\ln y}{1-y^3}\,\mathrm{d}y - \int_0^1 \frac{y^{\frac{7}{4}}\ln y}{1-y^3}\,\mathrm{d}y\right)\\
&= \frac{1}{4}\left(\int_0^1 \sum_{n=0}^{\infty} y^{3n+\frac{3}{4}}\ln y\,\mathrm{d}y - \int_0^1 \sum_{n=0}^{\infty} y^{3n+\frac{7}{4}}\ln y\,\mathrm{d}y\right)\\
&= \frac{1}{4}\sum_{n=0}^{\infty}\left(\int_0^1 y^{3n+\frac{3}{4}}\ln y\,\mathrm{d}y - \int_0^1 y^{3n+\frac{7}{4}}\ln y\,\mathrm{d}y\right).
\end{aligned}$$

Also, using the following relationship

$$\int_0^1 x^a \ln x\,\mathrm{d}x = -\frac{1}{(a+1)^2},\ a \geq 0, a \in \mathbb{R},$$

we obtain

$$A = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1}{\left(3n+\frac{11}{4}\right)^2} - \frac{1}{\left(3n+\frac{7}{4}\right)^2}\right) = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{\frac{1}{9}}{\left(n+\frac{11}{12}\right)^2} - \frac{\frac{1}{9}}{\left(n+\frac{7}{12}\right)^2}\right).$$

Now, using the definition of trigamma function $\psi_1(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2}$, we obtain the value of the A integral as

$$A = \frac{1}{36}\left(\psi_1\left(\frac{11}{12}\right) - \psi_1\left(\frac{7}{12}\right)\right).$$

For the integral B we make the substitution $x = \frac{1}{u}$, which results in

$$B = -\int_0^1 \frac{u^{-\frac{1}{2}}\ln y}{u^4+u^2+1}\,\mathrm{d}u.$$

Proceeding similarly as for the A integral, we get

$$B = \frac{1}{36}\left(\psi_1\left(\frac{1}{12}\right) - \psi_1\left(\frac{5}{12}\right)\right).$$

The third integral is found to be

$$I = A + B = \frac{1}{36}\left(\psi_1\left(\frac{1}{12}\right) - \psi_1\left(\frac{5}{12}\right) - \psi_1\left(\frac{7}{12}\right) + \psi_1\left(\frac{11}{12}\right)\right).$$

Finally, using trigamma's reflection formula

$$\psi_1(x) + \psi_1(1-x) = \frac{\pi^2}{\sin^2(\pi x)}$$

and the formulas

$$\sin\frac{5\pi}{12} = \sin\left(\frac{2\pi}{3} - \frac{\pi}{4}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}, \quad \sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\sqrt{6}-\sqrt{2}}{4},$$

we get

$$\psi_1\left(\frac{5}{12}\right)+\psi_1\left(\frac{7}{12}\right)=4\pi^2(2-\sqrt{3});\ \psi_1\left(\frac{1}{12}\right)+\psi_1\left(\frac{11}{12}\right)=4\pi^2(2+\sqrt{3}).$$

This gives the result for the original integral as

$$I=\frac{1}{36}\left(4\pi^2(2+\sqrt{3})-4\pi^2(2-\sqrt{3})\right)=\frac{2\sqrt{3}}{9}\pi^2.$$

Solution by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania. We solve the problem by complex contour integration.

Let $D=\mathbb{C}\setminus\{-iy \mid y\geq 0\}$. Then every $z\in D$ writes as $z=|z|e^{\theta i}$, with $\theta\in(-\pi/2,\pi/2)$. So, on D we may define a branch of the log function, given by $\log z=\log|z|+\theta i$, where $\theta\in(-\pi/2,\pi/2)$. On $(0,\infty)$ this coincides with the usual log function. Also, for every $z\in U$ and $\alpha\in\mathbb{C}$ we can define z^α as $e^{\alpha\log z}$.

We denote by I our integral, $I=\int_0^\infty f(x)\mathrm{d}x$, where $f:D\to\mathbb{C}$ is defined by formula $f(z)=\frac{z^{5/2}\log z}{z^4+z^2+1}$.

Now for every $x>0$ we have $-x=xe^{\pi i}$, so $\log(-x)=\log x+\pi i$. Consequently, $(-x)^{5/2}=e^{5/2\log x+5\pi/2\,i}=x^{5/2}i$ because $e^{5\pi/2\,i}=i$. Hence,

$$f(-x)=\frac{(-x)^{5/2}\log(-x)}{x^4+x^2+1}=\frac{ix^{5/2}(\log x+\pi i)}{x^4+x^2+1}=if(x)-\pi g,$$

with $g:D\to\mathbb{C}$, $g(z)=\frac{z^{5/2}}{z^4+z^2+1}$. We also have

$$g(-x)=\frac{(-x)^{5/2}}{x^4+x^2+1}=\frac{ix^{5/2}}{x^4+x^2+1}=ig(x).$$

Let $J=\int_0^\infty g(x)\mathrm{d}x$. Since we have

$$\int_{-\infty}^0 f(x)\mathrm{d}x=\int_0^\infty f(-x)\mathrm{d}x=\int_0^\infty(if(x)-\pi g(x))\mathrm{d}x=iI-\pi J,$$

$$\int_{-\infty}^0 g(x)\mathrm{d}x=\int_0^\infty g(-x)\mathrm{d}x=\int_0^\infty ig(x)\mathrm{d}x=iJ,$$

it follows that

$$\int_{-\infty}^\infty f(x)\mathrm{d}x=\int_{-\infty}^0 f(x)\mathrm{d}x+\int_0^\infty f(x)\mathrm{d}x=(1+i)I-\pi J,$$

$$\int_{-\infty}^\infty g(x)\mathrm{d}x=\int_{-\infty}^0 g(x)\mathrm{d}x+\int_0^\infty g(x)\mathrm{d}x=(1+i)J.$$

Now the zeros of $z^4+z^2+1=(z^6-1)/(z^2-1)$ are $e^{\theta i}$, where $\theta=\pm\pi/3,\pm2\pi/3$. Of these, the roots in the upper plane are $\alpha=e^{\pi i/3}=1/2+i\sqrt{3}/2$ and $\beta=e^{2\pi i/3}=-1/2+i\sqrt{3}/2$. We have $\log\alpha=\pi i/3$, $\log\beta=2\pi i/3$, $\alpha^3=-1$, $\beta^3=1$, $\alpha^{5/2}=e^{5\pi i/6}=-\sqrt{3}/2+i/2$ and $\beta^{5/2}=e^{5\pi i/3}=$

$1/2 - i\sqrt{3}/2$. This allows us to calculate the residues of f and g at α and β, which are their (simple) poles in the upper plane. Since $g(z) = \frac{z^{5/2}}{P(z)}$ and $f(z) = \frac{z^{5/2}\log z}{P(z)}$, where $P(z) = z^4 + z^2 + 1$, and α is a simple root of P, the residues of g and f at α are $\frac{\alpha^{5/2}}{P'(\alpha)}$ and $f(z) = \frac{\alpha^{5/2}\log\alpha}{P'(\alpha)}$, respectively. Since $P'(z) = 4z^3 + 2z$, we get

$$\mathrm{Res}(g,\alpha) = \frac{\alpha^{5/2}}{4\alpha^3 + 2\alpha} = \frac{-\sqrt{3}/2 + i/2}{4(-1) + 2(1/2 + i\sqrt{3}/2)} = \frac{\sqrt{3}}{6}$$

and

$$\mathrm{Res}(f,\alpha) = \log\alpha \mathrm{Res}(g,\alpha) = \frac{\sqrt{3}\pi i}{18}.$$

Similarly,

$$\mathrm{Res}(g,\beta) = \frac{\beta^{5/2}}{4\beta^3 + 2\beta} = \frac{1/2 - i\sqrt{3}/2}{4(-1) + 2(1/2 + i\sqrt{3}/2)} = -\frac{\sqrt{3}i}{6}$$

and

$$\mathrm{Res}(f,\beta) = \log\beta \mathrm{Res}(g,\beta) = \frac{\sqrt{3}\pi}{9}.$$

Now for $R \gg 0$ we integrate f and g over the path C_R consisting of the interval $[-R, -1/R]$, the semicircle S'_R joining $-1/R$ and $1/R$ in the upper plane, the interval $[1/R, R]$, and the semicircle S_R joining R and $-R$ in the upper plane. Since $R \gg 0$, we have that C_R contains inside both poles α and β of f and g. It follows that

$$\begin{aligned}
\int_{C_R} g(z)\,\mathrm{d}z &= 2\pi i\left(\mathrm{Res}(g,\alpha) + \mathrm{Res}(g,\beta)\right) = 2\pi i\left(\frac{\sqrt{3}}{6} - \frac{\sqrt{3}i}{6}\right) \\
&= \frac{\sqrt{3}\pi}{3}(1+i), \\
\int_{C_R} f(z)\,\mathrm{d}z &= 2\pi i\left(\mathrm{Res}(f,\alpha) + \mathrm{Res}(f,\beta)\right) = 2\pi i\left(\frac{\sqrt{3}\pi i}{18} + \frac{\sqrt{3}\pi}{9}\right) \\
&= \frac{\sqrt{3}\pi^2}{9}(-1+2i).
\end{aligned}$$

We now take limits as $R \to \infty$ in the equalities above. In both cases, the integral $\int_{C_R}$ writes as $\int_{-R}^{-1/R} + \int_{S'_R} + \int_{1/R}^{R} + \int_{S_R}$. As the path S'_R has a length of π/R and for $z \in S'_R$ we have $|z| = 1/R$, so that $g(z) = O(R^{-5/2})$ and $f(z) = O(R^{-5/2}\log R)$, it follows that $\int_{S'_R} g(z)\,\mathrm{d}z = O(R^{-7/2})$ and $\int_{S'_R} f(z)\,\mathrm{d}z = O(R^{-7/2}\log R)$. Also the path S_R has a length of πR and for $z \in S_R$ we have $|z| = R$ and so $g(z) = O(R^{-3/2})$ and $f(z) = O(R^{-3/2}\log R)$.

It follows that $\int_{S_R} g(z)\,\mathrm{d}z = O(R^{-1/2})$ and $\int_{S_R} f(z)\,\mathrm{d}z = O(R^{-1/2}\log R)$. Therefore,

$$\begin{aligned}\frac{\sqrt{3}\pi}{3}(1+i) &= \lim_{R\to\infty}\int_{C_R} g(z)\,\mathrm{d}z\\ &= \lim_{R\to\infty}\left(\int_{-R}^{-1/R}+\int_{S'_R}+\int_{1/R}^{R}+\int_{S_R}\right) g(z)\,\mathrm{d}z\\ &= \int_{-\infty}^{0} g(z)\,\mathrm{d}z + 0 + \int_{0}^{\infty} g(z)\,\mathrm{d}z + 0 = \int_{-\infty}^{\infty} g(z)\,\mathrm{d}z = (1+i)J,\end{aligned}$$

whence $J = \frac{\sqrt{3}\pi}{3}$.

Similarly, we have

$$\frac{\sqrt{3}\pi^2}{9}(-1+2i) = \lim_{R\to\infty}\int_{C_R} f(z)\,\mathrm{d}z = \int_{-\infty}^{\infty} f(z)\,\mathrm{d}z = (1+i)I - \pi J$$

and so $(1+i)I = \frac{\sqrt{3}\pi^2}{9}(-1+2i)+\pi J = \frac{\sqrt{3}\pi^2}{9}(-1+2i)+\frac{\sqrt{3}\pi^2}{3} = \frac{\sqrt{3}\pi^2}{9}(2+2i)$, which implies that $I = \frac{2\sqrt{3}\pi^2}{9}$. $\square$

519. Calculate

$$\sum_{n=1}^{\infty}(-1)^n n^2\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}-\ln 2+\frac{1}{4n}-\frac{1}{16n^2}\right).$$

`Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania.`

Solution by the authors. The series equals $\frac{\pi-3}{32}$.

We need the following formula, which can be proved by mathematical induction

$$\sum_{k=1}^{n}(-1)^k k^2 = (-1)^n\frac{n(n+1)}{2}, \quad n \geq 1. \tag{1}$$

We also use Abel's summation formula, which states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of real or complex numbers and $A_n = \sum_{k=1}^{n} a_k$, then $\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k(b_k - b_{k+1})$, or, its infinite version,

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} A_n b_{n+1} + \sum_{k=1}^{\infty} A_k(b_k - b_{k+1}). \tag{2}$$

We apply formula (2), with $a_n = (-1)^n n^2$ and $b_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} - \ln 2 + \frac{1}{4n} - \frac{1}{16n^2}$. By (1), we have $A_n = (-1)^n \dfrac{n(n+1)}{2}$. Since also

$$b_n - b_{n+1} = -\frac{1}{16n^2(n+1)^2(2n+1)}, \tag{3}$$

we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^n n^2 \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} - \ln 2 + \frac{1}{4n} - \frac{1}{16n^2}\right) \\
&\overset{(1)}{=} \lim_{n\to\infty} (-1)^n \frac{n(n+1)}{2} b_{n+1} + \sum_{n=1}^{\infty}(-1)^n \frac{n(n+1)}{2}(b_n - b_{n+1}) \\
&= -\frac{1}{32}\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)(2n+1)}.
\end{aligned}$$

Here we used the fact that $\lim\limits_{n\to\infty} n(n+1)b_{n+1} = 0$, which is proved by Cesàro-Stolz lemma, the $0/0$ case:

$$\begin{aligned}
\lim_{n\to\infty} n(n+1)b_{n+1} &= \lim_{n\to\infty} \frac{b_{n+1}}{\frac{1}{n(n+1)}} = \lim_{n\to\infty} \frac{b_n - b_{n+1}}{\frac{1}{(n-1)n} - \frac{1}{n(n+1)}} \\
&= \lim_{n\to\infty} \frac{b_n - b_{n+1}}{\frac{2}{(n-1)n(n+1)}} = \lim_{n\to\infty} -\frac{n-1}{32n(n+1)(2n+1)} = 0.
\end{aligned}$$

We have $\frac{1}{n(n+1)(2n+1)} = \frac{1}{n} + \frac{1}{n+1} - \frac{4}{2n+1}$, so our sum writes as

$$\begin{aligned}
-\frac{1}{32}\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)(2n+1)} &= \frac{1}{32}\left(4\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}\right) \\
&= \frac{1}{32}\left(4\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}\right) \\
&= \frac{1}{32}\left(4\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n} + \sum_{m=2}^{\infty}\frac{(-1)^m}{m}\right) \\
&= \frac{1}{32}\left(4\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1} + 1\right) \\
&= \frac{\pi - 3}{32}
\end{aligned}$$

and the problem is solved.

Here we used the well-known Madhava–Leibniz series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

□

520. Let $(x_n)_{n\geq 0}$ be a sequence with $x_0 \in (0, \pi/2)$ and

$$x_{n+1} = \begin{cases} \sin x_n & \text{if } n \text{ is even,} \\ \cos x_n & \text{if } n \text{ is odd.} \end{cases}$$

Prove that $x_{2n} \to a$ and $x_{2n+1} \to b$ when $n \to \infty$, where a and b are two constants that are independent of the choice of x_0.

Also determine if the series

$$\sum_{n=1}^{\infty} |x_{2n} - a|^{\alpha} \quad \text{and} \quad \sum_{n=1}^{\infty} |x_{2n+1} - b|^{\alpha}$$

are convergent for $\alpha > 0$.

Proposed by Radu Strugariu, Gheorghe Asachi Technical University of Iaşi, Romania.

Solution by the author. If $x_n \in (0, \pi/2)$, then both $\sin x_n$ and $\cos x_n$ belong to $(0, 1)$ and so $x_{n+1} \in (0, 1) \subset (0, \pi/2)$, regardless of the parity of n. Since $x_0 \in (0, \pi/2)$, we have, by induction, $x_n \in (0, \pi/2)$ $\forall n \geq 0$.

We note that $x_{2n+2} = \cos(\sin x_{2n})$, so, if $x_{2n} \to a$ as $n \to \infty$, then, by taking limits, we have $a = \cos(\sin a)$. Similarly, from $x_{2n+3} = \sin(\cos x_{2n+1})$ we obtain that if $x_{2n+1} \to b$, then $b = \sin(\cos b)$.

We claim that each of the equations $\cos(\sin x) = x$ and $\sin(\cos x) = x$ has a unique solution in the interval $(0, \pi/2)$. For convenience, we denote by $p, q : [0, \pi/2] \to [0, 1] \subset [0, \pi/2]$ the maps given by $p(x) = \cos(\sin x)$ and $q(x) = \sin(\cos x)$. Since $\cos : [0, \pi/2] \to [0, 1] \subset [0, \pi/2]$ is decreasing and the function $\sin : [0, \pi/2] \to [0, 1] \subset [0, \pi/2]$ is increasing, it follows that both p and q are decreasing. Since the map $x \mapsto -x$ is also decreasing, we have that both functions $g, h : (0, \pi/2) \to \mathbb{R}$, given by $g(x) = p(x) - x$ and $h(x) = q(x) - x$ are decreasing. But $g(0) = 1 > 0$, $h(0) = \sin 1 > 0$, $g(1) = \cos(\sin 1) - 1 < 0$, and $h(1) = \sin(\cos 1) - 1 < 0$. Since g and h are (strictly) decreasing, this implies that each of the equations $g(x) = 0$ and $h(x) = 0$ has unique solution, say, a and b, respectively. Moreover, $a, b \in (0, 1)$.

For every integer $n \geq 0$ we have $x_{2n+2} = p(x_{2n})$ and $x_{2n+3} = q(x_{2n+1})$. We put $y_n = x_{2n}$ and $z_n = x_{2n+1}$, so that $y_{n+1} = p(y_n)$ and $z_{n+1} = q(z_n)$.

For every $n \geq 0$ we have $y_{n+2} = p \circ p(y_n)$. As p is decreasing, $p \circ p$ is increasing. It follows that both sequences $y_0, y_2, y_4, \ldots$ and $y_1, y_3, y_5, \ldots$ are monotone. (If $y_0 \leq y_2$ then (y_{2n}) is increasing; if $y_0 \geq y_2$ then (y_{2n}) is increasing. Similarly for (y_{2n+1}).) Consequently, we have $y_{2n} \to \ell_1$ and $y_{2n+1} \to \ell_2$, with $\ell_1, \ell_2 \in [0, \pi/2]$. By taking limits in $y_{2n+2} = p \circ p(y_{2n})$, we get $\ell_1 = p \circ p(\ell_1)$. Similarly, $\ell_2 = p \circ p(\ell_2)$, so ℓ_1, ℓ_2 are solutions of $G(x) = 0$, where $G : [0, \pi/2] \to \mathbb{R}$, $G(x) = p \circ p(x) - x$. But for every $x \in (0, \pi/2)$ we have $G'(x) = \sin(\sin(\cos(\sin x))) \cos(\cos(\sin x)) \sin(\sin x) \cos x - 1 < 0$, so $G(x)$ is strictly decreasing. It follows that G has at most one zero. But an obvious zero for G is a. We have $p(a) = a$, so that $p \circ p(a) - a = 0$. Hence, $\ell_1 = \ell_2 = a$, which means $y_n \to a$.

By a similar proof, $z_n \to b$. This time $q \circ q$ is increasing, so that (z_{2n}) and (z_{2n+1}) are monotone. We denote by ℓ_1' and ℓ_2' the limits of the two sequences. Then both ℓ_1' and ℓ_2' are zeros of H, where $H : [0, \pi/2] \to \mathbb{R}$, $H(x) = q \circ q(x) - x$. Again, by taking derivatives, we get $H'(x) < 0$ for all $x \in (0, \pi/2)$. Thus, H is strictly decreasing, so it has at most one zero. From $q(b) = b$ one obtains $H(b) = q \circ q(b) - b = 0$ and therefore $\ell_1' = \ell_2' = b$.

As $x_{2n} \to a$, one also has

$$\lim_{n\to\infty} \frac{x_{2n+2} - a}{x_{2n} - a} = \lim_{n\to\infty} \frac{p(x_{2n}) - p(a)}{x_{2n} - a} = p'(a) = -\sin(\sin a)\cos a.$$

Note that $|p'(a)| < 1$.

It follows that for every $\alpha > 0$ we have

$$\lim_{n\to\infty} \frac{|x_{2n+2} - a|^\alpha}{|x_{2n} - a|^\alpha} = |p'(a)|^\alpha < 1$$

and so $\sum_{n\geq 0} |x_{2n} - a|^\alpha$ is convergent.

Similarly, $|q'(b)| = |-\cos(\cos b)\sin b| < 1$ implies

$$\lim_{n\to\infty} \frac{|x_{2n+3} - a|^\alpha}{|x_{2n+1} - a|^\alpha} = |q'(b)|^\alpha < 1,$$

whence one obtains that the series $\sum_{n\geq 0} |x_{2n+1} - b|^\alpha$ is convergent for every $\alpha > 0$. $\square$