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ARTICLES

The Kepler problem and shocks in the direction of motion

Eleutherius Symeonidis[1)], Wolfgang Bothe[2)]

Abstract. Considering the case of an elliptic orbit as a solution to the classical Kepler problem we subject one of the two bodies to a shock in its moving direction. What happens to the center of mass of the two-body system? And how does the orbit change with respect to this center of mass? These questions are answered in a thoroughly mathematical treatment.

Keywords: Kepler problem, two-body problem, shock, elliptic orbit.

MSC: Primary 70F05; Secondary 34A12.

1. Introduction

In the year 1609, Johannes Kepler published the first of his three laws on the planetary motion around the Sun. According to it, the orbit of every planet is an ellipse, and the Sun lies at one of its foci. In 1687, Isaac Newton published his work *Philosophiae Naturalis Principia Mathematica*, where he formulated the inverse-square law of universal gravitation. In modern language, this law states that every two objects of masses m_1 and m_2 attract each other by a force

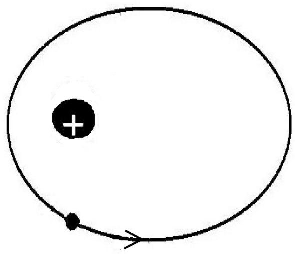
$$F = G \cdot \frac{m_1 \cdot m_2}{r^2}\,,$$

where r denotes the distance between the centers of masses of the objects and G is the so-called gravitational constant. If the masses are measured in kilograms (kg), the distance in meters (m), and the force in newtons (N), then the internationally accepted approximate value is $G \approx 6.674 \cdot 10^{-11}\,\mathrm{N\,m^2kg^{-2}}$. The problem of determining the motion of two objects under Newton's gravitational law is nowadays called *Kepler problem.* This gravitational law confirms Kepler's first law in the case where the mass of one of the objects (here:

[1)]Katholische Universität Eichstätt-Ingolstadt, Mathematisch-Geographische Fakultät, 85071 Eichstätt, Germany, `e.symeonidis@ku.de`

[2)]Rebdorfer Str. 84, 85072 Eichstätt, Germany, `ute_koe_bo@web.de`

the Sun) is large in comparison with the other mass. In general, under the assumption of periodic motions, the orbits remain elliptic (or circular), but one of the foci is occupied by rather the center of mass of the two-body system. The following figure shows an elliptic orbit of the smaller body. Therein, the plus sign indicates the position of the center of mass.



In this article we consider the Kepler problem in the periodic case and study the perturbation of the motion, if one of the bodies is subjected to a shock in the direction of its movement. More precisely, we consider a shock that results in an instant leap in the magnitude of the body's velocity, but not in its direction. As causes of shocks one may think of external strokes or of internal thrusters as in the case of satellites, for instance.

To keep the article self-contained, we give a short derivation of the equations of motion by using the Euler-Lagrange equations. (For a background knowledge, the reader is referred to any textbook on classical mechanics, e. g. [1], or [2].) Therein we take the center of mass of the two-body system as the origin (i.e. the location of the observer) in the reference frame, and we take advantage of the resulting fact that the motion takes place on a fixed plane. (This fact, being a consequence of the conservation of the angular momentum, we use here without proof.)

2. The equations of motion

We use polar coordinates to describe the motion of the two bodies on the plane on which it takes place. The origin is occupied by the center of mass of the system and a direction is fixed as a reference for the angles. Let (R, ϑ) describe the position of one of the bodies (the reader may think of the bigger one), (r, φ) the position of the other body. We may take

$$\vartheta = \varphi + \pi \,. \tag{1}$$

Let M, m be the masses of the bodies at (R, θ), (r, φ), respectively. Since the center of mass lies at the origin, it holds that

$$R = \frac{rm}{M} \,. \tag{2}$$

When the bodies move, their coordinates are functions of the time t. Their velocities v_M and v_m are the first derivatives of (the coordinates of) their position vectors, that is,

$$v_M = \left(\frac{\mathrm{d}}{\mathrm{d}t}(R\cos\vartheta), \frac{\mathrm{d}}{\mathrm{d}t}(R\sin\vartheta)\right), \quad v_m = \left(\frac{\mathrm{d}}{\mathrm{d}t}(r\cos\varphi), \frac{\mathrm{d}}{\mathrm{d}t}(r\sin\varphi)\right).$$

By (2) and (1), the total kinetic energy of the system only depends on r, φ, and their time derivatives, and it amounts to

$$T(r,\varphi,\dot r,\dot\varphi) = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2$$

$$= \frac{1}{2}M\left[\left(\overbrace{R\cos\vartheta}^{\cdot}\right)^2 + \left(\overbrace{R\sin\vartheta}^{\cdot}\right)^2\right] + \frac{1}{2}m\left[\left(\overbrace{r\cos\varphi}^{\cdot}\right)^2 + \left(\overbrace{r\sin\varphi}^{\cdot}\right)^2\right],$$

having used the standard dot notation for the time derivatives. By (1) and (2) it holds $\dot\vartheta = \dot\varphi$ and $\dot R = \frac{\dot r m}{M}$, so calculation gives

$$T(r,\varphi,\dot r,\dot\varphi) = \frac{1}{2}M(\dot R^2 + R^2\dot\varphi^2) + \frac{1}{2}m(\dot r^2 + r^2\dot\varphi^2) = \frac{(\dot r^2 + r^2\dot\varphi^2)m}{2}\left(1 + \frac{m}{M}\right).$$

Since the distance of the two bodies is $R + r$, the potential energy of the system amounts to

$$V(r,\varphi,\dot r,\dot\varphi) = -\frac{GMm}{R+r} = -\frac{GM^2m}{(M+m)r}.$$

Therefore, the Lagrangian $L = T - V$ (see, e. g., [1] or [2]) of this two-body system is given by the equation

$$L(r,\varphi,\dot r,\dot\varphi) = \frac{m(M+m)(\dot r^2 + r^2\dot\varphi^2)}{2M} + \frac{GM^2m}{(M+m)r}. \tag{3}$$

The motion of the system is now determined by Lagrange's equations. From (3) we obtain:

$$0 = \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r}\right)(r,\varphi,\dot r,\dot\varphi) = \frac{m(M+m)\ddot r}{M} - \frac{m(M+m)\dot\varphi^2 r}{M} + \frac{GM^2m}{(M+m)r^2}$$

$$\iff \frac{(M+m)\ddot r}{M} - \frac{(M+m)\dot\varphi^2 r}{M} + \frac{GM^2}{(M+m)r^2} = 0, \tag{4}$$

and

$$0 = \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\varphi} - \frac{\partial L}{\partial \varphi}\right)(r,\varphi,\dot r,\dot\varphi) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{m(M+m)r^2\dot\varphi}{M} \iff \dot\varphi = \frac{\sigma}{r^2} \tag{5}$$

with a constant $\sigma \geq 0$ without loss of generality. Inserting (5) in (4) we arrive at a differential equation of second order for r:

$$\frac{(M+m)\ddot r}{M} = \frac{(M+m)\sigma^2}{Mr^3} - \frac{GM^2}{(M+m)r^2}. \tag{6}$$

Our intention will be to study how the orbit of a body changes after a shock. For this purpose, it is advisable to relate the function r to the angle φ rather than to the time t. Ignoring the trivial case $\sigma = 0$, this is done by changing the dependent variable to ρ by setting $r(t) =: \rho(\varphi(t))$. Denoting by primes the derivatives of ρ and using (5) we have

$$\dot{r} = \rho' \cdot \dot{\varphi}\,, \qquad \ddot{r} = \rho'' \cdot \dot{\varphi}^2 + \rho' \cdot \ddot{\varphi} = \frac{\rho''\sigma^2}{r^4} - \frac{2(\rho')^2\sigma^2}{r^5}\,.$$

After inserting these expressions in (6) and simplifying, we are led to the equation

$$\frac{\rho\rho'' - 2(\rho')^2}{\rho^3} = \frac{1}{\rho} - \frac{GM^3}{(M+m)^2\sigma^2}\,. \tag{7}$$

The final change of variables

$$\rho = \frac{1}{\tau}\,, \quad \rho' = -\frac{\tau'}{\tau^2}\,, \quad \rho'' = \frac{2(\tau')^2 - \tau''\tau}{\tau^3}$$

transforms (7) into the linear equation

$$\tau'' + \tau - \frac{GM^3}{(M+m)^2\sigma^2} = 0 \tag{8}$$

with the general solution

$$\tau(\varphi) = \frac{GM^3}{(M+m)^2\sigma^2} + \alpha\cos(\varphi + \beta)\,, \tag{9}$$

where $\alpha, \beta \in \mathbb{R}$. By adjusting the direction of the ray $\varphi = 0$, we can assume $\beta = 0$ and

$$0 \leq \alpha < \frac{GM^3}{(M+m)^2\sigma^2} \tag{10}$$

without loss of generality. (If the upper bound condition for α is dropped, then the angle φ cannot assume every value, since τ has to be positive; this fact, on the basis of the monotonicity of $\varphi(t)$ (see (5)), would render a non-periodic orbit, which is not the case of consideration in this article.) Then, $\varphi = 0$ is the maximum point of τ and the minimum point of ρ and r (unless $\alpha = 0$, in which case these functions are constant). If the body of mass M is the Sun, this position is the perihelion of the other body's orbit (unless $\alpha = 0$, in which case both bodies move on circles around their center of mass). Under these assumptions, (9) leads to the equation

$$\rho(\varphi) = \frac{1}{A + \alpha\cos\varphi}\,, \tag{11}$$

where

$$A := \frac{GM^3}{(M+m)^2\sigma^2} > \alpha\,. \tag{12}$$

In Cartesian coordinates, (11) takes the form

$$\left(\frac{A^2-\alpha^2}{A}\right)^2\left(x+\frac{\alpha}{A^2-\alpha^2}\right)^2+\left(A^2-\alpha^2\right)y^2=1\,.$$

This is the equation of an ellipse (or a circle, if $\alpha=0$) with eccentricity $\frac{\alpha}{A}$ and foci at $\left(-\frac{2\alpha}{A^2-\alpha^2},0\right)$ and $(0,0)$. (Since the origin is occupied by the center of mass of the system, we have derived Kepler's first law of planetary motion.)

3. The effect of shocks

As we have specified in the introduction, we consider here shocks in the direction of one body's movement. Mathematically, this amounts to a leap in just the magnitude of the body's velocity, not in its direction. If the motion is described by $(x(t),y(t))$ in Cartesian coordinates and the shock occurs at the time t_0, then we agree that the velocity instantly changes from $(\dot{x}(t_0),\dot{y}(t_0))$ to $(\dot{x}(t_0)(1+\delta),\dot{y}(t_0)(1+\delta))$. We assume $\delta>0$, but we shall also discuss the case $-1<\delta<0$ in the next section.

In the (τ,φ) coordinate system that we have been using, similarly, this shock amounts to the leap from

$$\left(\dot{\overbrace{\tau\circ\varphi}}(t_0),\dot{\varphi}(t_0)\right)\quad\text{to}\quad\left(\dot{\overbrace{\tau\circ\varphi}}(t_0),\dot{\varphi}(t_0)\right)(1+\delta),$$

due to the linearity of the differential of the coordinate transform. Since, on the other hand,

$$\dot{\overbrace{\tau\circ\varphi}}(t_0)=\tau'(\varphi(t_0))\cdot\dot{\varphi}(t_0)\,,$$

we see that $\tau'(\varphi(t_0))$ is not affected by the shock. As for the constant σ, it follows from (5) that σ must be replaced by $\sigma_\delta:=\sigma(1+\delta)$ after the shock (there is no jump in r). Recalling (8) and setting $\varphi_0:=\varphi(t_0)$ we are thus led to the initial value problem

$$\begin{aligned}\tau''+\tau &= \frac{GM^3}{(M+m)^2\sigma_\delta^2},\\ \tau(\varphi_0) &= \frac{GM^3}{(M+m)^2\sigma^2}+\alpha\cos\varphi_0,\\ \tau'(\varphi_0) &= -\alpha\sin\varphi_0\,.\end{aligned}\tag{13}$$

Remark 1. At this point we note that the leap in the velocity v_m of the body of mass m automatically causes a leap in the other body's velocity expression v_M in the ratio $\frac{m}{M}$ (see (1), (2), and the subsequent equations for v_M and v_m). This does not mean that the velocity of the body of mass M instantly changes in reality too, but it is due to our assumption of the center of mass to lie at the origin of the reference frame, before as well as after the shock. The shock affects the movement of the center of mass, but this is out of our consideration. See also the closing paragraph of this article.

Let the solution of (13) have the form

$$\tau(\varphi) = \frac{GM^3}{(M+m)^2\sigma_\delta^2} + \alpha_\delta \cos(\varphi+\beta)\,,$$

where we may assume $\alpha_\delta \geq 0$ without loss of generality. Then, the initial conditions amount to

$$\frac{GM^3}{(M+m)^2\sigma_\delta^2} + \alpha_\delta \cos(\varphi_0+\beta) = \frac{GM^3}{(M+m)^2\sigma^2} + \alpha\cos\varphi_0 \tag{14}$$

and

$$\alpha_\delta \sin(\varphi_0+\beta) = \alpha \sin\varphi_0\,. \tag{15}$$

We now assume $\alpha > 0$. (For $\alpha = 0$ see Remark 2 below.) In solving the last two equations for α_δ and β we first treat two special cases.

Case $\varphi_0 = 0$ (perihelion).
Here, (15) implies $\alpha_\delta = 0$ or $\beta \in \pi\mathbb{Z}$. However, $\alpha_\delta = 0$ would contradict (14), since $\sigma_\delta = \sigma(1+\delta) > \sigma$. So, $\beta \in \pi\mathbb{Z}$, actually $\beta \in 2\pi\mathbb{Z}$, for otherwise (14) would again fail (α_δ was assumed nonnegative). In fact, we can set $\beta = 0$. It then follows from (14) and the definition of σ_δ that

$$\alpha_\delta = \frac{GM^3(\delta^2+2\delta)}{(M+m)^2\sigma^2(1+\delta)^2} + \alpha > \alpha\,,$$

and

$$A_\delta := \frac{GM^3}{(M+m)^2\sigma_\delta^2} < \frac{GM^3}{(M+m)^2\sigma^2} = A \tag{16}$$

(see (12)). If the shock (measured by its parameter δ) is so big that the upper bound condition $\alpha_\delta < A_\delta$ (recall (10)) does not hold, then the shocked body continues on a non-elliptic orbit. Otherwise, the elliptic form is preserved, also its orientation and the position of the perihelion, whereas its semi-major axis $\frac{A_\delta}{A_\delta^2-\alpha_\delta^2}$ and eccentricity $\frac{\alpha_\delta}{A_\delta}$ become bigger.

Case $\varphi_0 = \pi$ (aphelion).
Again, (15) implies $\alpha_\delta = 0$ or $\beta \in \pi\mathbb{Z}$. Here, $\alpha_\delta = 0$ is possible, and this happens when $\delta = \delta_0$, where δ_0 is the solution of the equation

$$\frac{GM^3}{(M+m)^2\sigma^2} - \alpha = \frac{GM^3}{(M+m)^2\sigma^2(1+\delta_0)^2}\,.$$

In this case, the orbit is a circle.

If $\delta < \delta_0$, by an analogous reasoning with the previous case we can take $\beta = 0$ and

$$0 < \alpha_\delta = \alpha + A_\delta - A < \alpha$$

from (14), having used (16). Then, the equations

$$\begin{aligned}\frac{\alpha_\delta}{A_\delta} &= \frac{(\alpha + A_\delta - A)(1+\delta)^2}{A} = 1 - (1+\delta)^2 + \frac{\alpha(1+\delta)^2}{A} \\ &= \frac{\alpha}{A} - \left(\delta^2 + 2\delta\right)\left(1 - \frac{\alpha}{A}\right) < \frac{\alpha}{A}\end{aligned}$$

and

$$\begin{aligned}\frac{A_\delta}{A_\delta^2 - \alpha_\delta^2} &= \frac{A(1+\delta)^2}{A^2 - [\alpha_\delta(1+\delta)^2]^2} = \frac{A}{2A(A-\alpha) - (A-\alpha)^2(1+\delta)^2} \\ &= \frac{A}{A^2 - \alpha^2 - (A-\alpha)^2(\delta^2 + 2\delta)} > \frac{A}{A^2 - \alpha^2}\end{aligned}$$

show that the eccentricity becomes smaller, the semi-major axis bigger.

If $\delta > \delta_0$, then (14) and (15) urge us to take $\beta \in \pi + 2\pi\mathbb{Z}$ ($\beta = \pi$ without loss of generality) and

$$\alpha_\delta = A - \alpha - A_\delta > 0\,.$$

This means that the aphelion becomes perihelion of the new elliptic orbit, as long as $\alpha_\delta < A_\delta$. Otherwise, the shocked body continues on a non-elliptic orbit.

In all other cases (15) gives

$$\alpha_\delta = \frac{\alpha \sin\varphi_0}{\sin(\varphi_0 + \beta)} \neq 0 \tag{17}$$

(so the new orbit cannot be circular), and if this value is inserted in (14), we obtain

$$\frac{GM^3}{(M+m)^2\sigma_\delta^2} + \alpha \sin\varphi_0 \cot(\varphi_0 + \beta) = \frac{GM^3}{(M+m)^2\sigma^2} + \alpha\cos\varphi_0$$

$$\iff \cot(\varphi_0 + \beta) = \frac{GM^3(\delta^2 + 2\delta)}{(M+m)^2\sigma^2\alpha\sin\varphi_0 \cdot (1+\delta)^2} + \cot\varphi_0\,.$$

This equation determines β, and then α_δ follows from (17). Since $\delta > 0$, $\beta \neq 0$, so the new orbit, if it remains elliptic (depending on whether $\alpha_\delta < A_\delta$ or not), has a different orientation. In fact, from the last equation and the monotonicity of the cotangent it follows that for $0 < \varphi_0 < \pi$ it holds that $0 < \varphi_0 + \beta < \varphi_0$, which means that the major axis of the original elliptic orbit turns in the direction of the shock.

Remark 2. If the original orbit is circular, $\alpha = 0$ in (13). In this case we take $\beta = -\varphi_0$, so that (14) has a positive solution α_δ. This means that the new orbit, if it is elliptic (if $\alpha_\delta < A_\delta$), has its perihelion at the point where the shock occurred.

Remark 3. The total energy E of the system is strongly connected with the semi-major axis a of the ellipse. From the formulas of Section 2, using $\dot{r} = \rho' \cdot \dot{\varphi}$, (5), and (11), it namely follows:

$$E = T + V =$$
$$\frac{(M+m)m}{2M}\alpha^2\sigma^2\sin^2\varphi + \frac{(M+m)m}{2M}\sigma^2(A+\alpha\cos\varphi)^2 - \frac{GM^2m}{M+m}(A+\alpha\cos\varphi)$$
$$= \frac{(M+m)m}{2M}\sigma^2(\alpha^2+A^2) - \frac{GM^2m}{M+m}A = \frac{(M+m)m}{2M}\sigma^2(\alpha^2-A^2)$$
$$= \frac{GM^2m}{2(M+m)} \cdot \frac{\alpha^2-A^2}{A} = -\frac{GM^2m}{2(M+m)\cdot a}\,.$$

As the total energy obviously increases by the shock ($\delta > 0$), this relation shows that the semi-major axis increases too.

4. Conclusions

As we have shown in the previous section, the effect of shocks in the direction of one body's movement and such that the body's velocity increases ($\delta > 0$) can be described as follows.

If the body moves on a circular orbit and the new orbit remains closed, then the latter is elliptic and has its perihelion where the shock occurred.

If the body moves on an elliptic orbit and the shock occurs at the perihelion, then that point remains perihelion of the perturbed orbit, as long as the latter remains closed. If the shock occurs at the aphelion, then, similarly, that point remains aphelion of the perturbed (closed) orbit, but for a certain magnitude of the shock the latter can become circular.

If the body moves on an elliptic orbit and the shock occurs between the perihelion and the aphelion, then the perturbed orbit, if closed, is elliptic with a different orientation of its axes.

Finally, if the shock reduces the body's velocity ($-1 < \delta < 0$), then it suffices to interchange *perihelion* and *aphelion* for the above statements to remain valid.

The following table summarizes the results in the different cases.

	semi-major axis	axes of ellipse	circle possible
$\delta > 0$ at perihelion	increases	same	no
$\delta > 0$ at aphelion	increases	same	yes
$\delta > 0$ elsewhere	increases	change	no
$-1 < \delta < 0$ at perihelion	decreases	same	yes
$-1 < \delta < 0$ at aphelion	decreases	same	no
$-1 < \delta < 0$ elsewhere	decreases	change	no

At this point we consider it important to remind the reader that our observations are made from the viewpoint of the center of mass of the two-body system. Naturally, the shock also affects the overall momentum of the center of mass, but this effect is not considered here.

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Several inequalities of Erdős–Mordell type

Izabella Abraham[1)], Nicuşor Minculete[2)], Eugen Păltănea[3)]

Abstract. The paper presents many geometric identities and inequalities inspired by the famous inequality due to Erdős and Mordell. We highlight so-called *dual inequalities*. The order of duality for Erdős–Mordell's inequality and Oppenheim's inequality is studied by five methods. Other interesting inequalities are obtained by duality method.

Keywords: Erdős–Mordell's inequality, Oppenheim's inequality.

MSC: 26D05, 42A05, 51M16, 26D15.

1. Introduction

Let M be an interior point of a triangle ABC. Denote the distances from M to the vertices A, B, C by R_1, R_2, R_3 and the distances from M to the sides BC, CA, AB by r_1, r_2, r_3, respectively. Consider also the following notations: h_a, h_b, h_c are the lengths of the triangle's altitudes, O is the center of the circumscribed circle, R is the radius of the circumscribed circle, I is the center of the inscribed circle, r is the radius of the inscribed circle, H is the orthocenter, G is the centroid, s is the semiperimeter and Δ is the area of the triangle ABC. The following remarkable inequality:

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3) \tag{1}$$

was proposed by Erdős in 1935, [5]. The inequality was proved in 1937 by Mordell and Barrow in [15]. Several other mathematicians, such as Bankoff [3], Avez [1], Komornik [7], Lee [8] and Dergiades [4], have proved this result

[1)] Department of Mathematics, Transilvania University of Braşov, Romania, izabella.abraham@student.unitbv.ro

[2)] Department of Mathematics, Transilvania University of Braşov, Romania, minculete.nicusor@unitbv.ro

[3)] Department of Mathematics, Transilvania University of Braşov, Romania, epaltanea@unitbv.ro

by various methods. In [16,17], Oppenheim proved the following related inequalities:

$$R_1R_2R_3 \geq (r_1+r_2)(r_2+r_3)(r_3+r_1) \tag{2}$$

and

$$R_1R_2+R_2R_3+R_3R_1 \geq (r_1+r_2)(r_1+r_3)+(r_2+r_3)(r_2+r_1)+(r_3+r_1)(r_3+r_2). \tag{3}$$

In [10], Liu proves the relevant inequality

$$R_2+R_3 \geq 2r_1+\frac{(r_2+r_3)^2}{R_1}. \tag{4}$$

The above inequality and the analogous inequalities can help to obtain (3). In [9], Liu obtains the following inequalities:

$$\frac{(r_2+r_3)^2}{R_1}+\frac{(r_3+r_1)^2}{R_2}+\frac{(r_1+r_2)^2}{R_3} \leq R_1+R_2+R_3, \tag{5}$$

$$\frac{(r_2+r_3)^2}{R_1+r_2+r_3}+\frac{(r_3+r_1)^2}{R_2+r_3+r_1}+\frac{(r_1+r_2)^2}{R_3+r_1+r_2} \leq r_1+r_2+r_3, \tag{6}$$

$$\begin{aligned}\frac{(kr_1+r_2+r_3)^2}{R_1+kr_1}+\frac{(kr_2+r_3+r_1)^2}{R_2+kr_2}+\frac{(kr_3+r_1+r_2)^2}{R_3+kr_3}&\\ \leq \frac{k+2}{2}(R_1+R_2+R_3),&\end{aligned} \tag{7}$$

for all $k \geq 0$. Other inequalities of the Erdös–Mordell type in a triangle are given by Bottema in [2], Liu in [11], Tsintsifas in [14], Satnoianu in [19] and Wu and Debnath in [21]. In [12,13], we find Hayashi's inequality:

$$\frac{R_1R_2}{ab}+\frac{R_2R_3}{bc}+\frac{R_3R_1}{ca} \geq 1,$$

where the equality is achieved if and only if M is the orthocenter of the acute triangle ABC, or M is one of the vertices of the triangle ABC. Recently, the topic is treated in [13].

The paper is organized as follows. In Section 2, we propose an extension of Erdös–Mordell's inequality. In Section 3, we highlight an interesting identity in a triangle (Theorem 3.2) with multiple consequences. Theorem 3.4 groups eight derived inequalities of Erdös–Mordell type. Section 4 contains the main ideas and results of this paper. Assume a triangle ABC and a point M. Using a certain geometric construction, called *method*, which involves the points A, B, C and M, we obtain another triangle, say $A_1B_1C_1$. Then, by the same kind of construction, starting from the points A_1, B_1, C_1 and M we find the triangle $A_2B_2C_2$ and so on. Suppose that (I_0) is an inequality of Erdös–Mordell type for the triangle ABC and the point M. We successively apply (I_0) for the point M and the constructed triangles $A_1B_1C_1$, $A_2B_2C_2$, ... In this way, we obtain a sequence of *dual inequalities*

(I_1), (I_2), ... We say that the initial inequality (I_0) has the order k of duality (with respect to the considered method) if this sequence of inequalities has the period k. We illustrate the dual inequalities and we establish the order of duality for Erdös–Mordell's inequality and for some related inequalities by using five methods. Finally, in Section 5, we show that the particular choice of the point M can lead to remarkable classical inequalities.

2. An improvement of Erdös–Mordell's inequality

To highlight an improvement of Erdös–Mordell's inequality (1) we need the next lemma.

Lemma 2.1. *With the above notations, we have*

$$R_1^2 = \frac{r_2^2 + r_3^2 + 2r_2r_3\cos A}{\sin^2 A}.$$

Analogous relations hold for R_2 and R_3.

Proof. We will prove the relation for R_3. Let P, Q and R be the projections of M onto the sides BC, CA and AB, respectively (see Fig. 1).

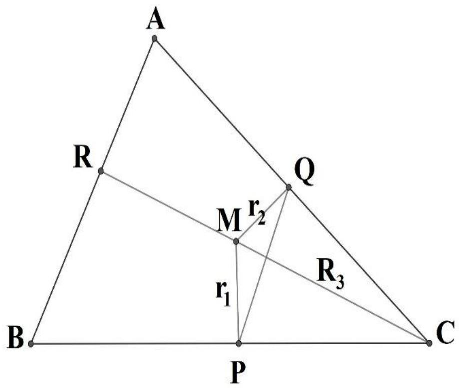

Fig. 1

By the law of cosines in the triangle MPQ, we have $PQ^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\widehat{PMQ}) = r_1^2 + r_2^2 + 2r_1r_2\cos C$. Since $MPCQ$ is a cyclic quadrilateral, we obtain $PQ = R_3\sin C$ (law of sines). So $R_3^2 = \dfrac{r_1^2 + r_2^2 + 2r_1r_2\cos C}{\sin^2 C}$. Analogous relations for R_1 and R_2 are similarly obtained. □

Remarks.

1. The identity $\cos A = \sin B\sin C - \cos B\cos C$ implies the relation

$$r_2^2 + r_3^2 + 2r_2r_3\cos A = (r_2\sin C + r_3\sin B)^2 + (r_2\cos C - r_3\cos B)^2. \quad (8)$$

Then, from Lemma 2.1, we obtain

$$R_1^2\sin^2 A = r_2^2 + r_3^2 + 2r_2r_3\cos A \geq (r_2\sin C + r_3\sin B)^2.$$

Therefore

$$R_1 \sin A \geq r_2 \sin C + r_3 \sin B. \tag{9}$$

Similarly, $R_2 \sin B \geq r_3 \sin A + r_1 \sin C$ and $R_3 \sin C \geq r_1 \sin B + r_2 \sin A$. It is easy to see that inequality (9) and its analogues prove the inequality of Erdős and Mordell.

2. A beautiful improvement of (1) is provided by the paper [16]. Denote by T_1, T_2 and T_3 the distances from M to the tangents to the circumcircle of the triangle ABC at A, B and C, respectively. Clearly $R_k \geq T_k,\ k = 1, 2, 3$. So, $R_1 + R_2 + R_3 \geq T_1 + T_2 + T_3$. Applying Ptolemy's theorem, the following relations are obtained:

$$T_1 = \frac{r_2 \sin C + r_3 \sin B}{\sin A},\ T_2 = \frac{r_3 \sin A + r_1 \sin C}{\sin B} \text{ and } T_3 = \frac{r_1 \sin B + r_2 \sin A}{\sin C}.$$

Therefore

$$T_1+T_2+T_3 = r_1 \left(\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B}\right) + r_2 \left(\frac{\sin C}{\sin A} + \frac{\sin A}{\sin C}\right) + r_3 \left(\frac{\sin A}{\sin B} + \frac{\sin B}{\sin A}\right).$$

Hence, $T_1 + T_2 + T_3 \geq 2\,(r_1 + r_2 + r_3)$ is a strengthened version of the Erdös–Mordell inequality.

3. From (9) and the analogous inequalities, by applying the law of sines, we obtain

$$aR_1 \geq cr_2 + br_3,\ bR_2 \geq ar_3 + cr_1,\ cR_3 \geq br_1 + ar_2. \tag{10}$$

By summing these inequalities, we deduce:

$$aR_1 + bR_2 + cR_3 \geq r_1(b + c) + r_2(c + a) + r_3(a + b) > ar_1 + br_2 + cr_3 = 2\Delta.$$

The following theorem provides an elementary improvement of Erdös–Mordell's inequality.

Theorem 2.1.

$$R_1 + R_2 + R_3 \geq 2\,(r_1 + r_2 + r_3) + \sum \frac{(r_2 \cos C - r_3 \cos B)^2}{2R_1 \sin^2 A}. \tag{11}$$

Proof. From Lemma 2.1 and the identity (8), we obtain

$$R_1^2 \sin^2 A - (r_2 \sin C + r_3 \sin B)^2 = (r_2 \cos C - r_3 \cos B)^2.$$

Then, by using the inequality (9), we find

$$\begin{aligned} R_1 \sin A - (r_2 \sin C + r_3 \sin B) &= \frac{(r_2 \cos C - r_3 \cos B)^2}{R_1 \sin A + (r_2 \sin C + r_3 \sin B)} \\ &\geq \frac{(r_2 \cos C - r_3 \cos B)^2}{2R_1 \sin A}. \end{aligned}$$

Therefore

$$R_1 \geq \frac{r_2 \sin C + r_3 \sin B}{\sin A} + \frac{(r_2 \cos C - r_3 \cos B)^2}{2R_1 \sin^2 A}.$$

Analogous relations hold for R_2 and R_3. By summing these relations we obtain

$$R_1+R_2+R_3 \geq \sum r_1\left(\frac{\sin B}{\sin C}+\frac{\sin C}{\sin B}\right)+\sum \frac{(r_2\cos C - r_3\cos B)^2}{2R_1\sin^2 A}$$
$$\geq 2(r_1+r_2+r_3)+\sum \frac{(r_2\cos C - r_3\cos B)^2}{2R_1\sin^2 A}.$$

So, we get (11). □

3. Some geometric inequalities of Erdős–Mordell type

The Erdős–Mordell's inequality has inspired a constellation of related results. We will refer to some of them, by keeping the notations from the introduction.

Theorem 3.1 (Carlitz)**.**

$$R_1^2\sin^2 A+R_2^2\sin^2 B+R_3^2\sin^2 C \leq 3\left(r_1^2+r_2^2+r_3^2\right). \tag{12}$$

Proof. Lemma 2.1 leads to the identity

$$R_1^2\sin^2 A+R_2^2\sin^2 B+R_3^2\sin^2 C = 2\left(r_1^2+r_2^2+r_3^2\right)+2\sum r_2r_3\cos A.$$

Then, from Barrow's inequality (see [20]), we get

$$2\left(r_2r_3\cos A+r_3r_1\cos B+r_1r_2\cos C\right)\leq r_1^2+r_2^2+r_3^2.$$

It follows (12). □

Remark. The equality in (12) holds if $\dfrac{r_1}{\sin A}=\dfrac{r_2}{\sin B}=\dfrac{r_3}{\sin C}$, meaning that M is the point of Lemoine of the triangle ABC.

By using Cauchy's inequality, we derive the following result.

Corollary 3.1.

$$R_1\sin A+R_2\sin B+R_3\sin C\leq 3\sqrt{r_1^2+r_2^2+r_3^2}. \tag{13}$$

We will highlight new identities in the triangle ABC.

Theorem 3.2. *With the usual notations, the following identities hold*

$$a^2R_1^2r_1+b^2R_2^2r_2+c^2R_3^2r_3 = 4R^2(r_1+r_2)(r_2+r_3)(r_3+r_1)+8Rrr_1r_2r_3 \tag{14}$$

and

$$R_1^2=\frac{1}{4}\left[\left(\frac{r_2+r_3}{\sin\frac{A}{2}}\right)^2+\left(\frac{r_2-r_3}{\cos\frac{A}{2}}\right)^2\right]. \tag{15}$$

Proof. Using Lemma 2.1, the law of sines and the identity $\sum \cos A = 1 + \frac{r}{R}$, we find

$$\sum a^2 R_1^2 r_1 = 4R^2 \sum r_1 \left(R_1^2 \sin^2 A\right) = 4R^2 \sum r_1 \left(r_2^2 + r_3^2 + 2r_2 r_3 \cos A\right)$$
$$= 4R^2 \sum \left(r_1 r_2^2 + r_1 r_3^2\right) + 8R^2 r_1 r_2 r_3 \sum \cos A$$
$$= 4R^2 \left[(r_1 + r_2)(r_2 + r_3)(r_3 + r_1) - 2r_1 r_2 r_3\right] + 8R^2 r_1 r_2 r_3 \left(1 + \frac{r}{R}\right)$$
$$= 4R^2 (r_1 + r_2)(r_2 + r_3)(r_3 + r_1) + 8Rr r_1 r_2 r_3.$$

For the second relation, we also use Lemma 2.1. So, we have

$$R_1^2 - \frac{1}{4}\left(\frac{r_2 + r_3}{\sin \frac{A}{2}}\right)^2 = \frac{r_2^2 + r_3^2 + 2r_2 r_3 \cos A}{\sin^2 A} - \frac{(r_2 + r_3)^2 \cos^2 \frac{A}{2}}{4 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}}$$
$$= \frac{\left(r_2^2 + r_3^2\right)\left(1 - \cos^2 \frac{A}{2}\right) + 2r_2 r_3 \left(\cos A - \cos^2 \frac{A}{2}\right)}{4 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}} = \frac{1}{4}\left(\frac{r_2 - r_3}{\cos \frac{A}{2}}\right)^2.$$

Therefore, the identity (15) is proved. □

Remark. Assume that M is the centroid G of the triangle ABC. Then the identity (14) becomes

$$2\Delta \sum a m_a^2 = R\left(h_a + h_b\right)\left(h_b + h_c\right)\left(h_c + h_a\right) + 2Rr h_a h_b h_c.$$

Corollary 3.2.

$$2R_1 \sin \frac{A}{2} \geq r_2 + r_3, \;\; 2R_2 \sin \frac{B}{2} \geq r_3 + r_1, \;\; 2R_3 \sin \frac{C}{2} \geq r_1 + r_2. \tag{16}$$

Proof. We use (15) and the analogous identities. □

Corollary 3.3.

$$a^2 R_1^2 r_1 + b^2 R_2^2 r_2 + c^2 R_3^2 r_3 \geq 8Rr_1 r_2 r_3 (4R + r). \tag{17}$$

Proof. We apply the identity (14) and the inequality $(x + y)(y + z)(z + x) \geq 8xyz$, for all $x, y, z > 0$. □

Remark. If M is the center I of the inscribed circle of the triangle ABC then clearly $(r_1 + r_2)(r_2 + r_3)(r_3 + r_1) = 8r_1 r_2 r_3 = 8r^3$. So, we have equality in (17). In this way, we find the known identity

$$a^2 IA^2 + b^2 IB^2 + c^2 IC^2 = 8Rr^2 (4R + r).$$

Corollary 3.4.

$$(r_1 + r_2 + r_3)\left[(r_1 + r_2)(r_2 + r_3)(r_3 + r_1) + \frac{2rr_1 r_2 r_3}{R}\right] \geq \left(\sum R_1 r_1 \sin A\right)^2. \tag{18}$$

Proof. Using Jensen's inequality for the convex function $f(x) = x^2$, we obtain

$$\frac{r_1(aR_1)^2 + r_2(bR_2)^2 + r_3(cR_3)^2}{r_1 + r_2 + r_2} \geq \left(\frac{r_1(aR_1) + r_2(bR_2) + r_3(cR_3)}{r_1 + r_2 + r_3}\right)^2. \quad (19)$$

From (14), we have

$$\frac{r_1(aR_1)^2 + r_2(bR_2)^2 + r_3(cR_3)^2}{r_1 + r_2 + r_2} = \frac{4R^2}{r_1 + r_2 + r_2}\left[(r_1 + r_2)(r_2 + r_3)(r_3 + r_1) + \frac{2rr_1r_2r_3}{R}\right].$$

On the other hand, by the law of sines, we get

$$\left(\frac{r_1(aR_1) + r_2(bR_2) + r_3(cR_3)}{r_1 + r_2 + r_3}\right)^2 = \frac{4R^2\left(R_1r_1\sin A + R_2r_2\sin B + R_3r_3\sin C\right)^2}{(r_1 + r_2 + r_3)^2}.$$

By replacing this in (19), we easily find the desired inequality (18). □

The following elementary inequalities will help us to establish important relations.

Lemma 3.1.

$$R_1 + r_1 \geq h_a.$$

Analogous inequalities can be expressed for h_b and h_c.

Proof. See *Fig. 2*. We have $R_1 + r_1 = AM + MP \geq AP \geq AD = h_a$.

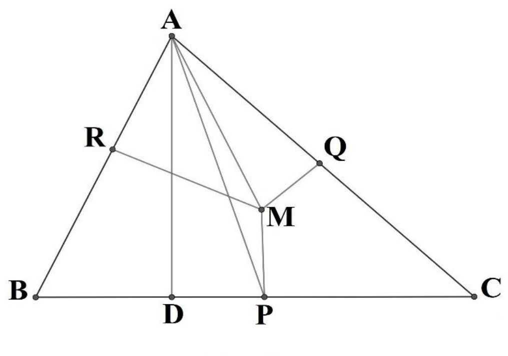

Fig. 2

The analogous relations for h_b and h_c are similarly obtained. □

Theorem 3.3. *The following inequalities hold:*

$$aR_1 + bR_2 + cR_3 \geq 4\Delta; \quad (20)$$

$$3R\sqrt{r_1^2 + r_2^2 + r_3^2} \geq 2\Delta. \quad (21)$$

Proof. We have $2\Delta = ar_1 + br_2 + cr_3 = ah_a = bh_b = ch_c$. From Lemma 3.1, we obtain

$$\sum aR_1 \geq \sum a(h_a - r_1) = \sum ah_a - \sum ar_1 = 6\Delta - 2\Delta = 4\Delta.$$

Multiplying by $2R$ the inequality (13) (Corollary 3.1), we obtain

$$aR_1 + bR_2 + cR_3 \leq 6R\sqrt{r_1^2 + r_2^2 + r_3^2}.$$

Therefore, by using the proved inequality (20), we get the inequality (21). □

Remarks.

1. If M is the center O of the circumscribed circle of the triangle ABC, then (20) is equivalent with Euler's inequality $R \geq 2r$.

2. If M coincides with the center I of the inscribed circle, then (21) is equivalent with Mitrinović's inequality $3\sqrt{3}R \geq 2s$ (see [20]).

The following theorem highlights eight inequalities of Erdös–Mordell type, which are essentially based on the Corollary 3.2 of Theorem 3.2.

Theorem 3.4.

$$R_1R_2R_3 \geq \frac{R}{2r}\left(r_1 + r_2\right)\left(r_2 + r_3\right)\left(r_3 + r_1\right); \tag{22}$$

$$R_1 \cos\frac{A}{2} + R_2 \cos\frac{B}{2} + R_3 \cos\frac{C}{2} \geq s; \tag{23}$$

$$R_1 \sin\frac{A}{2} + R_2 \sin\frac{B}{2} + R_3 \sin\frac{C}{2} \geq r_1 + r_2 + r_3; \tag{24}$$

$$\frac{r_2 + r_3}{R_1} + \frac{r_3 + r_1}{R_2} + \frac{r_1 + r_2}{R_3} \leq 3; \tag{25}$$

$$R_1r_1 \sin\frac{A}{2} + R_2r_2 \sin\frac{B}{2} + R_3r_3 \sin\frac{C}{2} \geq r_1r_2 + r_2r_3 + r_3r_1; \tag{26}$$

$$(2R - r)\left(R_1^2r_1^2 + R_2^2r_2^2 + R_3^2r_3^2\right) \geq 2R\left(r_1r_2 + r_2r_3 + r_3r_1\right)^2; \tag{27}$$

$$R_1 + R_2 + R_3 \geq 4\left(r_1 \sin\frac{A}{2} + r_2 \sin\frac{B}{2} + r_3 \sin\frac{C}{2}\right); \tag{28}$$

$$\frac{R_1R_2 + R_2R_3 + R_3R_1}{2}\sqrt{\frac{R_1 + R_2 + R_3}{R_1R_2R_3}} \geq \sum\left(r_2 + r_3\right)\cos\frac{A}{2}. \tag{29}$$

Proof.

1. By multiplying the inequalities from (16), we find

$$8R_1R_2R_3 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \geq \left(r_1 + r_2\right)\left(r_2 + r_3\right)\left(r_3 + r_1\right).$$

Then, by using the identity $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, we get the first inequality (22).

2. We can rewrite the first inequality of (16) as $R_1 \cos \frac{A}{2} \geq \frac{r_2 + r_3}{2} \cot \frac{A}{2}$. Since $\cot \frac{A}{2} = \frac{s-a}{r}$, we get

$$R_1 \cos \frac{A}{2} \geq \frac{(r_2 + r_3)(s-a)}{2r} = \frac{(r_1 + r_2 + r_3)(s-a) - r_1(s-a)}{2r}.$$

Therefore

$$\sum R_1 \cos \frac{A}{2} \geq \sum \frac{(r_1 + r_2 + r_3)(s-a) - r_1(s-a)}{2r} = \frac{ar_1 + br_2 + cr_3}{2r} = \frac{2\Delta}{2r} = s.$$

Thus, the inequality (23) is proved.
3. Inequality (24) is obtained by summing the inequalities from (16).
4. For the inequality (25), we also apply (16). We obtain

$$\sum \frac{r_2 + r_3}{R_1} \leq 2 \sum \sin \frac{A}{2} \leq 6 \sin \frac{A+B+C}{6} = 6 \sin \frac{\pi}{6} = 3,$$

due to the concavity of the sine function over the interval $(0, \pi/2)$.
5. Using again (16), we find

$$\sum R_1 r_1 \sin \frac{A}{2} \geq \sum \frac{r_1 (r_2 + r_3)}{2} = \sum r_1 r_2.$$

6. From the inequality (26), by using the Cauchy-Bunyakovsky-Schwarz inequality and the identity $\sum \sin^2 \frac{A}{2} = 1 - 2 \prod \sin \frac{A}{2} = \frac{2R - r}{2R}$ we obtain

$$2R \left(\sum r_1 r_2\right)^2 \leq 2R \left(\sum R_1 r_1 \sin \frac{A}{2}\right)^2 \leq 2R \sum R_1^2 r_1^2 \sum \sin^2 \frac{A}{2} = (2R - r) \sum R_1^2 r_1^2.$$

Then, the relation (27) is proved.
7. To obtain (28), we apply Lemma 3.1. We deduce $a(R_1 + r_1) \geq ah_a = 2\Delta = ar_1 + br_2 + cr_3$. Therefore

$$R_1 \geq \frac{br_2 + cr_3}{a}, \; R_2 \geq \frac{cr_3 + ar_1}{b}, \; R_3 \geq \frac{ar_1 + br_2}{c}. \tag{30}$$

Thus we get $\sum R_1 \geq \sum ar_1 \left(\frac{1}{b} + \frac{1}{c}\right) \geq \sum \frac{4ar_1}{b+c} \geq 4 \sum r_1 \sin \frac{A}{2}$, where the well-known inequalities of Ballieu are finally applied.
8. From (16), we get $\sum (r_2 + r_3) \cos \frac{A}{2} \leq \sum 2R_1 \sin \frac{A}{2} \cos \frac{A}{2} = \sum R_1 \sin A$. Then, using Klamkin's inequality (see [6]),

$$x \sin A + y \sin B + z \sin C \leq \frac{xy + yz + zx}{2} \sqrt{\frac{x+y+z}{xyz}}, \text{ for all } x, y, z > 0,$$

we get the desired inequality (29). □

Remark. The above inequality (22) improves Oppenheim's inequality (2).

4. Five methods for generating Erdős–Mordell type inequalities

In order to determine new significant geometric inequalities, we will carefully analyze the relations between R_i, r_i and the elements of the triangle ABC. We will describe five methods that would help us find new inequalities from well-known ones. These derived inequalities will be called *dual inequalities.* First, we present an elementary lemma.

Lemma 4.1. *If B and C are the projections of a point M onto the sides of an angle A, then $BC = AM \sin A$.*

Proof. We apply the law of sines (see *Fig.* 3).

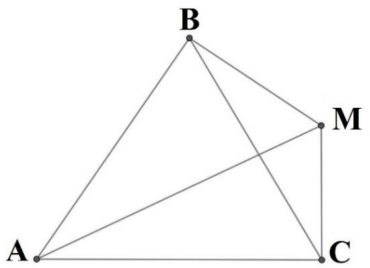

Fig. 3

$\square$

The first method consists of constructing a triangle $A'B'C'$ from the triangle ABC as follows: let M be a point in the interior of a given triangle ABC; the perpendiculars at points A, B and C on MA, MB and MC, respectively, meet at points A', B' and C' (see *Fig.* 4). The triangle $A'B'C'$ is called the *antipedal triangle* associated with the point M with respect to the triangle ABC.

Using the above Lemma 4.1 we obtain

$$a = BC = MA' \sin(\widehat{BA'C}) = MA' \sin(\widehat{BMC}).$$

Hence

$$MA' = \frac{a}{\sin(\widehat{BMC})} = \frac{aR_2R_3}{R_2R_3 \sin(\widehat{BMC})} = \frac{aR_2R_3}{r_1 a} = \frac{R_2R_3}{r_1}.$$

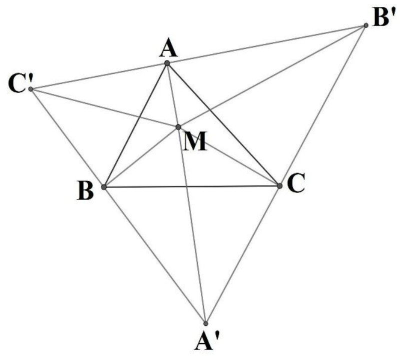

Fig. 4

Analogously, $MB' = \dfrac{R_3R_1}{r_2}$ and $MC' = \dfrac{R_1R_2}{r_3}$. Therefore, the distances from M to the vertices and the sides of the antipedal triangle $A'B'C'$ are the following:

$$R_1' = \frac{R_2R_3}{r_1},\ R_2' = \frac{R_3R_1}{r_2},\ R_3' = \frac{R_1R_2}{r_3},$$

and

$$r_1' = R_1,\ r_2' = R_2,\ r_3' = R_3.$$

The second method consists of constructing a triangle PQR from the triangle ABC as follows: let M be an interior point of the triangle ABC; denote by $P,\ Q,\ R$ the projections of M on the sides BC, CA, AB, and by $r_1',\ r_2',\ r_3'$ the distances from M to the sides RQ, QP, PQ, respectively (see *Fig.* 5).

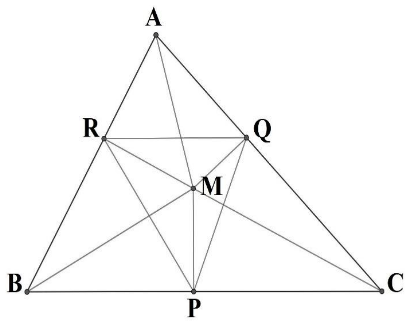

Fig. 5

PQR is called *the pedal triangle* associated with point M with respect to the triangle ABC. Since $RQ = R_1 \sin A$ and $RQ \cdot r_1' = r_2r_3 \sin A$, we have $R_1r_1' = r_2r_3$, that is $r_1' = \dfrac{r_2r_3}{R_1}$. Consequently, the distances from M to the vertices and the sides of the pedal triangle PQR are the following:

$$R_1' = r_1,\ R_2' = r_2,\ R_3' = r_3, \tag{31}$$

and

$$r_1' = \frac{r_2 r_3}{R_1},\ r_2' = \frac{r_3 r_1}{R_2},\ r_3' = \frac{r_1 r_2}{R_3}. \tag{32}$$

Now, we present the third method. Let PQR be the pedal triangle associated with the point M with respect to the triangle ABC. Let $I_{M,k} : \mathcal{P} \setminus \{M\} \to \mathcal{P} \setminus \{M\}$ be the inversion of pole M and power $k > 0$, where $\mathcal{P}$ is the plane of the triangle ABC. Denote $A_1 = I_{M,k}(P)$, $B_1 = I_{M,k}(Q)$, $C_1 = I_{M,k}(R)$ and $A' = I_{M,k}(A)$. We define the following points of intersection: $\{P_1\} = AM \cap B_1C_1$, $\{Q_1\} = BM \cap C_1A_1$ and $\{V_1\} = CM \cap A_1B_1$ (*Fig.* 6).

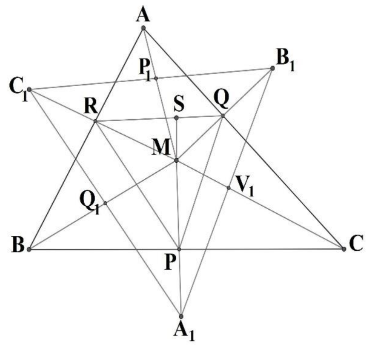

Fig. 6

Since $MR \perp AB$ and $MQ \perp CA$, we have $C_1A' \perp AM$ and $B_1A' \perp AM$. Then $P_1 = A'$ and $AM \perp B_1C_1$. Similarly, $BM \perp C_1A_1$ and $CM \perp A_1B_1$. Then the distances from M to the vertices and the sides of the triangle $A_1B_1C_1$ are as follows:

$$R_1' = MA_1 = \frac{k}{r_1},\ R_2' = MB_1 = \frac{k}{r_2},\ R_3' = MC_1 = \frac{k}{r_3},$$

and

$$r_1' = MP_1 = \frac{k}{R_1},\ r_2' = MQ_1 = \frac{k}{R_2},\ r_3' = MV_1 = \frac{k}{R_3}.$$

Note that

$$a_1 = B_1C_1 = \frac{kR_1 \sin A}{r_2 r_3},\ b_1 = C_1A_1 = \frac{kR_2 \sin B}{r_3 r_1},\ c_1 = A_1B_1 = \frac{kR_3 \sin C}{r_1 r_2}.$$

The fourth method is based on the following construction: assume that Γ is the circumscribed circle (with the center O) of the triangle ABC; denote $(AM \cap \Gamma = \{A_1\}$, $(BM \cap \Gamma = \{B_1\}$ and $(CM \cap \Gamma = \{C_1\}$. From the power of a point theorem, we have

$$MA \cdot MA_1 = MB \cdot MB_1 = MC \cdot MC_1 = R^2 - d^2,$$

with $d = MO$. Hence $\{A_1, B_1, C_1\} = I_{M,u}(\{A, B, C\})$, where $I_{M,u}$ is the inversion of pole M and power $u = d^2 - R^2 < 0$. Denote $k = -u = R^2 - d^2$. Let PQR be the associated pedal triangle and consider the projection S of M on B_1C_1 (see *Fig.* 7). Since $\triangle MSB_1 \sim \triangle MPC$, we obtain $\dfrac{MS}{MP} = \dfrac{MB_1}{MC}$, so that $MS = \dfrac{kr_1}{R_2R_3}$.

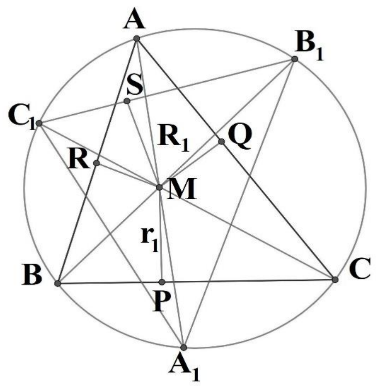

Fig. 7

Therefore, the distances from M to the vertices and the sides of the triangle $A_1B_1C_1$ are the following:

$$R_1' = MA_1 = \frac{k}{R_1},\ R_2' = MB_1 = \frac{k}{R_2},\ R_3' = MC_1 = \frac{k}{R_3},$$

and

$$r_1' = \frac{kr_1}{R_2R_3},\ r_2' = \frac{kr_2}{R_3R_1},\ r_3' = \frac{kr_3}{R_1R_2}.$$

The last method consists in the following construction of the triangle $A_1B_1C_1$. Let γ be the circumscribed circle of the pedal triangle PQR. Let us define $\{P'\} = (PM \cap \gamma$, $\{Q'\} = (QM \cap \gamma$, and $\{R'\} = (RM \cap \gamma$. The parallel lines through points P', Q', R' at the sides of the triangle ABC intersect at points A_1, B_1, C_1. Let S be the projection of M on PR (see *Fig.* 8).

From the power of a point theorem, we have $MP \cdot MP' = MQ \cdot MQ' = MR \cdot MR' =: k$. Since $\triangle MSP \sim \triangle MP'B_1$, we get $\dfrac{MS}{MP'} = \dfrac{MP}{MB_1}$. Thus

$$MB_1 = \frac{MP \cdot MP'}{MS} = \frac{k}{(r_3r_1)/R_2} = \frac{kR_2}{r_3r_1}.$$

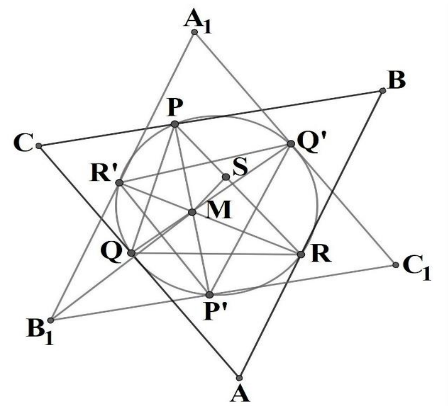

Fig. 8

Therefore, the distances from M to the vertices and the sides of the triangle $A_1B_1C_1$ are the following:

$$R_1' = MA_1 = \frac{kR_1}{r_2r_3},\ R_2' = MB_1 = \frac{kR_2}{r_3r_1},\ R_3' = MC_1 = \frac{kR_3}{r_1r_2},$$

and

$$r_1' = MP' = \frac{k}{r_1},\ r_2' = MQ' = \frac{k}{r_2},\ r_3' = MR' = \frac{k}{r_3}.$$

An inequality of Erdős–Mordell type in the triangle ABC can be applied in any of the triangles highlighted by the previous five constructions. In this way, we obtain a so-called "dual inequality". If, by applying this method $n+1$ times we find the initial inequality, then we say that the inequality has the order n of duality. Below are some examples.

Let us apply the first inequality of (30) in the pedal triangle PQR (see the second method). We have

$$R_1' \geq \frac{RP \cdot r_2' + PQ \cdot r_3'}{QR}.$$

From (31), (32) and Lemma 4.1, we obtain

$$r_1 \geq \frac{R_2 \sin B \cdot (r_3r_1)/R_2 + R_3 \sin C \cdot (r_1r_2)/R_3}{R_1 \sin A}.$$

By applying the law of sines, we find that the above inequality becomes

$$R_1 \geq \frac{cr_2 + br_3}{a},$$

which is called the dual inequality of $R_1 \geq \dfrac{br_2 + cr_3}{a}$. Thus, the dual inequalities of (30) with respect to the pedal triangle are the inequalities (10). Now, if we apply (10) in the pedal triangle PQR, we find $aR_1R_2R_3 \geq bR_2^2r_2 + cR_3^2r_3$,

$$bR_1R_2R_3 \geq cR_3^2r_3 + aR_1^2r_1,\ cR_1R_2R_3 \geq aR_1^2r_1 + bR_2^2r_2. \tag{33}$$

Finally, applying the inequalities (33) in PQR we obtain the initial inequalities (30). In summary, we have established the following scheme of implications:

$$(30)\Rightarrow(10)\Rightarrow(33)\Rightarrow(30).$$

We say that the inequalities (30) have the order 2 of duality with respect to the pedal triangle.

In the following, we study the order of duality for the Erdős-Mordell inequality with respect to all the five proposed methods.

Theorem 4.1. *The following inequalities are obtained by duality from the Erdős–Mordell inequality (1):*

$$\frac{R_2R_3}{r_1}+\frac{R_3R_1}{r_2}+\frac{R_1R_2}{r_3}\geq 2\left(R_1+R_2+R_3\right), \tag{34}$$

$$\frac{R_1R_2R_3}{r_1r_2r_3}\left(r_1+r_2+r_3\right)\geq 2\left(\frac{R_2R_3}{r_1}+\frac{R_3R_1}{r_2}+\frac{R_1R_2}{r_3}\right), \tag{35}$$

$$r_1+r_2+r_3\geq 2\left(\frac{r_2r_3}{R_1}+\frac{r_3r_1}{R_2}+\frac{r_1r_2}{R_3}\right), \tag{36}$$

$$\frac{1}{R_1r_1}+\frac{1}{R_2r_2}+\frac{1}{R_3r_3}\geq 2\left(\frac{1}{R_1R_2}+\frac{1}{R_2R_3}+\frac{1}{R_3R_1}\right), \tag{37}$$

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\geq 2\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right), \tag{38}$$

$$\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\geq 2\left(\frac{r_1}{R_2R_3}+\frac{r_2}{R_3R_1}+\frac{r_3}{R_1R_2}\right). \tag{39}$$

Proof. Denote by (E-M) the inequality (1). By the duality with respect to the above described methods, we successively obtain the inequalities from the statement. The details are omitted.

1. Starting from the inequality (E-M) and successively applying the first method, we obtain the chain of inequalities:

$$\text{(E-M)}\Rightarrow(34)\Rightarrow(35)\Rightarrow\text{(E-M)}.$$

So, (E-M) has the order 2 of duality with respect to the first method.

2. For the second method, we obtain the chain:

$$\text{(E-M)}\Rightarrow(36)\Rightarrow(37)\Rightarrow\text{(E-M)}.$$

Therefore, (E-M) also has the order 2 of duality with respect to the second method.

3. (E-M) has the order 1 of duality with respect to the last three methods. Explicitly, the following chains occur:

$$\text{method 3: (E-M)}\Rightarrow(38)\Rightarrow\text{(E-M)}$$
$$\text{method 4: (E-M)}\Rightarrow(39)\Rightarrow\text{(E-M)}.$$

□

Remark. In [9], the following inequality is proved:

$$\frac{R_1}{r_2r_3}+\frac{R_2}{r_3r_1}+\frac{R_3}{r_1r_2}\geq 2\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right). \tag{40}$$

By method 5, we obtain the chain: (E-M)⇒(40)⇒(E-M).

Next, we study the order of duality for Oppenheim's inequality (2).

Theorem 4.2. *The following inequalities are obtained by duality from Oppenheim's inequality* (2)*:*

$$\frac{(R_1R_2R_3)^2}{r_1r_2r_3}\geq (R_1+R_2)\,(R_2+R_3)\,(R_3+R_1)\,, \tag{41}$$

$$\frac{1}{(r_1r_2r_3)^2}\geq \prod\left(\frac{1}{R_1r_1}+\frac{1}{R_2r_2}\right), \tag{42}$$

$$\frac{1}{r_1r_2r_3}\geq \prod\left(\frac{1}{R_1}+\frac{1}{R_2}\right), \tag{43}$$

$$(R_1R_2R_3)^2\geq \prod\,(R_1r_1+R_2r_2)\,. \tag{44}$$

Proof. Denote by (O) the inequality (2) of Oppenheim. Starting from (O), the following chains of inequalities occur (demo details are omitted)

method 1: (O)⇒(41)⇒(42)⇒(O)
method 2: (O)⇒(40)⇒(39)⇒(O)
method 3: (O)⇒(43)⇒(O)
method 4: (O)⇒(44)⇒(O)
method 5: (O)⇒(O) (the inequality (O) is invariant)

Therefore, (O) has the order 2 of duality with respect to the first two methods, the order 1 of duality with respect to the following two methods and (O) is invariant with respect to the last method. □

We further highlight other interesting inequalities obtained by applying the duality technique.

Theorem 4.3. *The following dual inequalities are obtained from Oppenheim's inequality* (3)

$$\sum\frac{1}{r_1}\geq \sum\frac{1}{r_1}\left(\frac{r_1}{R_2}+\frac{r_2}{R_1}\right)\left(\frac{r_1}{R_3}+\frac{r_3}{R_1}\right), \tag{45}$$

$$\sum\frac{R_1}{r_2r_3}\geq \sum R_1\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\left(\frac{1}{R_1}+\frac{1}{R_3}\right). \tag{46}$$

Proof. Applying the inequality (3) in the pedal triangle PQR (the second method) we find (45). Then, using the same method for (45), we obtain (46). Note that, from (46), we get (3), that is the second inequality of Oppenheim has the order 2 of duality with respect to the pedal triangle. □

Theorem 4.4. *The following dual inequalities are obtained from the inequality* (4) *of Liu*

$$r_2+r_3\geq 2\frac{r_2r_3}{R_1}+r_1\left(\frac{r_2}{R_3}+\frac{r_3}{R_2}\right)^2, \tag{47}$$

$$R_2r_2+R_3r_3\geq 2r_2r_3+r_1r_2r_3\frac{(R_2+R_3)^2}{R_1R_2R_3}. \tag{48}$$

Proof. The inequality (4) applied in triangle PQR yields (47). By applying (47) in the same triangle, it follows (48). We mention that, by applying the inequality (48) in the pedal triangle, we get the initial inequality (4). □

To further illustrate the power of the duality technique, we will refer to a series of inequalities derived from Liu's inequalities (4)–(6).

Theorem 4.5. *The following inequalities are obtained by duality from the inequalities* (5), (6), *and* (7) *of Liu*

$$\sum r_1\frac{(R_2+R_3)^2}{R_2R_3}\leq\sum\frac{R_2R_3}{r_1}, \tag{49}$$

$$\sum r_1\left(\frac{r_2}{R_3}+\frac{r_3}{R_2}\right)^2\leq\sum r_1, \tag{50}$$

$$\sum\frac{r_1\left(\frac{r_2}{R_3}+\frac{r_3}{R_2}\right)^2}{1+\frac{r_2}{R_3}+\frac{r_3}{R_2}}\leq\sum\frac{r_2r_3}{R_1}, \tag{51}$$

$$\sum\frac{r_1(R_2+R_3)^2}{r_1(R_2+R_3)+R_2R_3}\leq\sum R_1, \tag{52}$$

$$\sum\frac{\left(\frac{kr_2r_3}{R_1}+\frac{r_3r_1}{R_2}+\frac{r_1r_2}{R_3}\right)^2}{r_1+\frac{kr_2r_3}{R_1}}\leq\frac{k+2}{2}\sum r_1,\ \ k\geq 0, \tag{53}$$

$$\sum\frac{\left(\frac{k}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)^2}{\frac{1}{R_1r_1}+\frac{k}{R_2R_3}}\leq\frac{k+2}{2}\sum\frac{1}{R_1r_1},\ \ k\geq 0. \tag{54}$$

Proof. The first two inequalities (49) and (50) are obtained from (5) by duality with respect to the antipedal triangle $A'B'C'$. The next two inequalities (51) and (52) are derived from (6) by duality with respect to the pedal triangle PQR. The last two inequalities (53) and (54) are the dual inequalities of (7) with respect to the pedal triangle PQR. All inequalities (4)-(6) of Liu have the duality order 2 with respect to the above mentioned triangles. □

5. Remarkable inequalities

At the end of our study, we believe it is of interest to point out a series of remarkable inequalities as particular cases. Thus, if $M=I$ the inequalities

(E-M), (38), (2) and (3) become the following well-known inequalities:

$$R \geq 2r; \quad 3\frac{r}{R} \leq \sum \sin\frac{A}{2} \leq \frac{3}{2}; \quad \sum \sin^{-1}\frac{A}{2} \geq 6.$$

If $M = O$, the inequalities (E-M), (38) and (39) yield

$$R \geq 2r; \quad \sum \frac{1}{\cos A} \geq 6; \quad \sum \cos A \leq \frac{3}{2}.$$

Finally, we mention that if $M = G$ then (40) becomes $\sum \frac{m_a}{a} \geq \frac{s}{R}$.

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An Osgood criterion for a class of second order ordinary differential equations

George Stoica[1)]

Abstract. We extend Osgood's criterion concerning uniqueness of solutions and their maximal intervals to a particular class of second order ordinary differential equations.

Keywords: Osgood's criterion, uniqueness of solutions, maximal interval.
MSC: 34C11, 34A12.

W.F. Osgood proved in [2] the following result: Consider the following ordinary differential equation

$$(1)\qquad \begin{cases} x'(t) = f(x(t)) \text{ for } t > 0, \\ x(0) = a, \end{cases}$$

with a continuous function $f : \mathbb{R} \to \mathbb{R}^+$ that is positive on (a, ∞). If we denote

$$F_a(u) := \int_a^u \frac{\mathrm{d}s}{f(s)} \text{ for } u > a,$$

then equation (1) has at most one solution, given by

$$x(t) = F_a^{-1}(t)$$

and defined on the maximal interval

$$0 < t < F_a(\infty).$$

For instance, if $f(x) = x^2$, then $x(t) := \dfrac{1}{1/a - t}$ is the unique solution of (1) provided that $a \neq 0$; and $F_a(\infty) < \infty$ if and only if $a > 0$. That is, if $a > 0$, the unique solution is defined on the interval $(0, 1/a)$; and if $a < 0$, the unique solution is defined on the interval $(0, \infty)$.

Finding an equivalent of Osgood's criterion for ordinary differential equations of order higher than one is still an open problem, although progress has been made recently, see e.g., [1]. In the sequel, we shall solve this issue for a particular class of second order ordinary differential equations, as follows.

Proposition 1. *Consider the following ordinary differential equation*

$$(2)\qquad \begin{cases} x''(t) = f(x(t)) \textit{ for } t > 0, \\ x(0) = a \textit{ and } x'(0) = b, \end{cases}$$

[1)] 5 Deveber Terrace, Saint John NB, E2K 2B5 Canada, gstoica2015@gmail.com

with a continuous function $f:\mathbb{R}\to\mathbb{R}^+$ *that is positive on* (a,∞)*. If we denote*

$$F_{a,b}(u) := \int_a^u \left[b^2 + 2\int_a^s f(w)\,\mathrm{d}w\right]^{-1/2} \mathrm{d}s \text{ for } u > a,$$

then equation (2) *has at most one solution, given by*

$$x(t) = F_{a,b}^{-1}(t)$$

and defined on the maximal interval

$$0 < t < F_{a,b}(\infty).$$

Proof. We start by integrating the equation $x''(t)\cdot x'(t) = f(x(t))\cdot x'(t)$, to obtain

$$\frac{1}{2}x'(t)^2 = \int_0^t f(x(s))\,\mathrm{d}x(s) + c,$$

where in the right-hand side we have a Riemann–Stieltjes integral, whose existence is justifed by the fact that x' is increasing, as $x''(t) = f(x(t)) \geq 0$ for $t > 0$.

From $x'(0) = b$ it follows that $c = b^2/2$. Therefore, using the change of variables $u = x(s)$ for Riemann–Stieltjes integrals, we obtain

$$x'(t)^2 - b^2 = 2\int_0^t f(x(s))\,\mathrm{d}x(s) = 2\int_a^{x(t)} f(u)\,\mathrm{d}u.$$

The above change of variables is justified (cf. [3, Theorem 6.17]), as f is locally integrable (being continuous) and x does not change sign (as f is nonnegative).

The previous equation can be written as:

$$x'(t) = g_{a,b}(x(t)),$$

with

$$g_{a,b}(x(t)) = \left[b^2 + 2\int_a^{x(t)} f(u)\,\mathrm{d}u\right]^{1/2}.$$

By Osgood's criterion applied to the above *first order* ordinary differential equation, the conclusion now follows. □

Examples. (i) If $f(x) = 6x^2$, $a = 1$ and $b = 2$, then

$$F_{1,2}(\infty) = \int_1^\infty \left[4 + 2\int_1^x 6t^2\mathrm{d}t\right]^{-1/2} \mathrm{d}x = \int_1^\infty (4x^3)^{-1/2}\mathrm{d}x = 1.$$

Therefore equation (2) has the unique solution $f(x) = \dfrac{1}{(1-x)^2}$ on the interval $(0,1)$.

(ii) If $f(x) = 12\sqrt{x}$, $a = 1$ and $b = 4$, then

$$F_{1,4}(\infty) = \int_1^\infty \left[16 + 24\int_1^x \sqrt{t}\mathrm{d}t\right]^{-1/2} \mathrm{d}x = \int_1^\infty \left(16x^{3/2}\right)^{-1/2} \mathrm{d}x = \infty.$$

Therefore equation (2) has the unique solution $f(x) = (x+1)^4$ on the interval $(0, \infty)$. $\square$

Remarks. (i) Although similarly defined, let us notice that the expressions $F_a(u)$ and $F_{a,b}(u)$ do not have the same effect on the corresponding maximal intervals of the solutions. Indeed, let

$$f(x) = |x|(\log_+ |x|)^\delta \text{ with } \delta > 0,$$

where $\log_+(z) := \log\max\{z, e\}, z \geq 0$, and e is Euler's number. Then $F_a(u) < \infty$ if and only if $\delta > 1$, whereas $F_{a,b}(u) < \infty$ if and only if $\delta > 2$. Thus, a higher power of δ is required in the maximal interval for the solution to (2), compared to the power of δ required by the maximal interval for the solution to (1).

(ii) Note that both $F_a(u)$ and $F_{a,b}(u)$ have the same type of behavior with respect to initial data, in the sense that, if $F_a(u) < \infty$ for some a, then $F_a(u) < \infty$ for all a; and, similarly, if $F_{a,b}(u) < \infty$ for some a and $b > 0$, then $F_{a,b}(u) < \infty$ for all a and $b > 0$. Let us prove the latter.

If $b_2 \geq b_1$, then $F_{a,b_2}(u) \leq F_{a,b_1}(u)$; and

$$F_{a,b_1}(u) \leq \frac{b_2}{b_1} F_{a,b_2}(u),$$

so $F_{a,b_1}(u) < \infty$ if and only if $F_{a,b_2}(u) < \infty$.

If $a_2 \geq a_1$, then $F_{a_2,b}(u) \leq F_{a_1,b}(u)$; and

$$F_{a_1,b}(u) \leq \int_{a_1}^{a_2} \left[b^2 + 2\int_{a_1}^{s} f(u)\,\mathrm{d}u\right]^{-1/2} \mathrm{d}s + F_{a_2,b}(u),$$

so $F_{a_1,b}(u) < \infty$ if and only if $F_{a_2,b}(u) < \infty$.

(iii) Instead of working with ordinary differential equations, we can compute the solutions and their maximal intervals for other types of equations. Indeed, (1) is equivalent to the following integral equation

$$x(t) = a + \int_0^t f(x(s))\mathrm{d}s \text{ for } t > 0$$

and (2) is equivalent to either the integro-differential equation

$$x'(t) = b + \int_0^t f(x(s))\mathrm{d}s \text{ for } t > 0, \quad x(0) = a,$$

or to the integral equation

$$x(t) = a + bt + \int_0^t (t - s)f(x(s))\mathrm{d}s \text{ for } t > 0.$$

All the results given in this note apply ad litteram to these types of equations.

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Corrigendum to "Asymptotic evaluations for some sequences of triple integrals" [G. M. A. 39(118) (2021), no. 1–2, 15–35]

DUMITRU POPA[1)]

Abstract. Small corrections to the mentioned article are pointed out.

Keywords: Riemann integral, multiple Riemann integral, uniform convergence, asymptotic expansion of a sequence.

MSC: Primary 26A42, 28A20; Secondary 40A05, 40A25.

On page 17, line 27, instead of

$$a_n(x) < \delta_\varepsilon,\ b_n(x) < \delta_\varepsilon,\ c_n(x) < \delta_\varepsilon, \forall x \in A.$$

it will be written

$$|a_n(x)| < \delta_\varepsilon,\ |b_n(x)| < \delta_\varepsilon,\ |c_n(x)| < \delta_\varepsilon, \forall x \in A$$

and on lines 28–29, instead of $\max(a_n(x), b_n(x), c_n(x)) < \delta_\varepsilon$ it will be written $\max(|a_n(x)|, |b_n(x)|, |c_n(x)|) < \delta_\varepsilon$.

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[1]Department of Mathematics, Ovidius University of Constanţa, Bd. Mamaia 124, 900527 Constanţa, Romania, `dpopa@univ-ovidius.ro`

Traian Lalescu National Mathematics Contest for university students, 2021 edition

Eugen Păltănea[1], Vasile Pop[2], Radu Strugariu[3]

Abstract. We present solutions for the problems of the 2021 edition of the Traian Lalescu Mathematics Contest for University Students, hosted by the Transilvania University of Braşov, November 25–27, 2021.

Keywords: group, subgroup, endomorphism, matrix, characteristic polynomial, rank, trace, determinant, kernel, eigenvalues, eigenvectors, convex function, sequences of real numbers, series of real numbers, Riemann integral, function of class C^2.

MSC: 15A24, 40A05, 26A42, 11E57, 15A03, 15A18, 15A21, 20K27, 26A24.

Despite all the difficulties generated by the COVID-19 pandemic during the whole year 2021, the organization of the national phase of the Traian Lalescu Mathematics Contest for University Students, or, in other words, of the National Mathematics Olympiad for University Students, was successful, with physical presence both of the students and of the university teachers, members of the competition committee.

The organization this year in November instead of May avoided the critical period, and a crucial role was played by Mrs. Smaranda Lalescu, the niece of the great mathematician and the president of the Traian Lalescu Foundation, supported, as usual, by Assoc. Prof. Antonela Toma, from the part of the Ministry of Education. Also, a very important role in the organization was played by Assoc. Prof. Nicuşor Minculete from Transilvania University of Braşov, who connected two important events: the National Contest Traian Lalescu and the Student Scientific Communications Session in the areas of Mathematics and Computer Science.

Despite all the restrictions imposed by the pandemic situation, the contest took place in very good conditions, in compliance with all the necessary rules.

The presence of the students in the Traian Lalescu contest and in the Session of Scientific Communications was surprisingly large:

- Section A – Faculties of Mathematics: 20 students, from University of Bucharest, Alexandru Ioan Cuza University of Iaşi, Babeş-Bolyai University of Cluj–Napoca, West University of Timişoara, University of Craiova, and Transilvania University of Braşov;
- Section B (B1 and B2) – Technical Faculties, electrical profile: 35 students, from Politehnica University of Bucharest, Alexandru Ioan Cuza University of Iaşi, Gheorghe Asachi Technical University of

[1] Transilvania University of Braşov, Romania, epaltanea@unitbv.ro
[2] Technical University of Cluj–Napoca, Romania, vasile.pop@math.utcluj.ro
[3] Gheorghe Asachi Technical University of Iaşi, Romania, rstrugariu@tuiasi.ro

Iaşi, Babeş-Bolyai University of Cluj–Napoca, Technical University of Cluj-Napoca, University of Craiova, and Transilvania University of Braşov;
- Section C (C1 and C2) – Technical Faculties, nonelectrical profile: 22 students, from Politehnica University of Bucharest, Technical University of Civil Engineering Bucharest, Gheorghe Asachi Technical University of Iaşi, Technical University of Cluj–Napoca, and Transilvania University of Braşov;
- Section D – Technical Faculties, Special Mathematics: 6 students, from Politehnica University of Bucharest, University of Civil Engineering Bucharest, Politehnica University of Timişoara, and Gheorghe Asachi Technical University of Iaşi;
- Scientific Communications, Mathematics Section – 13 students;
- Scientific Communications, Computer Science Section – 18 students.

This event would not have been possible without the financial and logistic support of the boards of Transilvania University of Braşov and of the Faculty of Mathematics and Computer Science, and of all the staff who struggled to offer the best conditions to the participants.

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We present in the sequel the statements and solutions of the problems given at Sections A and B of the contest. The maximum score, that is 40 points, was obtained by *Marian Daniel Vasile* (West University of Timişoara), Section A.

1. SECTION A

Problem 1. *Let $(G, \cdot)$ be a non-abelian group with the center $Z(G)$. Suppose that $f : G \to G$ is an endomorphism with the property that there are two positive co-prime integers m and n such that $f((xy)^m) = f((yx)^m)$ and $f((xy)^n) = f((yx)^n)$, for all $x, y \in G \setminus Z(G)$. Prove that* $\operatorname{Im}(f)$ *is an abelian subgroup of* G.

Mihai Chiş, West University of Timişoara

Solution. Assume $t = f(x) \in \operatorname{Im}(f)$ and $u = f(y) \in \operatorname{Im}(f)$, where $x, y \in G$.
If $x \in Z(G)$, then $tu = f(x)f(y) = f(xy) = f(yx) = f(y)f(x) = ut$. Similarly, if $y \in Z(G)$, then $tu = ut$. Suppose now $x, y \in G \setminus Z(G)$. Since $(m, n) = 1$, there are two integer numbers k and ℓ such that $km + \ell n = 1$.

Hence

$$\begin{aligned} tu &= f(x)f(y) = f(xy) = f((xy)^{km+\ell n}) = f((xy)^m)^k f((xy)^n)^\ell \\ &= f((yx)^m)^k f((yx)^n)^\ell = f((yx)^{km+\ell n}) = f(yx) = f(y)f(x) = ut. \end{aligned}$$

Therefore, $\mathrm{Im}(f)$ is an abelian subgroup of G.

Problem 2. *Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_1 = 1$ and $x_{n+1} = \dfrac{n}{1+x_n}$, for all $n \geq 1$. Find $\lim\limits_{n\to\infty}(x_n - \sqrt{n})$.*

Tiberiu Trif, Babeş–Bolyai University of Cluj–Napoca

Solution. For a positive integer n, let $a_n = (\sqrt{4n+1}-1)/2$ be the unique fixed point of the function $f_n : (0,\infty) \to (0,\infty)$, $f_n(x) = \dfrac{n}{1+x}$. The following statement can be obtained by induction:

$$\frac{\sqrt{4n-3}-1}{2} = a_{n-1} < x_n < a_n = \frac{\sqrt{4n+1}-1}{2}, \text{ for all } n \geq 5.$$

Then we have

$$-\frac{1}{2} + \frac{\sqrt{4n-3}-2\sqrt{n}}{2} < x_n - \sqrt{n} < -\frac{1}{2} + \frac{\sqrt{4n+1}-2\sqrt{n}}{2}, \text{ for } n \geq 5.$$

From the Squeeze Theorem for sequences, we obtain $\lim\limits_{n\to\infty}\left(x_n - \sqrt{n}\right) = -\dfrac{1}{2}$.

Problem 3. *Let $A \in \mathcal{M}_n(\mathbb{C})$, $n \geq 2$, be a matrix with $\mathrm{Tr}(A) \neq 0$. Suppose there is $k \in \mathbb{N}^*$ such that $A^{k+1} = \mathrm{Tr}(A^k)A$. Prove that*

$$A^{m+1} = \mathrm{Tr}(A^m)A \textit{ for all } m \geq 1.$$

Mihai Opincariu, Avram Iancu National College, Brad
Vasile Pop, Technical University of Cluj–Napoca

Solution. Denote $a = \mathrm{Tr}(A^k)$. If $a = 0$ then $A^{k+1} = O_n$ and A has only the eigenvalue 0. Hence $\mathrm{Tr}(A) = 0$, contradiction. Thus, $a \neq 0$. For any eigenvalue $\lambda \in \mathrm{Spec}(A)$ we have $\lambda^{k+1} = a\lambda$. Then $\lambda^k = a$, for all $\lambda \in \mathrm{Spec}(A) \setminus \{0\}$. From the relation

$$a = \mathrm{Tr}(A^k) = \sum_{\lambda\in\mathrm{Spec}(A)} \lambda^k = \sum_{\lambda\in\mathrm{Spec}(A)\setminus\{0\}} \lambda^k$$

it follows that the matrix A has only one non-zero eigenvalue $t \in \mathbb{C}^*$ and $a = t^k$. Therefore, the characteristic polynomial of A is $p_A = X^{n-1}(X-t)$. On the other hand, for the polynomial $p = X(X^k - t^k)$ we have $p(A) = O_n$. The greatest common divisor of the polynomials p_A and p is $X(X-t)$. So $A(A - tI_n) = O_n$.

Let m be a positive integer. Since $\mathrm{Tr}(A^m) = t^m$, we have

$$A^{m+1} - \mathrm{Tr}(A^m)A = A(A^m - t^m I_n) = A(A - tI_n)\left(\sum_{i=0}^{m-1} t^{m-1-i}A^i\right) = O_n,$$

that is $A^{m+1} = \mathrm{Tr}(A^m)A$.

Remark. $m_A = X(X-t)$ is the minimal polynomial of A (Frobenius' theorem).

Problem 4. *Let $f : [0,\infty) \to (0,\infty)$ be an integrable function on all compact intervals from $[0,\infty)$, such that there is a convex and strictly decreasing function $g : (0,\infty) \to \mathbb{R}$ with the property $g(f(x)) = \int_0^x f(t)\,\mathrm{d}t$, for all $x \geq 0$. Prove that f is a convex function.*

Eugen Păltănea, Transilvania University of Braşov

Solution. The function $F : [0,\infty) \to \mathbb{R},\ F(x) = \int_0^x f(t)\,\mathrm{d}t,\ \forall\, x \geq 0$, is well-defined, positive, and strictly increasing. Since g is strictly decreasing, the following limits exist: $b = \lim_{x \searrow 0} g(x) = \sup_{x>0} g(x)$ and $a = \lim_{x\to\infty} g(x) = \inf_{x>0} g(x)$. The convex function g in continuous on the open interval $(0,\infty)$. So we obtain $g((0,\infty)) = (a,b)$. Therefore, the restriction $g_1 : (0,\infty) \to (a,b)$ of the function g is bijective. Its inverse $g_1^{-1} : (a,b) \to (0,\infty)$ is continuos and strictly decreasing on (a,b). In addition, the function g_1^{-1} is convex. Thus, for arbitrary $u, v \in (a,b)$, with $g_1^{-1}(u) = x$ and $g_1^{-1}(v) = y$, and for arbitrary $\lambda \in [0,1]$, we have $(1-\lambda)g(x) + \lambda g(y) \geq g((1-\lambda)x + \lambda y)$. Hence

$$\begin{aligned} g_1^{-1}((1-\lambda)u + \lambda v) &= g_1^{-1}((1-\lambda)g(x) + \lambda g(y)) \leq g_1^{-1}(g((1-\lambda)x + \lambda y)) \\ &= (1-\lambda)x + \lambda y = (1-\lambda)g_1^{-1}(u) + \lambda g_1^{-1}(v). \end{aligned}$$

Since $f(x) = g_1^{-1}(F(x)),\ \forall\, x \geq 0$, the function f is strictly decreasing and continuous. Then $F' = f$ and F is concave on $[0,\infty)$.

Assume $x, y \in [0,\infty)$ and $\lambda \in [0,1]$. We have

$$F((1-\lambda)x + \lambda y) \geq (1-\lambda)F(x) + \lambda F(y).$$

Since g_1^{-1} is strictly decreasing and convex, we get

$$\begin{aligned} f((1-\lambda)x + \lambda y) &= g_1^{-1}(F((1-\lambda)x + \lambda y)) \leq g_1^{-1}((1-\lambda)F(x) + \lambda F(y)) \\ &\leq (1-\lambda)g_1^{-1}(F(x)) + \lambda g_1^{-1}(F(y)) = (1-\lambda)f(x) + \lambda f(y). \end{aligned}$$

In conclusion, f is a convex function on $[0,\infty)$.

2. Section B1

Problem 1. *For a matrix $M \in \mathcal{M}_n(\mathbb{C})$, we denote*

$$\ker M = \{X \in \mathcal{M}_{n,1}(\mathbb{C}) : MX = O_{n,1}\}.$$

Prove that, for any $A, B, C \in \mathcal{M}_n(\mathbb{C})$, the following assertions are equivalent:
(i) $AB = A, \quad BC = B, \quad CA = C$;
(ii) $A^2 = A, \quad B^2 = B, \quad C^2 = C, \quad \ker A = \ker B = \ker C$.

Mihai Opincariu, Avram Iancu National College, Brad
Vasile Pop, Technical University of Cluj–Napoca

Solution. Let us prove first $(i) \Rightarrow (ii)$.

From $AB = A$, it follows $ABC = AC$. From $BC = B$, it follows $ABC = AB$, but $AB = A$, hence $AC = A$.

Furthermore, from $AC = A$ we have $ACA = A^2$, and from $CA = C$ it follows $ACA = AC = A$. Hence, $A^2 = A$.

Using the symmetry of the relations, we obtain similarly that $B^2 = B$ and $C^2 = C$.

If $X \in \ker B$, then $BX = O_{n,1}$, hence $\underbrace{AB}_{A} X = O_{n,1}$, hence $AX = O_{n,1}$ and $X \in \ker A$. We have obtained $\ker B \subset \ker A$. By symmetry, we obtain $\ker C \subset \ker B$ and $\ker A \subset \ker C$, hence $\ker A = \ker B = \ker C$.

We will prove now $(ii) \Rightarrow (i)$.

Since $B^2 = B$, it follows that $B(B - I_n)X = (B^2 - B)X = O_{n,1}$, for all $X \in \mathcal{M}_{n,1}(\mathbb{C})$, so $(B - I_n)X \in \ker B = \ker A$, hence $(AB - A)X = A(B - I_n)X = O_{n,1}$, for all $X \in \mathcal{M}_{n,1}(\mathbb{C})$. In conclusion, $AB - A = O_{n,1}$, hence $AB = A$. From symmetry, we get the other relations.

Remark. From (i) or (ii), one can obtain that $\operatorname{Tr} A = \operatorname{Tr} B = \operatorname{Tr} C$. For example, following the argument of $(i) \Rightarrow (ii)$, we have $AC = A$ and $CA = C$, hence $\operatorname{Tr} A = \operatorname{Tr}(AC) = \operatorname{Tr}(CA) = \operatorname{Tr} C$. Analogously, $\operatorname{Tr} C = \operatorname{Tr} B$.

Problem 2. *We consider the function f of class C^2 such that $f'(0) \neq 0$ and we denote by $c_n \in \left[0, \frac{1}{n}\right]$ the point for which*

$$n \int_0^{\frac{1}{n}} f(x)\,\mathrm{d}x = f(c_n).$$

Compute

$$\lim_{n\to\infty} n\left(nc_n - \frac{1}{2}\right).$$

Radu Strugariu, Gheorghe Asachi Technical University of Iaşi

Solution. We will first prove that $\lim\limits_{n\to\infty} nc_n = \dfrac{1}{2}$.

Using Taylor's formula, we have

$$f(c_n) = f(0) + f'(0)\,c_n + o(c_n),$$

hence

$$\lim_{n\to\infty} nc_n = \lim_{n\to\infty} n\left(\frac{f(c_n) - f(0)}{f'(0)} - \frac{o(c_n)}{f'(0)}\right).$$

Observe that

$$f(c_n) - f(0) = n\int_0^{\frac{1}{n}} (f(x) - f(0))\,\mathrm{d}x$$

and, since $c_n \in \left[0, \frac{1}{n}\right]$, $\lim_{n\to\infty} n\cdot o(c_n) = 0$. We get

$$\lim_{n\to\infty} nc_n = \lim_{n\to\infty} n\left(\frac{f(c_n) - f(0)}{f'(0)}\right) = \frac{1}{f'(0)}\cdot\lim_{n\to\infty}\frac{\int_0^{\frac{1}{n}}(f(x) - f(0))\,\mathrm{d}x}{\frac{1}{n^2}}.$$

Since the function $g(x) = f(x) - f(0)$ is continuous, it has antiderivative $G(x) = \displaystyle\int_0^x g(t)\mathrm{d}t$. Then

$$\lim_{n\to\infty} nc_n = \frac{1}{f'(0)}\cdot\lim_{n\to\infty}\frac{G\left(\frac{1}{n}\right)}{\frac{1}{n^2}} = \frac{1}{f'(0)}\cdot\lim_{x\to 0}\frac{G(x)}{x^2}.$$

Using the l'Hospital's Rule, we get

$$\lim_{x\to 0}\frac{G(x)}{x^2} = \lim_{x\to 0}\frac{f(x) - f(0)}{2x} = \frac{f'(0)}{2},$$

whence $\lim\limits_{n\to\infty} nc_n = \frac{1}{2}$.

Next, we will prove that

$$\lim_{n\to\infty} n\left(nc_n - \frac{1}{2}\right) = \frac{f''(0)}{24f'(0)}.$$

Again from Taylor's formula, we have

$$f(c_n) = f(0) + f'(0)\,c_n + \frac{f''(0)}{2}c_n^2 + o\left(c_n^2\right),$$

hence

$$\lim_{n\to\infty} n\left(nc_n - \frac{1}{2}\right) = \lim_{n\to\infty} n^2\left(\frac{f(c_n) - f(0) - o\left(c_n^2\right)}{f'(0)} - \frac{f''(0)}{2f'(0)}c_n^2\right) - \frac{n}{2}.$$

Using that $\lim\limits_{n\to\infty} nc_n = \dfrac{1}{2}$, we have $\lim\limits_{n\to\infty} n^2\dfrac{f''(0)}{2f'(0)}c_n^2 = \dfrac{f''(0)}{8f'(0)}$, and, since we know that $\lim_{n\to\infty} n^2\cdot o\left(c_n^2\right) = 0$, we get

$$\lim_{n\to\infty} n\left(nc_n - \frac{1}{2}\right) = \lim_{n\to\infty}\frac{n^3}{f'(0)}\left(G\left(\frac{1}{n}\right) - \frac{f'(0)}{2n^2}\right) = \frac{1}{f'(0)}\lim_{x\to 0}\frac{G(x) - \frac{f'(0)}{2}x^2}{x^3}.$$

Observe that

$$\begin{aligned}
G(0) &= 0,\\
G'(x) &= f(x) - f(0) \Longrightarrow G'(0) = 0,\\
G''(x) &= f'(x) \qquad\quad\ \ \Longrightarrow G'(0) = f'(0),\\
G'''(x) &= f''(x) \qquad\quad\ \Longrightarrow G''(0) = f''(0).
\end{aligned}$$

Using now the Taylor's formula for G, we have

$$\begin{aligned}
G(x) &= G(0) + G'(0)x + \frac{G''(0)}{2}x^2 + \frac{G'''(0)}{6}x^3 + o\left(x^3\right),\\
G(x) &= \frac{f'(0)}{2}x^2 + \frac{f''(0)}{6}x^3 + o\left(x^3\right),
\end{aligned}$$

whence

$$\lim_{x\to 0}\frac{G(x) - \frac{f'(0)}{2}x^2}{x^3} = \lim_{x\to 0}\frac{\frac{f''(0)}{6}x^3 + o\left(x^3\right)}{x^3} = \frac{f''(0)}{6}.$$

The desired limit is hence

$$\lim_{n\to\infty} n\left(nc_n - \frac{1}{2}\right) = \frac{f''(0)}{6f'(0)} - \frac{f''(0)}{8f'(0)} = \frac{f''(0)}{24f'(0)}.$$

Remark. The fact that $\lim\limits_{n\to\infty} nc_n = \dfrac{1}{2}$ if f is of class C^1 with $f'(0) \neq 0$ is a well-known result.

Solution proposed by Marian Panţiruc. We can write, using integration by parts:

$$\begin{aligned}
f(c_n) &= nxf(x)\Big|_0^{1/n} - n\int_0^{1/n} xf'(x)\,\mathrm{d}x\\
&= f\left(\frac{1}{n}\right) - \frac{n}{2}x^2f'(x)\Big|_0^{1/n} + \frac{n}{2}\int_0^{1/n} x^2f''(x)\,\mathrm{d}x.
\end{aligned}$$

For the last integral, one can use now the 2nd mean theorem for integrals, since f'' is continuous and $x \mapsto x^2$ is positive, hence

$$\int_0^{1/n} x^2f''(x)\,\mathrm{d}x = f''(\alpha_n)\cdot\int_0^{1/n} x^2\,\mathrm{d}x = \frac{f''(\alpha_n)}{3n^3},$$

for some $\alpha_n \in \left[0, \frac{1}{n}\right]$. It follows that

$$f(c_n) = f\left(\frac{1}{n}\right) - \frac{1}{2n}f'\left(\frac{1}{n}\right) + \frac{f''(\alpha_n)}{6n^2}, \tag{1}$$

and then

$$\frac{f(c_n) - f(0)}{c_n}\cdot nc_n = \frac{f\left(\frac{1}{n}\right) - f(0)}{\frac{1}{n}} - \frac{1}{2}f'\left(\frac{1}{n}\right) + \frac{f''(\alpha_n)}{6n}.$$

Letting $n \to \infty$, we obtain that the limit $\lim\limits_{n\to\infty} nc_n$ exists and

$$f'(0) \cdot \lim_{n\to\infty} nc_n = f'(0) - \frac{1}{2}f'(0) \Rightarrow \lim_{n\to\infty} nc_n = \frac{1}{2}.$$

By refining the relation (1), we can write

$$\frac{f(c_n) - f\left(\frac{1}{2n}\right)}{c_n - \frac{1}{2n}} \cdot \left(c_n - \frac{1}{2n}\right) = f\left(\frac{1}{n}\right) - f\left(\frac{1}{2n}\right) - \frac{1}{2n}f'\left(\frac{1}{n}\right) + \frac{f''(\alpha_n)}{6n^2},$$

and multiplying with n^2 we obtain

$$\frac{f(c_n) - f(\frac{1}{2n})}{c_n - \frac{1}{2n}} \cdot n\left(nc_n - \frac{1}{2}\right) = \frac{f\left(\frac{1}{n}\right) - f\left(\frac{1}{2n}\right) - \frac{1}{2n}f'\left(\frac{1}{n}\right)}{\frac{1}{n^2}} + \frac{f''(\alpha_n)}{6}. \quad (2)$$

For the fraction in the left-hand side of this equality, one can apply the Lagrange Theorem to find some $\theta_n \in \left[c_n, \frac{1}{2n}\right]$ such that

$$\frac{f(c_n) - f(\frac{1}{2n})}{c_n - \frac{1}{2n}} = f'(\theta_n) \stackrel{n\to\infty}{\longrightarrow} f'(0).$$

Obviously, $f''(\alpha_n) \to f''(0)$.

Now, for the first term in the right hand side we can write

$$\ell := \lim_{n\to\infty} \frac{f(\frac{1}{n}) - f(\frac{1}{2n}) - \frac{1}{2n}f'(\frac{1}{n})}{\frac{1}{n^2}} = \lim_{x\to 0}\left(\frac{f(x) - f(\frac{x}{2})}{x^2} - \frac{f'(x)}{2x}\right),$$

so by the l'Hospital's Rule,

$$\ell = \lim_{x\to 0}\left(\frac{f'(x) - \frac{1}{2}f'(\frac{x}{2})}{2x} - \frac{f''(x)}{2}\right) = -\frac{f''(0)}{8}.$$

Returning to (2), it follows that the limit $\lim\limits_{n\to\infty} n\left(nc_n - \frac{1}{2}\right)$ exists and

$$f'(0) \cdot \lim_{n\to\infty} n\left(nc_n - \frac{1}{2}\right) = -\frac{f''(0)}{8} + \frac{f''(0)}{6} = \frac{f''(0)}{24}.$$

Problem 3. *Consider $A, B \in \mathcal{M}_n(\mathbb{C})$ such that*

$$(A + I_n)(B - I_n) = O_n, \qquad (B + I_n)(A - I_n) = O_n.$$

a) *Prove that $A^2 = B^2 = I_n$.*

b) *Prove that the function*

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) = \det((1 - z)A + zB), \quad z \in \mathbb{C},$$

is constant.

Mihai Opincariu, Avram Iancu National College, Brad
Vasile Pop, Technical University of Cluj–Napoca

Solution. a) We write the given relations as:

$$AB = A - B + I_n, \tag{3}$$

$$BA = B - A + I_n. \tag{4}$$

From (3) we obtain

$$ABA = A^2 - BA + A,$$

while from (4)

$$ABA = AB - A^2 + A,$$

hence

$$A^2 - BA = AB - A^2,$$

or

$$2A^2 = AB + BA = 2I_n.$$

It follows that $A^2 = I_n$ and, by symmetry, $B^2 = I_n$.

b) We obtain

$$\begin{aligned}((1-z)A + zB)^2 &= (1-z)^2 A^2 + z^2 B^2 + z(1-z)(A \cdot B + B \cdot A) \\ &= (1-z)^2 I_n + z^2 I_n + z(1-z) \cdot 2I_n = I_n,\end{aligned}$$

hence $(f(z))^2 = \det I_n = 1$. It follows $f(z) \in \{-1, 1\}$ for any $z \in \mathbb{C}$.

Since f is a polynomial function in z which has no roots in $\mathbb{C}$, it follows that f is a nonzero constant.

Remark 1. Since $\mathbb{C}$ is a connected set, and f is a continuous function (as sum and product of continuous functions), it follows that $f(\mathbb{C})$ is connected, hence f is constant.

Remark 2. Another justification (which uses the continuity of real functions) can be given using the functions $g_z : [0,1] \to \mathbb{R}$,

$$g_z(t) = \operatorname{Re} f(tz) = \operatorname{Re}\det\big((1-tz)A + tzB\big), \quad t \in [0,1],$$

for $z \in \mathbb{C}$. As shown, g_z is continuous and takes the values -1 or 1, hence it must be constant (otherwise it would take the value 0, which is impossible). It follows that

$$\operatorname{Re} f(z) = g_z(1) = g_z(0) = \operatorname{Re} f(0), \quad \text{for any } z \in \mathbb{C},$$

hence $\operatorname{Re} f$ is constant. Since $\operatorname{Im} f = 0$, it follows that f is constant.

Problem 4. *For the integers $n, m \in \mathbb{N}^*$, we denote*

$$p_m(n) = \begin{cases} \text{the first } m \text{ digits of } n, & \text{if } n \text{ has at least } m \text{ digits,} \\ 0, & \text{if } n \text{ has at most } m-1 \text{ digits.} \end{cases}$$

a) *Consider* $k \in \mathbb{N}^*$, *fixed. Determine the nature of the series*

$$\sum_{n=1}^{\infty} \frac{\sin(p_k(n))}{n^\alpha}, \tag{5}$$

where α *is a real parameter.*

b) *For* $n \in \mathbb{N}^*$, *denote* $k_n = \lfloor \lg n \rfloor$ *the integer part of the logarithm in base ten. Determine the nature of the series*

$$\sum_{n=1}^{\infty} \frac{\sin(p_{k_n}(n))}{n^\alpha}, \tag{6}$$

where α *is a real parameter.*

Radu Strugariu, Gheorghe Asachi Technical University of Iaşi

Solution. a) We will prove that

$$\sum_{n=1}^{\infty} \frac{\sin(p_k(n))}{n^\alpha} \begin{cases} \text{converges,} & \text{if } \alpha > 1, \\ \text{diverges,} & \text{if } \alpha \leq 1. \end{cases}$$

Observe that, for $\alpha > 1$,

$$\sum_{n=1}^{\infty} \left| \frac{\sin(p_k(n))}{n^\alpha} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^\alpha},$$

so the series (5) is absolutely convergent, hence convergent.

Consider now $\alpha = 1$. If

$$\overline{c_1 \ldots c_k \underbrace{0 \ldots 0}_{p \text{ times}}} \leq n \leq \overline{c_1 \ldots c_k \underbrace{9 \ldots 9}_{p \text{ times}}}$$

or, equivalently, if

$$\overline{c_1 \ldots c_k} \cdot 10^p \leq n \leq \overline{c_1 \ldots c_k} \cdot 10^p + 10^{p+1} - 1,$$

then we have

$$\sin(p_k(n)) = \sin\left(\overline{c_1 \ldots c_k}\right) \neq 0.$$

Next, if we denote by $(S_n)_n$ the sequence of partial sums, then we get

$$\begin{aligned} &\left| S_{\overline{c_1 \ldots c_k} \cdot 10^p + 10^{p+1} - 1} - S_{\overline{c_1 \ldots c_k} \cdot 10^p} \right| \\ &= \left|\sin\left(\overline{c_1 \ldots c_k}\right)\right| \cdot \left(\frac{1}{\overline{c_1 \ldots c_k} \cdot 10^p + 1} + \cdots + \frac{1}{\overline{c_1 \ldots c_k} \cdot 10^p + 10^{p+1} - 1} \right) \\ &\geq \left|\sin\left(\overline{c_1 \ldots c_k}\right)\right| \cdot \frac{10^{p+1} - 1}{\overline{c_1 \ldots c_k} \cdot 10^p + 10^{p+1} - 1} \\ &\xrightarrow{p \to \infty} \left|\sin\left(\overline{c_1 \ldots c_k}\right)\right| \cdot \frac{10}{\overline{c_1 \ldots c_k} + 10} > 0. \end{aligned}$$

Hence, $(S_n)_n$ is not a Cauchy sequence, so the series (5) diverges.

The case $\alpha \in (0,1)$ can be solved similarly, since in this case the difference $\left|S_{\overline{c_1\ldots c_k}\cdot 10^p+10^{p+1}-1} - S_{\overline{c_1\ldots c_k}\cdot 10^p}\right|$ has the same lower bound as in the previous formula.

For $\alpha = 0$, we observe that the sequence $\big(\sin\big(p_k(n)\big)\big)_n$ has the subsequences $\big(\sin\big(\overline{c_1\ldots c_{k-1}c_k}\big)\big)_n$ and $\big(\sin\big(\overline{c_1\ldots c_{k-1}(c_k+1)}\,\big)\big)_n$, hence its limit does not exist. It follows that the series (5) diverges in this case.

If $\alpha < 0$, since $\sin\big(\overline{c_1\ldots c_{k-1}c_k}\big) \neq 0$, the general term contains a subsequence which converges to $+\infty$ or to $-\infty$, so again, the series (5) diverges.

b) We will prove that

$$\sum_{n=1}^{\infty} \frac{\sin\left(p_{k_n}(n)\right)}{n^\alpha} \begin{cases} \text{converges}, & \text{if } \alpha > 0, \\ \text{diverges}, & \text{if } \alpha \leq 0. \end{cases}$$

Again, if $\alpha > 1$, the series (6) is absolutely convergent, hence convergent.

Observe that if n has m digits, then we have

$$10^{m-1} \leq n < 10^m \Longleftrightarrow m-1 \leq \lg n < m,$$

hence

$$k_n = \lfloor \lg n \rfloor = m-1.$$

Then, if we denote $a_n = \sin\left(p_{k_n}(n)\right)$ and let $(T_n)_n$ be the sequence of partial sums associated to $(a_n)_n$, we have

$$\begin{aligned} T_{10n-1} &= \underbrace{\sin 0 + \cdots + \sin 0}_{9 \text{ times}} + \underbrace{\sin 1 + \cdots + \sin 1}_{10 \text{ times}} + \cdots \\ &\quad + \underbrace{\sin(n-1) + \cdots + \sin(n-1)}_{10 \text{ times}} \\ &= 10\left(\sin 1 + \cdots + \sin(n-1)\right). \end{aligned}$$

It is well-known that the sequence $(\sin 1 + \cdots + \sin(n-1))_n$ is bounded. Since

$$T_{10n+i} = T_{10n-1} + (i+1)\cdot \sin n, \quad i = \overline{0,8},$$

it follows that the sequence $(T_n)_n$ is bounded. Then, we have that the series (6) converges for any $\alpha \in (0,1]$ by the Dirichlet test.

For $\alpha = 0$, observe that the general term of the series (6) satisfies

$$a_{10n} = \sin n,$$

and the sequence $(\sin n)_n$ does not have limit for $n \to \infty$, hence the series (6) diverges.

For $\alpha < 0$, we have

$$a_{10n} = (10n)^{-\alpha} \sin n.$$

Since $\left((10n)^{-\alpha}\sin n\right)_n$ does not have limit for $n \to \infty$ (we can use, for example, the fact that $(\sin n)_n$ densely covers $[-1,1]$), it follows that the series (6) diverges.

Remark. The series (6) has the same nature if one takes, for fixed $r \in \mathbb{N}^*$, $k_n = \max\{\lfloor \lg n \rfloor - r, 0\}$.

3. Section B2

Problem 1. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a matrix with real eigenvalues such that*

$$\operatorname{Tr}\left(A^2\right) = \operatorname{Tr}\left(A^4\right) = \operatorname{Tr}\left(A^6\right).$$

Prove that $\operatorname{Tr}\left(A^{2020}\right) = \operatorname{Tr}\left(A^2\right)$ *and* $\operatorname{Tr}\left(A^{2021}\right) = \operatorname{Tr}\left(A^3\right)$.

Vasile Pop, Technical University of Cluj–Napoca

Solution. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be the eigenvalues of the matrix A. Then $\lambda_1^k, \ldots, \lambda_n^k$ are the eigenvalues of the matrix A^k (for any $k \geq 1$), and in the conditions of the problem we have

$$\sum_{i=1}^n \lambda_i^2 = \sum_{i=1}^n \lambda_i^4 = \sum_{i=1}^n \lambda_i^6.$$

It follows that $\sum\limits_{i=1}^n \left(\lambda_i^2 - 2\lambda_i^4 + \lambda_i^6\right) = 0$, hence $\sum\limits_{i=1}^n \lambda_i^2\left(1 - \lambda_i^2\right)^2 = 0$. It follows that $\lambda_i \in \{-1, 0, 1\}$ for all $i = \overline{1, n}$.

We obtain

$$\operatorname{Tr}\left(A^{2020}\right) = \sum_{i=1}^n \lambda_i^{2020} = \sum_{i=1}^n \lambda_i^2 = \operatorname{Tr}\left(A^2\right),$$

$$\operatorname{Tr}\left(A^{2021}\right) = \sum_{i=1}^n \lambda_i^{2021} = \sum_{i=1}^n \lambda_i^3 = \operatorname{Tr}\left(A^3\right) = \operatorname{Tr}(A).$$

Problem 2. *We consider the function f of class C^2 such that $f'(0) \neq 0$ and we denote by $c_n \in \left[0, \frac{1}{n}\right]$ the point for which*

$$n \int_0^{\frac{1}{n}} f(x)\,\mathrm{d}x = f(c_n).$$

Compute

$$\lim_{n\to\infty} n\left(nc_n - \frac{1}{2}\right).$$

Radu Strugariu, Gheorghe Asachi Technical University of Iaşi

Solution. This problem was also given in Section B1.

Problem 3. *Consider $C \in \mathcal{M}_n(\mathbb{C})$, $C = A + \mathrm{i}B$, $A, B \in \mathcal{M}_n(\mathbb{R})$, with* $\operatorname{rank} C = 1$. *Prove that:*

a) $\operatorname{Tr}(A^2 + B^2) \geq 0$;

b) *if* $\operatorname{Tr}(A^2 + B^2) = 0$, *then* $A^2 + B^2 = O_n$.

Mihai Opincariu, Avram Iancu National College, Brad
Vasile Pop, Technical University of Cluj–Napoca

Solution. a) Since $\operatorname{rank} C = 1$, there exist a column matrix $C_1 + \mathrm{i}C_2$, with $C_1, C_2 \in \mathcal{M}_{n,1}(\mathbb{R})$, and a row matrix $L_1 + \mathrm{i}L_2$, with $L_1, L_2 \in \mathcal{M}_{1,n}(\mathbb{R})$, such that

$$C = (C_1 + \mathrm{i}C_2)\,(L_1 + \mathrm{i}L_2)\,,$$

hence

$$A = C_1L_1 - C_2L_2, \quad B = C_1L_2 + C_2L_1.$$

Then

$$\begin{aligned} A^2 &= C_1\underbrace{(L_1C_1)}_{\alpha_{11}}L_1 + C_2\underbrace{(L_2C_2)}_{\alpha_{22}}L_2 - C_1\underbrace{(L_1C_2)}_{\alpha_{12}}L_2 - C_2\underbrace{(L_2C_1)}_{\alpha_{21}}L_1 \\ &= \alpha_{11}C_1L_1 + \alpha_{22}C_2L_2 - \alpha_{12}C_1L_2 - \alpha_{21}C_2L_1, \end{aligned}$$

$$\begin{aligned} B^2 &= C_1\,(L_2C_1)\,L_2 + C_2\,(L_1C_2)\,L_1 + C_1\,(L_2C_2)\,L_1 + C_2\,(L_1C_1)\,L_2 \\ &= \alpha_{22}C_1L_2 + \alpha_{11}C_2L_2 + \alpha_{21}C_1L_2 + \alpha_{12}C_2L_2, \end{aligned}$$

$$A^2 + B^2 = (\alpha_{11} + \alpha_{22})\,(C_1 \cdot L_1 + C_2 \cdot L_2) + (\alpha_{21} - \alpha_{12})\,(C_1L_2 - C_2L_1)\,.$$

Since $\operatorname{Tr}(C_iL_j) = \operatorname{Tr}(L_jC_i) = \alpha_{ij}$, we get

$$\operatorname{Tr}(A^2 + B^2) = (\alpha_{11} + \alpha_{22})^2 + (\alpha_{21} - \alpha_{12})^2 \geq 0.$$

b) If $\operatorname{Tr}(A^2 + B^2) = 0$, then $\alpha_{11} + \alpha_{22} = \alpha_{21} - \alpha_{12} = 0$, which implies $A^2 + B^2 = O_n$.

Problem 4. *For the integers $n, m \in \mathbb{N}^*$, we denote*

$$p_m(n) = \begin{cases} \textit{the first } m \textit{ digits of } n, & \textit{if } n \textit{ has at least } m \textit{ digits,} \\ 0, & \textit{if } n \textit{ has at most } m-1 \textit{ digits.} \end{cases}$$

a) *Consider $k \in \mathbb{N}^*$, fixed. Determine the nature of the series*

$$\sum_{n=1}^{\infty} \frac{\sin(p_k(n))}{n^\alpha},$$

where α is a real parameter.
b) *For $n \in \mathbb{N}^*$, denote $k_n = [\lg n]$. Determine the nature of the series*

$$\sum_{n=1}^{\infty} \frac{\sin(p_{k_n}(n))}{n^\alpha},$$

where α is a real parameter.

Radu Strugariu, Gheorghe Asachi Technical University of Iaşi

Solution. This problem was also given in Section B1.

MATHEMATICAL NOTES

Two proofs of the basis theorem for Weyl algebra

CONSTANTIN-NICOLAE BELI[1)]

Abstract. The Weyl algebra $A_1(R)$ over a ring R is the universal unital R-algebra generated by x, y, with the relation $[y, x] := yx - xy = 1$. The basis theorem for the Weyl algebra states that $x^a y^b$ are linear independent and so they are a basis for $A_1(R)$. The "standard" proof in the general case uses Bergman's Diamond Lemma. In this note we present two alternative proofs.

Keywords: Weyl algebra, Diamond lemma.
MSC: 16S32

The Weyl algebra $A_1(R)$ over a ring R is the universal unital R-algebra generated by x, y, with the relation $[y, x] := yx - xy = 1$.

By using the derivation formula

$$[t, z_1 \cdots z_m] = \sum_{i=1}^{m} z_1 \cdots z_{i-1} [t, z_i] z_{i+1} \cdots z_m$$

for $t = y$ and $z_1 = \cdots = z_m = x$, one gets

$$yx^a - x^a y = y[y, x^a] = \sum_{i=1}^{a} x^{i-1} [y, x] x^{a-i} = \sum_{i=1}^{a} x^{i-1} x^{a-i} = a x^{a-1},$$

i.e., $yx^a = x^a y + a x^{a-1}$.

This can be used to show that $A_1(R)$ is spanned by the products $x^a y^b$, $a, b \geq 0$. For this we use induction on s to prove that every product $z_1 \cdots z_s$, with $z_i \in \{x, y\}$, is a linear combination of $x^a y^b$ with $a, b \geq 0$. By the induction hypothesis, $z_2 \cdots z_s$ is a linear combination of $x^a y^b$, so $z_1 \cdots z_s$ is a linear combination of $z_1 x^a y^b$. If $z_1 = x$ then $z_1 x^a y^b = x^{a+1} y^b$. If $z_1 = y$ then $z_1 x^a y^b = y x^a y^b = (x^a y + a x^{a-1}) y = x^a y^{b+1} + a x^{a-1} y^b$.

The basis theorem for the Weyl algebra states that $x^a y^b$ are linearly independent and so they are a basis for $A_1(R)$.

If R is field of characteristic zero, this can be done by defining a morphism of algebras from $A_1(R)$ to the algebra of differential operators on $R[X]$, given by $x \mapsto X$ and $y \mapsto \frac{\partial}{\partial X}$. Here by X we mean the multiplication by X, i.e., $P \mapsto XP$, and by $\frac{\partial}{\partial X}$ the derivative, $P \mapsto P'$, for every $P \in R[X]$. For every $P \in R[X]$ we have $\frac{\partial}{\partial X} XP = (XP)' = XP' + P = (X \frac{\partial}{\partial X} + 1)P$, so $\frac{\partial}{\partial X} X = X \frac{\partial}{\partial X} + 1$, i.e., $[\frac{\partial}{\partial X}, X] = 1$. So the map $x \mapsto X$, $y \mapsto \frac{\partial}{\partial X}$ preserves the relation $[y, x] = 1$, and so indeed it defines a morphism of algebras. The image of $x^a y^b$ via this morphism is $X^a \frac{\partial^b}{\partial X^b}$. Since the operators $X^a \frac{\partial^b}{\partial X^b}$, $a, b \geq 0$, are linearly independent, so are $x^a y^b$, $a, b \geq 0$.

[1)] IMAR, Bucureşti, Romania, `raspopitu1@yahoo.com`

In positive characteristic the operators $X^a \frac{\partial^b}{\partial X^b}$ are no longer linearly independent, so the above proof doesn't work. (In characteristic p we have $\frac{\partial^p}{\partial X^p} = 0$.) The "standard" proof in the general case uses Bergman's Diamond Lemma; see [1].

In this note we present two alternative proofs.

First proof.

We start with the relation $yx^a = x^a y + x^{a-1}$ to prove by induction on b the well known formula

$$y^b x^a = \sum_n n! \binom{a}{n} \binom{b}{n} x^{a-n} y^{b-n}.$$

(Here $n \in \mathbb{Z}$ but if $n < 0$ or $n > \min\{a, b\}$ then $\binom{a}{n}\binom{b}{n} = 0$. Hence in fact the sum goes only over $0 \le n \le \min\{a, b\}$.)

For the induction step $b \to b+1$ we write $y^{b+1} x^a = y(y^b x^a)$ as

$$\begin{aligned} \sum_n n! \binom{a}{n} \binom{b}{n} y x^{a-n} y^{b-n} &= \sum_n n! \binom{a}{n} \binom{b}{n} (x^{a-n} y + (a-n) x^{a-n-1}) y^{b-n} \\ &= S_1 + S_2, \end{aligned}$$

where

$$S_1 = \sum_n n! \binom{a}{n} \binom{b}{n} x^{a-n} y^{b-n+1}, \qquad S_2 = \sum_n n! \binom{a}{n} \binom{b}{n} (a-n) x^{a-n-1} y^{b-n}.$$

In S_2 we make the substitution $n = m - 1$ and we get

$$\begin{aligned} S_2 &= \sum_m (m-1)! \binom{a}{m-1} \binom{b}{m-1} (a-m+1) x^{a-m} y^{b-m+1} \\ &= \sum_m m! \binom{a}{m} \binom{b}{m-1} x^{a-m} y^{b-m+1}. \end{aligned}$$

(Here we used the fact that $\binom{a}{m-1}(a-m+1) = m\binom{a}{m}$.)

In conclusion, $y^{b+1} x^a = S_1 + S_2$ writes as

$$\begin{aligned} y^{b+1} x^a &= \sum_n n! \binom{a}{n} \binom{b}{n} x^{a-n} y^{b-n+1} + \sum_n n! \binom{a}{n} \binom{b}{n-1} x^{a-n} y^{b-n+1} \\ &= \sum_n n! \binom{a}{n} \binom{b+1}{n} x^{a-n} y^{b+1-n}. \end{aligned}$$

From this we get a multiplication table for the generators $x^a y^b$:

$$(x^c y^b)(x^a y^d) = x^c (y^b x^a) y^d = \sum_n n! \binom{a}{n} \binom{b}{n} x^{a+c-n} y^{b+d-n}.$$

We now start our proof. On the free R-module $C(R)$ generated by $z_{a,b}$, $a, b \geq 0$, we define an R-algebra structure which is similar with that of $A_1(R)$, i.e.,

$$z_{c,b} z_{a,d} = \sum_{0 \leq n \leq a,b} n! \binom{a}{n} \binom{b}{n} z_{a+c-n,b+d-n}.$$

The unit of this algebra is $z_{0,0}$ so we will write $z_{0,0} = 1$.

Let $f : \{x, y\} \longrightarrow C(R)$ be given by $x \mapsto z_{1,0}$, $y \mapsto z_{0,1}$. We have $z_{1,0} z_{0,1} = z_{1,1}$ and $z_{0,1} z_{1,0} = z_{1,1} + z_{0,0} = z_{1,1} + 1$, so that $[z_{0,1}, z_{1,0}] = z_{0,1} z_{1,0} - z_{1,0} z_{0,1} = 1$. Hence the relation $[y, x] = 1$ defining $A_1(R)$ is preserved in $C(R)$. It follows that f extends to a morphism of algebras $f : A_1(R) \longrightarrow C(R)$.

For every $k, l \geq 0$ we have $z_{k,0} z_{l_0} = z_{k+l,0}$ and $z_{0,k} z_{0,l} = z_{0,k+l}$. Consequently, for $a, b \geq 0$ we have $z_{1,0}^a = z_{a,0}$ and $z_{0,1}^b = z_{0,b}$ and therefore $f(x^a y^b) = z_{1,0}^a z_{0,1}^b = z_{a,0} z_{0,b} = z_{a,b}$. Since $f(x^a y^b) = z_{a,b}$, with $a, b \geq 0$, are a basis of $C(R)$, they are linearly independent, and so are $x^a y^b$ with $a, b \geq 0$. This concludes the proof.

The difficult part is to prove that $C(R)$ defined above is indeed an algebra. We already have that it has a unit $1 = z_{0,0}$. We still need to prove the associativity. We consider three elements $z_{\alpha,d}$, $z_{c,b}$ and $z_{a,\beta}$ of the basis. We have

$$\begin{aligned} z_{\alpha,d}(z_{c,b} z_{a,\beta}) &= z_{\alpha,d} \sum_l l! \binom{a}{l} \binom{b}{l} z_{a+c-l,b+\beta-l} = \sum_l l! \binom{a}{l} \binom{b}{l} z_{\alpha,d} z_{a+c-l,b+\beta-l} \\ &= \sum_l l! \binom{a}{l} \binom{b}{l} \sum_k k! \binom{a+c-l}{k} \binom{d}{k} z_{\alpha+a+c-k-l,b+d+\beta-k-l} \\ &= \sum_n A_n z_{\alpha+a+c-n,b+d+\beta-n}, \end{aligned}$$

where

$$A_n = \sum_{k+l=n} k! l! \binom{a}{l} \binom{b}{l} \binom{d}{k} \binom{a+c-l}{k}.$$

By similar calculations, we get $z_{\alpha,d}(z_{c,b} z_{a,\beta}) = \sum_n B_n z_{\alpha+a+c-n,b+d+\beta-n}$, where

$$B_n = \sum_{k+l=n} k! l! \binom{a}{l} \binom{c}{k} \binom{d}{k} \binom{b+d-k}{l}.$$

So we are left to prove that $A_n = B_n$. But this is the subject of problem 467 from the issue 3–4/2017 of GMA. The proof, together with some comments regarding the connection with the Weyl algebra, appears in the issue 3–4/2018; see [2].

Second proof.

We start by noting that if $z = xy$ then $zx = xyx = x(xy+1) = x(z+1)$. By induction, $z^k x = x(z+1)^k$ for every $k \geq 0$. (For the induction step

$k \to k+1$ we have $z^{k+1}x = zz^k x = zx(z+1)^k = x(z+1)(z+1)^k = x(z+1)^k$.) By linearity, we have $P(z)x = xP(z+1)$ for every $P \in R[X]$.

Then by induction on a we get $P(z)x^a = x^a P(z+a)$ for $a \geq 0$, $P \in R[X]$. (For the induction step $a \to a+1$ we have $P(z)x^{a+1} = P(z)x^a x = x^a P(z+a)x = x^a x P(z+a+1) = x^{a+1}P(z+a+1)$.) This allows us to multiply the products $x^a P(z)$ in a very convenient way:

$$(x^a P(z))(x^b Q(z)) = x^a (P(z)x^b)Q(z) = x^a(x^b P(z+b))Q(z) = x^{a+b}P(z+b)Q(z)$$

for every $a, b \geq 0$ and $P, Q \in R[X]$.

However, the products $x^a P(z)$ do not span all $A_1(R)$, but only a subalgebra. Since $z = xy$, formally we have $y = x^{-1}z$. So we may consider the formal expressions $x^a P(z)$ with $a \in \mathbb{Z}$ and we extend the formula $(x^a P(z))(x^b Q(z)) = x^{a+b}P(z+b)Q(z)$ to the case when $a, b \in \mathbb{Z}$.

To make the matter rigorous, we consider a variable Z and we define $C(R)$ as the free right $R[Z]$-module with the basis X_a, i.e., $C(R) = \bigoplus_{a\in\mathbb{Z}} X_a R[Z]$. For every $a \in \mathbb{Z}$, $P \in R[X]$ the element $X_a P(Z) \in C(R)$ will correspond to $x^a P(z)$. In particular, $X_{-1}Z$ corresponds to $x^{-1}z = y$.

On $C(R)$ we define an algebra structure where the product is given by

$$(X_a P(Z))(X_b Q(Z)) = X_{a+b}P(Z+b)Q(Z), \quad \forall a, b \in \mathbb{Z}, P, Q \in R[X].$$

For the associativity we take $a, b, c \in \mathbb{Z}$ and $P, Q, S \in R[X]$ and we note that

$$\begin{aligned}\Big((X_a P(Z))(X_b Q(Z))\Big)(X_c S(Z)) &= (X_a P(Z))\Big((X_b Q(Z))(X_c S(Z))\Big) \\ &= X_{a+b+c}P(Z+b+c)Q(z+c)S(Z).\end{aligned}$$

Also X_0 is an obvious unit for $C(R)$, so we write $X_0 = 1$.

We define $f : \{x, y\} \longrightarrow C(R)$ by $x \mapsto X_1$ and $y \mapsto X_{-1}Z$. We have $X_1(X_{-1}Z) = X_0 Z$ and $(X_{-1}Z)X_1 = X_0(Z+1) = X_0 Z + X_0$, so $[(X_{-1}Z), X_0] = (X_{-1}Z)X_0 - X_0(X_{-1}Z) = X_0 = 1$. Hence f preserves the relation $[y, x] = 1$ defining $A_1(R)$, so it extends to a morphism of R-algebras $f : A_1(R) \longrightarrow C(R)$.

Suppose now that $x^a y^b$, with $a, b \geq 0$, are not linearly independent. Then we have a linear combination $\sum_{a,b\geq 0} \alpha_{a,b} x^a y^b = 0$, where almost all, but not all $\alpha_{a,b}$ are zero. By applying f, we get $\sum_{a,b\geq 0} \alpha_{a,b} f(x^a y^b) = 0$.

Since $X_0 = 1$ and $X_k X_l = X_{k+l}$, we have $X_1^a = X_a$ $\forall a \geq 0$. For $b \geq 0$ we claim that $(X_{-1}Z)^b = X_{-b}Q_b(Z)$, where $Q_b \in R[X]$, $Q_0 = 1$ and $Q_b = X(X-1)\cdots(X-b+1)$. For $b = 0$ we have $X_0 Q_0(Z) = X_0 = 1 = (X_{-1}Z)^0$. For the induction step $b \to b+1$ we have

$$\begin{aligned}(X_{-1}Z)^{b+1} &= (X_{-1}Z)(X_{-1}Z)^b = (X_{-1}Z)(X_{-b}Q_b(Z)) = X_{-b-1}(X-b)Q_b(Z) \\ &= X_{-b-1}Q_{b+1}(Z).\end{aligned}$$

Consequently $f(x^a y^b) = X_1^a (X_{-1}Z)^b = X_a X_{-b} Q_b(Z) = X_{a-b} Q_b(Z)$. It follows that

$$0 = \sum_{a,b \geq 0} \alpha_{a,b} X_{a-b} Q_b(Z) = \sum_{n \in \mathbb{Z}} X_n P_n(Z),$$

where $P_n = \sum_{a-b=n} \alpha_{a,b} Q_b$. Since X_n, $n \in \mathbb{Z}$, are a basis of $C(R)$ as an $R[Z]$-module, we have $P_n(Z) = 0$, so $P_n = 0 \ \forall n \in \mathbb{Z}$.

Let a_0, $b_0 \geq 0$ with $\alpha_{a_0,b_0} \neq 0$ be such that b_0 is maximal and let $n_0 = a_0 - b_0$. Then $P_{n_0} = \sum_{a-b=n_0} \alpha_{a,b} Q_b$ contains the term $\alpha_{a_0,b_0} P_{b_0}$, which is a polynomial of degree b_0 with the leading coefficient $\alpha_{a_0,b_0} \neq 0$. For every other term $\alpha_{a,b} Q_b$ of P_{n_0} with $\alpha_{a,b} \neq 0$ we have $a - b = a_0 - b_0 = n_0$ and $(a, b) \neq (a_0, b_0)$, which implies $b \neq b_0$. But since $\alpha_{a,b} \neq 0$, by the maximality of b_0, we have $b \leq b_0$. Hence $\deg \alpha_{a,b} Q_b = b < b_0$. In conclusion. P_{n_0} is a non-zero polynomial of degree b_0. Contradiction. Hence x^a, y^b with $a, b \geq 0$ are linearly independent.

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