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A new characterization of injective and surjective functions and group homomorphisms

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Abstract. A model of a morphism f between two objects is defined to be a factorization $f = \pi i$, where π is a surjective morphism and i is an injective morphism. In this note we shall prove that a morphism f is surjective (respectively injective) if and only if it has an initial (respectively final) model in some classes of objects and morphisms.

Keywords: Factorization of morphisms, injective/surjective maps, initial/final objects.

MSC: Primary 03E20; Secondary 20A99.

1. Introduction and preliminary remarks

In what follows we denote by $\mathcal{C}$ either

(i) the class of sets with morphisms between them being maps, or

(ii) the class of groups with morphisms between them being group homomorphisms.

The purpose of this paper is to study the link between injectivity and surjectivity of morphisms in $\mathcal{C}$ and the notion of a *model* defined below.

Definition 1. *For a morphism $f\colon A \to B$ in $\mathcal{C}$ we define a* model *of f to be a triple (X, i, π) such that $X \in \mathcal{C}$, $i\colon A \to X$ is an injective morphism, $\pi\colon X \to B$ is a surjective morphism, and $f = \pi i$, i.e., the following diagram commutes*

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$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\ & X \arrow[ur, "\pi"'] & \end{tikzcd}$$

A model (X, i, π) *of a morphism* f *is called an* initial model *if for any other model* (Y, j, p) *of* f *there is a unique morphism* $g\colon X \to Y$ *such that* $gi = j$ *and* $pg = \pi$.

A model (X, i, π) *of a morphism* f *is called a* final model *if for any other model* (Y, j, p) *of* f *there is a unique morphism* $g\colon Y \to X$ *such that* $gj = i$ *and* $\pi g = p$.

We now introduce a couple of definitions necessary for the proof.

Definition 2. *A* product *of* A *and* B *in* $\mathcal{C}$ *is a triple* $(A \times B, \pi_A, \pi_B)$*, where* $A \times B$ *is an object in* $\mathcal{C}$ *and* $\pi_A\colon A \times B \to A$, $\pi_B\colon A \times B \to B$ *are morphisms such that the following universal property is satisfied: for any object* H *in* $\mathcal{C}$ *and any morphisms* $f\colon H \to A$, $g\colon H \to B$, *there exists a unique morphism* $\varphi\colon H \to A \times B$ *such that* $\pi_A \varphi = f$ *and* $\pi_B \varphi = g$.

In both cases studied in this paper (sets and groups) the morphisms $\pi_A\colon A \times B \to A$ and $\pi_B\colon A \times B \to B$ are surjective and it is easy to see that there is at least one product for any two objects in $\mathcal{C}$: in (i) a product is the cartesian product of sets along with the canonical projections and in (ii) a product is the direct product of groups along with the canonical projections [1], p. 41.

Definition 3. *A* coproduct *of* A *and* B *in* $\mathcal{C}$ *is a triple* $(A \amalg B, i_A, i_B)$*, where* $A \amalg B$ *is an object in* $\mathcal{C}$ *and* $i_A\colon A \to A \amalg B$, $i_B\colon B \to A \amalg B$ *are morphisms such that the following universal property is satisfied: for any object* H *in* $\mathcal{C}$ *and any morphisms* $f\colon A \to H$, $g\colon B \to H$, *there exists a unique morphism* $\varphi\colon A \amalg B \to H$ *such that* $\varphi i_A = f$ *and* $\varphi i_B = g$.

In both cases studied in this paper (sets and groups) the morphisms $i_A\colon A \to A \amalg B$ and $i_B\colon B \to A \amalg B$ are injective and it is easy to see that there is at least one coproduct for any two objects in $\mathcal{C}$: in (i) a coproduct is the disjoint union along with the canonical injections and in (ii) a coproduct is the free product of groups along with the canonical injections [1, p. 59–60].

Here are a couple of remarks before we dive into the main result of this paper.

Remark 4. There is at least one model of $f\colon A \to B$, for any morphism f in $\mathcal{C}$. For example, apply the universal property of the coproduct to $(A \amalg B, i_A, i_B)$ and the object B with morphisms $f\colon A \to B$, $\mathrm{id}_B\colon B \to B$. We obtain a morphism $\varphi\colon A \amalg B \to B$ which satisfies the relations $f = \varphi i_A$ and

$\mathrm{id}_B = \varphi i_B$. From the second relation it is obvious that φ is surjective hence $(A \amalg B, i_A, \varphi)$ is a model for f.

Remark 5. In $\mathcal{C}$, a morphism is injective if and only if it is a monomorphism and it is surjective if and only if it is an epimorphism. For definitions and proofs of these results see [2], p. 24–25 and 27–30.

2. Initial and final models of a morphism in $\mathcal{C}$

The next theorem gives a new characterization of injective and surjective morphisms in $\mathcal{C}$.

Theorem 6. *Let $f\colon A \to B$ be a morphism in $\mathcal{C}$. Then:*

a) there is an initial model of f if and only if f is surjective;
b) there is a final model of f if and only if f is injective.

Proof. *a)* " $\Longleftarrow$ " Let f be a surjective morphism. In this case, (A, id_A, f) is obviously a model for f and we will now show that it is an initial model. Let (Y, j, p) be another model. We want there to be a unique morphism $g\colon A \to Y$ such that $g\,\mathrm{id}_A = j$ and $pg = f$. The existence is obvious since $g = j$ satisfies the required relations. The uniqueness is also given by the fact that $g\,\mathrm{id}_A = j$, since the unique morphism that satisfies this is $g = j$.

" $\Longrightarrow$ " Let $f\colon A \to B$ be a morphism in $\mathcal{C}$ such that there is an initial model of f, which we denote (X, i, π). Suppose f is not surjective, i.e., not an epimorphism. Then there is an object U in $\mathcal{C}$ and two morphisms $u_1, u_2\colon B \to U$ such that $u_1 f = u_2 f$ and $u_1 \neq u_2$. In this case, $u_1\pi \neq u_2\pi$ since π is an epimorphism.

Consider now a product of X and U, $(X \times U, \pi_X, \pi_U)$. We apply the universal property of the product to $(X \times U, \pi_X, \pi_U)$ and the morphisms $u_1 f\colon A \to U$ and $i\colon A \to X$. Thus, there is a unique morphism $q\colon A \to X \times U$ such that $\pi_X q = i$ and $\pi_U q = u_1 f$. Since $\pi_X q = i$ is injective, so is q, thus q is a monomorphism.

We now observe that $(X \times U, q, \pi\pi_X)$ is a model of f.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "i"'] \arrow[drrr, dashed, "q"] & & B \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_2"'] & U \\
& X \arrow[ur, "\pi"] & & X \times U \arrow[u, "\pi_U"'] \arrow[ll, "\pi_X"]
\end{tikzcd}$$

Next we apply the universal property of the product to $(X \times U, \pi_X, \pi_U)$ and the morphisms $u_1\pi\colon X \to U$ and $\mathrm{id}_X\colon X \to X$. This implies that there is a unique morphism $g_1\colon X \to X \times U$ such that $\pi_U g_1 = u_1\pi$ and $\pi_X g_1 = \mathrm{id}_X$. Similarly, we obtain a morphism $g_2\colon X \to X \times U$ such that $\pi_U g_2 = u_2\pi$ and $\pi_X g_2 = \mathrm{id}_X$. From these relations, it is obvious that $g_1 \neq g_2$ since $u_1\pi \neq u_2\pi$.

Finally, we prove that g_1 and g_2 are morphisms from the initial model (X, i, π) to the model $(X \times U, q, \pi\pi_X)$ and this is in contradiction with the assumption that (X, i, π) is initial, i.e.,

$$g_1 i = q, \quad \pi\pi_X g_1 = \pi, \quad g_2 i = q, \quad \pi\pi_X g_2 = \pi.$$

Indeed, $\pi_X g_1 = \mathrm{id}_X$ implies that $\pi\pi_X g_1 = \pi$ and similarly for g_2. Furthermore, we have the following relations which imply that $g_1 i = q$ based on the uniqueness of the morphism in the universal property of the product:

$$i = \mathrm{id}_X i = \pi_X g_1 i \quad \text{and} \quad i = \pi_X q,$$

$$u_1 f = u_1 \pi i = \pi_U g_1 i \quad \text{and} \quad u_1 f = \pi_U q.$$

In a similar manner it can be proved that $g_2 i = q$ and thus we reach a contradiction.

b) "$\Longleftarrow$" Let f be an injective morphism. In this case, (B, f, id_B) is obviously a model for f and we will now show that it is a final model. Let (Y, j, p) be another model. We want there to be a unique morphism $g \colon Y \to B$ such that $gj = f$ and $\mathrm{id}_B g = p$. The existence is obvious since $g = p$ satisfies the relations. The uniqueness is also given by the fact that $\mathrm{id}_B g = p$, since the unique morphism that satisfies this is $g = p$.

"$\Longrightarrow$" Let $f \colon A \to B$ be a morphism in $\mathcal{C}$ such that there is a final model of f, which we denote (X, i, π). Suppose f is not injective, i.e., not a monomorphism. Then there is an object U in $\mathcal{C}$ and two morphisms $u_1, u_2 \colon U \to A$ such that $fu_1 = fu_2$ and $u_1 \neq u_2$. In this case, $iu_1 \neq iu_2$ since i is a monomorphism.

Consider now a coproduct of X and U, $(X \amalg U, i_X, i_U)$. We apply the universal property of the coproduct to $(X \amalg U, i_X, i_U)$ and the morphisms $fu_1 \colon U \to B$ and $\pi \colon X \to B$. Thus, there is a unique morphism $q \colon X \amalg U \to B$ such that $qi_X = \pi$ and $qi_U = fu_1$. Since $qi_X = \pi$ is surjective, so is q, thus q is an epimorphism.

We now observe that $(X \amalg U, i_X i, q)$ is a model of f.

$$\begin{tikzcd}
U \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_2"'] \arrow[d, "i_U"'] & A \arrow[rr, "f"] \arrow[dr, "i"] & & B \\
X \amalg U \arrow[urrr, dash dot, "q"] & & X \arrow[ll, "i_X"] \arrow[ur, "\pi"'] &
\end{tikzcd}$$

Next we apply the universal property of the coproduct to $(X \amalg U, i_X, i_U)$ and the morphisms $iu_1 \colon U \to X$ and $\mathrm{id}_X \colon X \to X$. This implies that there is a unique morphism $g_1 \colon X \amalg U \to X$ such that $g_1 i_U = iu_1$ and $g_1 i_X = \mathrm{id}_X$. Similarly, we obtain a morphism $g_2 \colon X \amalg U \to X$ such that $g_2 i_U = iu_2$ and $g_2 i_X = \mathrm{id}_X$. From these relations, it is obvious that $g_1 \neq g_2$ since $iu_1 \neq iu_2$.

Finally, we prove that g_1 and g_2 are morphisms from the model $(X \amalg U, i_X i, q)$ to the final model (X, i, π) and this is in contradiction with the assumption that (X, i, π) is final, i.e.,

$$\pi g_1 = q, \quad g_1 i_X i = i, \quad \pi g_2 = q, \quad g_2 i_X i = i.$$

Indeed, $g_1 i_X = \mathrm{id}_X$ implies that $g_1 i_X i = i$ and similarly for g_2. Furthermore, we have the following relations which imply that $\pi g_1 = q$ based on the uniqueness of the morphism in the universal property of the coproduct:

$$\pi = \pi \mathrm{id}_X = \pi g_1 i_X \quad \text{and} \quad \pi = q i_X,$$
$$f u_1 = \pi i u_1 = \pi g_1 i_U \quad \text{and} \quad f u_1 = q i_U.$$

In a similar manner it can be proved that $\pi g_2 = q$ and thus we reach a contradiction. □

3. Final remarks

The reader may have noticed that in the proof given above the fact that $\mathcal{C}$ is either the class of sets or the class of groups is only used for the facts that products (coproducts) exist for any two objects in $\mathcal{C}$, injective (surjective) morphisms are monomorphisms (epimorphisms) and the two morphisms in the product (coproduct) are epimorphisms (monomorphisms). Indeed, the result above holds in a more general setting if we replace injective (surjective) morphisms in the theorem's statement and in the definition of a model with monomorphisms (epimorphisms), namely when $\mathcal{C}$ is a category with products (coproducts) for any two objects and in which the morphisms in any product (coproduct) are epimorphisms (monomorphisms).

Alternatively, either one of *a*) and *b*) can be obtained from the other by considering the dual category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$. If $f\colon A \to B$ is a morphism in $\mathcal{C}$ and (X, i, π) is a model of f in $\mathcal{C}$, then (X, π, i) is a model of f in $\mathcal{C}^{\mathrm{op}}$. Moreover, (X, i, π) is an initial model of f in $\mathcal{C}$ if and only if (X, π, i) is a final model of f in $\mathcal{C}^{\mathrm{op}}$. Using the fact that f is a monomorphism in $\mathcal{C}$ if and only if it is an epimorphism in $\mathcal{C}^{\mathrm{op}}$, we obtain the equivalence of *a*) and *b*).

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References

[1] S. Awodey, *Category Theory*, Oxford University Press, 2010.
[2] F. Borceaux, *Handbook of Categorical Algebra 1*, Cambridge University Press, 1994.
[3] G. Militaru, *Note de curs, Algebra* (Romanian) available at https://gigelmilitaru.wordpress.com/2020/08/28/algebra-anul-i-semestru-1-2020-2021

Two new proofs of Sandham–Yeung series

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The paper is dedicated to the 125th appearance of Gazeta Matematică.
La mulţi ani, Gazeta Matematică!!!

Abstract. In this paper we give two new proofs of the remarkable equality

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\frac{17}{4}\zeta(4),$$

where $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$ denotes the nth harmonic number. The first proof is based on evaluating the series $\sum\limits_{n=1}^{\infty}H_n^2\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}\right)$ by two different methods and the second proof follows from calculating, by a new method, the series $\sum\limits_{n=1}^{\infty}\frac{H_n^2}{(n+1)^2}$.

Keywords: Hölder continuous function, Lipschitz function.
MSC: Primary 40A05; Secondary 40C10.

1. Introduction and the main results

In this paper we give two new proofs of the remarkable formula

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\frac{17}{4}\zeta(4), \tag{1}$$

where $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$ denotes the nth harmonic number.

Formula (1) has an interesting history. It was the first quadratic series introduced in the literature by H. F. Sandham in 1948 as a problem in the American Mathematical Monthly [14]. Apparently, the series went unnoticed. Castellanos recorded it in his survey article [3, p. 86], attributed it rightly to Sandham, but with a wrong entry in the bibliography. De Doelder [5] evaluated the associated series $\sum\limits_{n=1}^{\infty}\frac{H_n^2}{(n+1)^2}=\frac{11\pi^4}{360}$ without any reference to Sandham's series. In April 1993 the series was discovered numerically by Enrico Au-Yeung, an undergraduate student in the Faculty of Mathematics in Waterloo, and proved rigorously by David Borwein and Jonathan Borwein in [2], who used Parseval's theorem to prove it. Formula (1) was rediscovered by Freitas as Proposition A.1 in the appendix section of [6]. Freitas proved

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it by calculating a double integral involving a logarithmic function. This formula is revived and brought into light by Vălean and Furdui [11], who proved it by calculating a special integral involving a quadratic logarithmic function. The series also appears as a problem in [7, problem 3.70, p. 150] and [13, problem 2.6.1. p. 110]. In [8] Furdui and Sîntămărian proved formula (1) as a consequence of evaluating the series $\sum\limits_{n=1}^{\infty} \frac{1}{n}\left(2\zeta(3) - \frac{H_1}{1^2} - \frac{H_2}{2^2} - \cdots - \frac{H_n}{n^2}\right)$ by two different ways. It is clear that this remarkable quadratic series has attracted lots of attention lately and has become a classic in the theory of nonlinear harmonic series.

The first proof of formula (1) is based on calculating the series

$$\sum_{n=1}^{\infty} H_n^2 \left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right) = 3\zeta(4) - 4\zeta(3) + 2\zeta(2)$$

in two different ways. The second proof follows from the series $\sum\limits_{n=1}^{\infty} \frac{H_n^2}{(n+1)^2} = \frac{11}{4}\zeta(4)$, which is calculated differently than it is in [5]. We record the results we prove in the next theorem.

Theorem 1. *(a)* **A harmonic series.**

The following identity holds

$$\sum_{n=1}^{\infty} H_n^2 \left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right) = 3\zeta(4) - 4\zeta(3) + 2\zeta(2).$$

(b) **The Sandham–Yeung series.**

The following identity holds

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 = \frac{17}{4}\zeta(4).$$

We collect some results we need in the proof of Theorem 1.

The Dilogarithm function $\mathrm{Li}_2(z)$ is defined, for $|z| \le 1$, by ([4, p. 176])

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\ln(1-t)}{t}\mathrm{d}t.$$

The generating function of the nth harmonic number is given by the formula (see [15, problem 3.54, (a)])

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x}, \quad x \in (-1, 1). \tag{2}$$

The previous formula can be proved by multiplying the power series of the functions $f(x) = \ln(1-x)$ and $g(x) = \frac{1}{1-x}$.

Integrating out the previous equality we have that the following power series formula holds true [15, problem 3.54, (c)]

$$\ln^2(1-x) = 2\sum_{n=1}^{\infty} \frac{H_n}{n+1} x^{n+1}, \quad x \in [-1,1). \tag{3}$$

The generating function of the sequence $\left(H_n^2\right)_{n\geq 1}$ is given by ([10, Theorem 9, p. 215])

$$\sum_{n=1}^{\infty} H_n^2 x^n = \frac{\ln^2(1-x) + \mathrm{Li}_2(x)}{1-x}, \quad x \in (-1,1). \tag{4}$$

Lemma 2. *The following equalities hold*:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$;

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{H_n}{n(n+1)} = \zeta(2)$;

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{\pi^4}{72} = \frac{5}{4}\zeta(4)$.

Proof. (a) For the proof of part (a) see [7, problem 3.55, p. 148]. For the sake of completeness we include below a new proof of this result.

First we show that if $f : [0,1] \to \mathbb{R}$ is a continuous function, then

$$\int_0^1 \int_0^1 f(xy)\mathrm{d}x\mathrm{d}y = -\int_0^1 f(x) \ln x \, \mathrm{d}x. \tag{5}$$

We have

$$\begin{aligned}\int_0^1 \int_0^1 f(xy)\mathrm{d}x\mathrm{d}y &= \int_0^1 \left(\int_0^1 f(xy)\mathrm{d}x\right) \mathrm{d}y \overset{xy=t}{=} \int_0^1 \left(\frac{1}{y}\int_0^y f(t)\mathrm{d}t\right) \mathrm{d}y \\ &= \ln y \int_0^y f(t)\mathrm{d}t \,\Big|_0^1 - \int_0^1 f(y) \ln y \, \mathrm{d}y = -\int_0^1 f(y) \ln y \, \mathrm{d}y,\end{aligned}$$

since $\lim\limits_{y\to 0} \ln y \int_0^y f(t)\mathrm{d}t = 0$.

Formula (5) is also valid in the case when f is a Riemann integrable function [9].

We have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2} &= \sum_{n=1}^{\infty}H_n\int_0^1\int_0^1(xy)^n\mathrm{d}x\mathrm{d}y = \int_0^1\int_0^1\left(\sum_{n=1}^{\infty}H_n(xy)^n\right)\mathrm{d}x\mathrm{d}y\\ &\overset{(2)}{=} -\int_0^1\int_0^1\frac{\ln(1-xy)}{1-xy}\mathrm{d}x\mathrm{d}y \overset{(5)}{=} \int_0^1\frac{\ln(1-x)\ln x}{1-x}\mathrm{d}x\\ &= \int_0^1\frac{\ln x\ln(1-x)}{x}\mathrm{d}x = -\int_0^1\frac{\ln x}{x}\sum_{n=1}^{\infty}\frac{x^n}{n}\mathrm{d}x\\ &= -\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^{n-1}\ln x\,\mathrm{d}x = \sum_{n=1}^{\infty}\frac{1}{n^3}\\ &= \zeta(3).\end{aligned}$$

It follows that

$$\zeta(3) = \sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2} = \sum_{n=1}^{\infty}\frac{H_{n+1}-\frac{1}{n+1}}{(n+1)^2} = \sum_{m=2}^{\infty}\frac{H_m-\frac{1}{m}}{m^2} = \sum_{m=1}^{\infty}\frac{H_m}{m^2} - \zeta(3),$$

and part (a) of Lemma 2 is proved.

(b) We have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n(n+1)} &= \sum_{n=1}^{\infty}\left(\frac{H_n}{n}-\frac{H_n}{n+1}\right) = \sum_{n=1}^{\infty}\left(\frac{H_n}{n}-\frac{H_{n+1}}{n+1}+\frac{1}{(n+1)^2}\right)\\ &= 1+\zeta(2)-1 = \zeta(2).\end{aligned}$$

(c) A proof of the formula $\sum\limits_{n=1}^{\infty}\frac{H_n}{n^3} = \frac{\pi^4}{72}$, which is a special linear Euler sum, is given in [7, problem 3.58, pp. 207–208] and it also appears in the literature as a problem proposed by M.S. Klamkin [12]. Another proof of this formula is given in [15, p. 247] and a proof based on symmetry appears in [8, pp. 4–5]. $\square$

Remark 3. *One can prove, by using the same technique as in part* (a) *of Lemma 2 that the following formula holds*

$$\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{(n+1)^2} = -\frac{\zeta(3)}{8},$$

from which we recover the known result $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^2} = \frac{5}{8}\zeta(3)$ *(*[7, problem 3.56, p. 148]*).*

Now we are ready to prove Theorem 1.

Proof. (a) One can prove, using integration by parts twice, that the following formula holds true $\int_0^1 x^k \ln^2 x \,\mathrm{d}x = \frac{2}{(k+1)^3}$, $k > -1$. It follows that

$$\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3} = \frac{1}{2}\int_0^1 \frac{x^n}{1-x}\ln^2 x \,\mathrm{d}x. \tag{6}$$

We have, based on formula (6), that

$$\begin{aligned}\sum_{n=1}^{\infty} H_n^2\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right) &= \frac{1}{2}\sum_{n=1}^{\infty} H_n^2 \int_0^1 \frac{x^n}{1-x}\ln^2 x \,\mathrm{d}x\\ &= \frac{1}{2}\int_0^1 \frac{\ln^2 x}{1-x}\left(\sum_{n=1}^{\infty} H_n^2 x^n\right)\mathrm{d}x\\ &\overset{(3)}{=} \frac{1}{2}\int_0^1 \frac{\ln^2 x}{1-x}\cdot\frac{\ln^2(1-x) + \mathrm{Li}_2(x)}{1-x}\mathrm{d}x\\ &= \frac{1}{2}\int_0^1 \frac{\ln^2 x \ln^2(1-x)}{(1-x)^2}\mathrm{d}x\\ &\qquad + \frac{1}{2}\int_0^1 \frac{\ln^2 x \,\mathrm{Li}_2(x)}{(1-x)^2}\mathrm{d}x\\ &= \frac{1}{2}I + \frac{1}{2}J,\end{aligned} \tag{7}$$

where

$$I = \int_0^1 \frac{\ln^2 x \ln^2(1-x)}{(1-x)^2}\mathrm{d}x \quad \text{and} \quad J = \int_0^1 \frac{\ln^2 x \,\mathrm{Li}_2(x)}{(1-x)^2}\mathrm{d}x.$$

We calculate the first integral and we have

$$\begin{aligned}I &= \int_0^1 \frac{\ln^2 x \ln^2(1-x)}{(1-x)^2}\mathrm{d}x \overset{1-x=y}{=} \int_0^1 \frac{\ln^2 y \ln^2(1-y)}{y^2}\mathrm{d}y\\ &\overset{(3)}{=} 2\int_0^1 \frac{\ln^2 y}{y^2}\left(\sum_{n=1}^{\infty}\frac{H_n}{n+1}y^{n+1}\right)\mathrm{d}y = 2\sum_{n=1}^{\infty}\frac{H_n}{n+1}\int_0^1 y^{n-1}\ln^2 y\,\mathrm{d}y\\ &= 4\sum_{n=1}^{\infty}\frac{H_n}{n^3(n+1)} = 4\sum_{n=1}^{\infty}\left(\frac{H_n}{n^3} - \frac{H_n}{n^2} + \frac{H_n}{n(n+1)}\right)\\ &\overset{\text{Lemma 2}}{=} 4\left(\frac{5}{4}\zeta(4) - 2\zeta(3) + \zeta(2)\right).\end{aligned} \tag{8}$$

We calculate the second integral and we have

$$\begin{aligned}
J &= \int_0^1 \frac{\ln^2 x \operatorname{Li}_2(x)}{(1-x)^2} \mathrm{d}x = \frac{\ln^2 x \operatorname{Li}_2(x)}{1-x} \bigg|_0^1 \\
&\quad - \int_0^1 \left(\frac{2 \ln x \operatorname{Li}_2(x)}{x} - \frac{\ln^2 x \ln(1-x)}{x} \right) \frac{1}{1-x} \mathrm{d}x \\
&= -2 \int_0^1 \frac{\ln x \operatorname{Li}_2(x)}{x} \mathrm{d}x - 2 \int_0^1 \frac{\ln x \operatorname{Li}_2(x)}{1-x} \mathrm{d}x \\
&\quad + \int_0^1 \frac{\ln^2 x \ln(1-x)}{x} \mathrm{d}x + \int_0^1 \frac{\ln^2 x \ln(1-x)}{1-x} \mathrm{d}x \\
&= -2A - 2B + C + D,
\end{aligned} \tag{9}$$

where

$$A = \int_0^1 \frac{\ln x \operatorname{Li}_2(x)}{x} \mathrm{d}x, \quad B = \int_0^1 \frac{\ln x \operatorname{Li}_2(x)}{1-x} \mathrm{d}x,$$

and

$$C = \int_0^1 \frac{\ln^2 x \ln(1-x)}{x} \mathrm{d}x, \quad D = \int_0^1 \frac{\ln^2 x \ln(1-x)}{1-x} \mathrm{d}x.$$

We calculate the integrals A, C, D and B, in this order. We have,

$$\begin{aligned}
A &= \int_0^1 \frac{\ln x \operatorname{Li}_2(x)}{x} \mathrm{d}x = \int_0^1 \frac{\ln x}{x} \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2} \right) \mathrm{d}x \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 x^{n-1} \ln x \, \mathrm{d}x = -\sum_{n=1}^{\infty} \frac{1}{n^4} = -\zeta(4).
\end{aligned} \tag{10}$$

On the other hand,

$$\begin{aligned}
C &= \int_0^1 \frac{\ln^2 x \ln(1-x)}{x} \mathrm{d}x = -\int_0^1 \frac{\ln^2 x}{x} \left(\sum_{n=1}^{\infty} \frac{x^n}{n} \right) \mathrm{d}x \\
&= -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{n-1} \ln^2 x \, \mathrm{d}x = -\sum_{n=1}^{\infty} \frac{2}{n^4} = -2\zeta(4).
\end{aligned} \tag{11}$$

We calculate the integral D and we have

$$\begin{aligned}
D &= \int_0^1 \frac{\ln^2 x \ln(1-x)}{1-x} \mathrm{d}x \overset{(2)}{=} -\int_0^1 \ln^2 x \left(\sum_{n=1}^{\infty} H_n x^n\right) \mathrm{d}x \\
&= -\sum_{n=1}^{\infty} H_n \int_0^1 x^n \ln^2 x \,\mathrm{d}x = -2\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^3} \\
&= -2\sum_{n=1}^{\infty} \frac{H_{n+1} - \frac{1}{n+1}}{(n+1)^3} = -2\sum_{n=1}^{\infty} \left(\frac{H_n}{n^3} - \frac{1}{n^4}\right) \\
&\overset{\text{Lemma 2, (c)}}{=} -\frac{1}{2}\zeta(4).
\end{aligned} \tag{12}$$

We calculate the integral B, using integration by parts, and we have

$$\begin{aligned}
B &= \int_0^1 \frac{\ln x \operatorname{Li}_2(x)}{1-x} \mathrm{d}x = -\ln x \ln(1-x) \operatorname{Li}_2(x)\Big|_0^1 \\
&\quad + \int_0^1 \left(\frac{\operatorname{Li}_2(x)}{x} - \frac{\ln x\, \ln(1-x)}{x}\right) \ln(1-x) \mathrm{d}x \\
&= \int_0^1 \frac{\ln(1-x) \operatorname{Li}_2(x)}{x} \mathrm{d}x - \int_0^1 \frac{\ln^2(1-x) \ln x}{x} \mathrm{d}x \\
&= -\frac{\operatorname{Li}_2^2(x)}{2}\Bigg|_0^1 - \int_0^1 \frac{\ln^2 x \ln(1-x)}{1-x} \mathrm{d}x \\
&= -\frac{\operatorname{Li}_2^2(1)}{2} - D \\
&= -\frac{3}{4}\zeta(4),
\end{aligned} \tag{13}$$

since $\operatorname{Li}_2(1) = \frac{\pi^2}{6}$.

A calculation, based on (9), (10), (11), (12) and (13), readily shows that $J = \zeta(4)$ and this implies together with (7) and (8) that

$$\sum_{n=1}^{\infty} H_n^2 \left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right) = 3\zeta(4) - 4\zeta(3) + 2\zeta(2).$$

(b) **The first proof.** The following equality can be proved by mathematical induction

$$\begin{aligned}
\sum_{k=1}^{n} H_k^2 &= (n+1)H_n^2 - (2n+1)H_n + 2n \\
&= (n+1)H_{n+1}^2 - H_{n+1} - 2(n+1)H_{n+1} + 2(n+1), \quad n \geq 1.
\end{aligned} \tag{14}$$

We also need *Abel's summation formula* ([1, p. 55], [7, Lemma A.1, p. 258]) which states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences of real numbers and $A_n = \sum\limits_{k=1}^{n} a_k$, then $\sum\limits_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum\limits_{k=1}^{n} A_k(b_k - b_{k+1})$. We will be using the *infinite version* of this formula

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} (A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k(b_k - b_{k+1}). \tag{15}$$

We calculate the series in part (a) of Theorem 1 by using formula (15), with

$$a_n = H_n^2 \quad \text{and} \quad b_n = \zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}$$

and we have, since $b_n - b_{n+1} = \frac{1}{(n+1)^3}$, that

$$\begin{aligned}
&\sum_{n=1}^{\infty} H_n^2 \left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right) \\
&\overset{(14)}{=} \lim_{n\to\infty} \left[(n+1)H_n^2 - (2n+1)H_n + 2n\right] \left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{(n+1)^3}\right) \\
&\quad + \sum_{n=1}^{\infty} \frac{(n+1)H_{n+1}^2 - H_{n+1} - 2(n+1)H_{n+1} + 2(n+1)}{(n+1)^3} \\
&= \sum_{n=1}^{\infty} \left[\frac{H_{n+1}^2}{(n+1)^2} - \frac{H_{n+1}}{(n+1)^3} - 2\frac{H_{n+1}}{(n+1)^2} + \frac{2}{(n+1)^2}\right] \\
&\overset{n+1=m}{=} \sum_{m=2}^{\infty} \left[\frac{H_m^2}{m^2} - \frac{H_m}{m^3} - 2\frac{H_m}{m^2} + \frac{2}{m^2}\right] \\
&= \sum_{m=1}^{\infty} \left[\frac{H_m^2}{m^2} - \frac{H_m}{m^3} - 2\frac{H_m}{m^2} + \frac{2}{m^2}\right] \\
&\overset{\text{Lemma 2, (a),(c)}}{=} \sum_{m=1}^{\infty} \left(\frac{H_m}{m}\right)^2 - \frac{5}{4}\zeta(4) - 4\zeta(3) + 2\zeta(2),
\end{aligned}$$

since

$$\lim_{n\to\infty} \left[(n+1)H_n^2 - (2n+1)H_n + 2n\right] \left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{(n+1)^3}\right) = 0.$$

It follows that

$$\sum_{m=1}^{\infty} \left(\frac{H_m}{m}\right)^2 - \frac{5}{4}\zeta(4) - 4\zeta(3) + 2\zeta(2) = 3\zeta(4) - 4\zeta(3) + 2\zeta(2)$$

and this implies that $\sum\limits_{m=1}^{\infty} \left(\frac{H_m}{m}\right)^2 = \frac{5}{4}\zeta(4) + 3\zeta(4) = \frac{17}{4}\zeta(4)$.

(b) **The second proof.** The second proof of formula (1) is based on calculating, by a new method, De Doelder's series $\sum\limits_{n=1}^{\infty} \frac{H_n^2}{(n+1)^2} = \frac{11}{4}\zeta(4)$.

We have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)^2}=\sum_{n=1}^{\infty}H_n^2\int_0^1\int_0^1 (xy)^n\mathrm{d}x\mathrm{d}y=\int_0^1\int_0^1\left(\sum_{n=1}^{\infty}H_n^2(xy)^n\right)\mathrm{d}x\mathrm{d}y\\
&\overset{(3)}{=}\int_0^1\int_0^1\frac{\ln^2(1-xy)+\mathrm{Li}_2(xy)}{1-xy}\mathrm{d}x\mathrm{d}y\overset{(5)}{=}-\int_0^1\frac{\ln^2(1-x)+\mathrm{Li}_2(x)}{1-x}\ln x\,\mathrm{d}x\\
&=-\int_0^1\frac{\ln^2(1-x)\ln x}{1-x}\mathrm{d}x-\int_0^1\frac{\mathrm{Li}_2(x)\ln x}{1-x}\mathrm{d}x\\
&=-\int_0^1\frac{\ln^2 x\ln(1-x)}{x}\mathrm{d}x-\int_0^1\frac{\mathrm{Li}_2(x)\ln x}{1-x}\mathrm{d}x\\
&=-C-B\\
&=\frac{11}{4}\zeta(4).
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{11}{4}\zeta(4)&=\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)^2}=\sum_{n=1}^{\infty}\frac{\left(H_{n+1}-\frac{1}{n+1}\right)^2}{(n+1)^2}\\
&=\sum_{n=1}^{\infty}\left(\frac{H_{n+1}^2}{(n+1)^2}-2\frac{H_{n+1}}{(n+1)^3}+\frac{1}{(n+1)^4}\right)\\
&=\sum_{m=1}^{\infty}\left(\frac{H_m^2}{m^2}-2\frac{H_m}{m^3}+\frac{1}{m^4}\right)\\
&\overset{\text{Lemma 2 (c)}}{=}\sum_{m=1}^{\infty}\frac{H_m^2}{m^2}-\frac{5}{2}\zeta(4)+\zeta(4)\\
&=\sum_{m=1}^{\infty}\frac{H_m^2}{m^2}-\frac{3}{2}\zeta(4),
\end{aligned}$$

and the result follows. □

REFERENCES

[1] D. D. Bonar, M. J. Koury, *Real Infinite Series*, Mathematical Association of America, Washington DC, 2006.

[2] D. Borwein, J. M. Borwein, On an intriguing integral and some series related to $\zeta(4)$, *Proc. Amer. Math. Soc.* **123**, 1995, 1191–1198.

[3] D. Castellanos, The ubiquitous π, Part I, *Math. Mag.* **61** (2), 1988, 67–98; The ubiquitous π, Part II, *Math. Mag.* **61** (3), 1988, 148–163.

[4] H. M. Srivastava, J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier, Amsterdam, 2012.

[5] P. J. De Doelder, On some series containing $\psi(x)-\psi(y)$ and $(\psi(x)-\psi(y))^2$ for certain values of x and y, *J. Comput. Appl. Math.* **37**, 1991, 125–141.

[6] P. Freitas, Integrals of polylogarithmic functions, recurrence relations, and associated Euler sums, *Math. Comp.* **74**, 2005, 1425–1440.
[7] O. Furdui, Limits, *Series and Fractional Part Integrals. Problems in Mathematical Analysis*, Springer, New York, 2013.
[8] O. Furdui, A Sîntămărian, A new proof of the quadratic series of Au–Yeung, *Gaz. Mat., Ser. A*, **37 (116)** (1–2), 2019, 1–6.
[9] M. L. Glasser, A note on Beukers's and related double integrals, *Amer. Math. Monthly* **126** (4), 2019, 361–363.
[10] I. Mező, Nonlinear Euler sums, *Pac. J. Math.* **272** (1), 2014, 201–226.
[11] C. Vălean, O. Furdui, Reviving the quadratic series of Au–Yeung, *J. Class. Anal.* **6** (2), 2015, 113–118.
[12] M. S. Klamkin, A summation problem, Advanced problem 4431, *Amer. Math. Monthly* **58**, 1951, 195.
[13] T.-L. T. Rădulescu, V. D. Rădulescu, T. Andreescu, *Problems in Real Analysis. Advanced Calculus on the Real Axis*, Springer, New York, 2009.
[14] H. F. Sandham, Advanced Problem 4305, *Amer. Math. Monthly* **55** (7), 1948, 431.
[15] A. Sîntămărian, O. Furdui, *Teme de analiză matematică. Exerciţii şi probleme*, Ediţia a VI-a, Editura Mega, Cluj-Napoca, 2019.

Asymptotic evaluations for some sequences of triple integrals

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Abstract. Let $f:[0,1]^3\to\mathbb{R}$ be a continuous function, a_n, b_n, $c_n:[0,1]^3\to[0,1]$ be three sequences of continuous functions and the sequence of triple integrals

$$I_n=\iiint\limits_{[0,1]^3} f\left(a_n\left(x,y,z\right),b_n\left(x,y,z\right),c_n\left(x,y,z\right)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

As a consequence of a general result we obtain asymptotic evaluations of the sequence $\left(I_n\right)_{n\in\mathbb{N}}$ in the case when f is differentiable at $(0,0,0)$ and twice differentiable at $(0,0,0)$. Many and various concrete examples are given.

Keywords: Riemann integral, multiple Riemann integral, uniform convergence, asymptotic expansion of a sequence.

MSC: Primary 26A42, 28A20; Secondary 40A05, 40A25.

1. Introduction

In the theory of integration the problem of finding various asymptotic estimates is of great importance. We recommend the reader to consult the books [2, 3, 4, 5]. The main purpose of this paper is the following: given a continuous function $f:[0,1]^3\to\mathbb{R}$ and three sequences of continuous

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functions $a_n, b_n, c_n : [0,1]^3 \to [0,1]$, we will find the asymptotic evaluation for the sequence of triple integrals

$$I_n = \iiint\limits_{[0,1]^3} f\left(a_n\left(x,y,z\right), b_n\left(x,y,z\right), c_n\left(x,y,z\right)\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

in the case when f is differentiable at $(0,0,0)$ and twice differentiable at $(0,0,0)$ in Theorems 5 and 18, respectively. In order to do this we first prove a general result, Theorem 4. We give also many and various concrete examples.

In the paper we will use the multiple Riemann integral. For details regarding the multiple Riemann integral we recommend the reader to consult the excellent treatment of this concept in the book of N. Boboc, see [1]. If k is a natural number and $A \subset \mathbb{R}^k$ is Jordan measurable, $\lambda_k(A)$ denote its Jordan measure, see [1]; we recall just that $\lambda_k([a_1,b_1] \times \cdots \times [a_k,b_k]) = (b_1 - a_1)\cdots(b_k - a_k)$, for $a_i \leq b_i$, $i = 1,\ldots, k$. If $A \subset \mathbb{R}^k$ is a compact Jordan measurable set and $f : A \to \mathbb{R}$ is a continuous function, we denote by $\int_A f(x)\,\mathrm{d}x$ the multiple Riemann integral. If $E \subseteq \mathbb{R}^3$ then $\overline{E}$ denotes the closure of E. The notation and notions used and not defined in this paper are standard, see [1, 2, 5].

Definition 1. *Let $E \subseteq \mathbb{R}^3$ be such that $(0,0,0) \in \overline{E}$, $f, g : E \to \mathbb{R}$ and $h : E \to [0,\infty)$. We write*

$$f(x,y,z) = g(x,y,z) + o(h(x,y,z)) \ \text{ for } \ (x,y,z) \to (0,0,0)$$

if and only if $\forall\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $\forall (x,y,z) \in E$ with $\max(|x|,|y|,|z|) < \delta_\varepsilon$ it follows that $|f(x,y,z) - g(x,y,z)| \leq \varepsilon h(x,y,z)$.

Let us note that since $(0,0,0) \in \overline{E}$ then, as is well-known, $\forall\varepsilon > 0$, we have $B((0,0,0),\varepsilon) \bigcap E \neq \varnothing$, and thus there exist points $(x,y,z) \in E$ with $\max(|x|,|y|,|z|) < \delta_\varepsilon$. Also if $E \subseteq \mathbb{R}^3$ is such that $(0,0,0) \in E$ and

$$f(x,y,z) = g(x,y,z) + o(h(x,y,z)) \ \text{ for } \ (x,y,z) \to (0,0,0)$$

then $f(0,0,0) = g(0,0,0)$. Indeed, $\forall\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for $(x,y,z) \in E$ with $\max(|x|,|y|,|z|) < \delta_\varepsilon$ it holds $|f(x,y,z) - g(x,y,z)| \leq \varepsilon h(x,y,z)$. In particular, since $(0,0,0) \in E$, we have $|f(0,0,0) - g(0,0,0)| \leq \varepsilon h(0,0,0)$ and since $\varepsilon > 0$ is arbitrary, passing to the limit for $\varepsilon \to 0$, $\varepsilon > 0$, we get $|f(0,0,0) - g(0,0,0)| \leq 0$, $f(0,0,0) = g(0,0,0)$.

Definition 2. *Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. If $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers we write $a_n = o(b_n)$ if and only if $\forall\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$ it follows that $|a_n| \leq \varepsilon|b_n|$. If $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ are two sequences of real numbers we write $x_n = y_n + o(b_n)$ if and only if $x_n - y_n = o(b_n)$.*

Let us note that if there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ we have $b_n \neq 0$ then the condition $a_n = o(b_n)$ is equivalent to the well known-condition $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$. The proof of the following remark is obvious and therefore is omitted.

Remark 3. *Let $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ be two sequences of real numbers with the property that there exists $M > 0$ such that $|b_n| \leq M|c_n|$, for all $n \in \mathbb{N}$. If $a_n = o(b_n)$, then $a_n = o(c_n)$*

2. A general result

We prove a general asymptotic evaluation for sequences of multiple Riemann integrals.

Theorem 4. *Let $E \subseteq \mathbb{R}^3$ be such that $(0,0,0) \in \overline{E}$, $f, g : E \to \mathbb{R}$ and $h : E \to [0,\infty)$ three continuous functions such that*

$$f(x,y,z) = g(x,y,z) + o(h(x,y,z)) \text{ for } (x,y,z) \to (0,0,0).$$

Let k be a natural number, $A \subset \mathbb{R}^k$ a compact Jordan measurable set and $a_n, b_n, c_n : A \to \mathbb{R}$ three sequences of continuous functions such that

$$(a_n(x), b_n(x), c_n(x)) \in E, \forall n \in \mathbb{N}, \forall x \in A$$

and moreover $\lim\limits_{n\to\infty} a_n = 0$, $\lim\limits_{n\to\infty} b_n = 0$, $\lim\limits_{n\to\infty} c_n = 0$ all uniformly on A. Then

$$\int_A f(a_n(x), b_n(x), c_n(x))\,\mathrm{d}x = \int_A g(a_n(x), b_n(x), c_n(x))\,\mathrm{d}x + o\left(\int_A h(a_n(x), b_n(x), c_n(x))\,\mathrm{d}x\right).$$

Proof. Let us note first that all the functions on the integrals are continuous hence are Riemann integrable, see [1]. Let $\varepsilon > 0$. By the hypothesis and the definition 1 there exists $\delta_\varepsilon > 0$ such that $\forall (x,y,z) \in E$ with $\max(|x|,|y|,|z|) < \delta_\varepsilon$ it follows that $|f(x,y,z) - g(x,y,z)| \leq \varepsilon h(x,y,z)$. Since $\lim\limits_{n\to\infty} a_n = 0$, $\lim\limits_{n\to\infty} b_n = 0$, $\lim\limits_{n\to\infty} c_n = 0$ all uniformly on A, it follows that there exists $n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$ we have

$$a_n(x) < \delta_\varepsilon, \quad b_n(x) < \delta_\varepsilon, \quad c_n(x) < \delta_\varepsilon, \forall x \in A.$$

Let $n \geq n_\varepsilon$. Then $(a_n(x), b_n(x), c_n(x)) \in E$, $\max(a_n(x), b_n(x), c_n(x)) < \delta_\varepsilon$ and hence $\forall x \in A$ we have

$$|f(a_n(x), b_n(x), c_n(x)) - g(a_n(x), b_n(x), c_n(x))| \leq \varepsilon h(a_n(x), b_n(x), c_n(x)).$$

By integration we get

$$\int_A |f(a_n(x), b_n(x), c_n(x)) - g(a_n(x), b_n(x), c_n(x))| \,\mathrm{d}x$$

$$\leq \varepsilon \int_A h(a_n(x), b_n(x), c_n(x)) \,\mathrm{d}x.$$

Since

$$\left| \int_A [f(a_n(x), b_n(x), c_n(x)) - g(a_n(x), b_n(x), c_n(x))] \,\mathrm{d}x \right|$$

$$\leq \int_A |f(a_n(x), b_n(x), c_n(x)) - g(a_n(x), b_n(x), c_n(x))| \,\mathrm{d}x,$$

by the linearity of the integral we obtain

$$\left| \int_A f(a_n(x), b_n(x), c_n(x)) \,\mathrm{d}x - \int_A g(a_n(x), b_n(x), c_n(x)) \,\mathrm{d}x \right|$$

$$\leq \varepsilon \int_A h(a_n(x), b_n(x), c_n(x)) \,\mathrm{d}x,$$

which ends the proof, see Definition 2. □

3. The case of differentiable functions

In the sequel we analyze the case of differentiable functions.

Theorem 5. *Let $f : [0,1]^3 \to \mathbb{R}$ be a continuous function which is differentiable at $(0,0,0)$. Let k be a natural number, $A \subset \mathbb{R}^k$ a compact Jordan measurable set, $a_n, b_n, c_n : A \to [0,1]$ be three sequences of continuous functions such that $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$, $\lim_{n\to\infty} c_n = 0$ all uniformly on A. Then*

$$\int_A f(a_n(x), b_n(x), c_n(x)) \,\mathrm{d}x = f(0,0,0)\,\lambda_k(A)$$
$$+\frac{\partial f}{\partial x}(0,0,0) \int_A a_n(x) \,\mathrm{d}x + \frac{\partial f}{\partial y}(0,0,0) \int_A b_n(x) \,\mathrm{d}x + \frac{\partial f}{\partial z}(0,0,0) \int_A c_n(x) \,\mathrm{d}x$$
$$+o\left(\int_A a_n(x) \,\mathrm{d}x + \int_A b_n(x) \,\mathrm{d}x + \int_A c_n(x) \,\mathrm{d}x\right).$$

Proof. Since f is differentiable at $(0,0,0)$ we have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{f(x,y,z) - f(0,0,0) - \frac{\partial f}{\partial x}(0,0,0)x - \frac{\partial f}{\partial y}(0,0,0)y - \frac{\partial f}{\partial z}(0,0,0)z}{|x|+|y|+|z|} = 0.$$

We deduce easily that $f(x,y,z) = f(0,0,0) + \frac{\partial f}{\partial x}(0,0,0)\,x + \frac{\partial f}{\partial y}(0,0,0)\,y + \frac{\partial f}{\partial z}(0,0,0)\,z + o(x+y+z)$ for $(x,y,z) \to (0,0,0)$. We apply Theorem 4 for $E = [0,1]^3$, $h(x,y,z) = x+y+z$. Let us note that in this case $E = \overline{E}$ (since E is closed) and $(0,0,0) \in E$. □

We will prove in the sequel some applications of Theorem 5.

Proposition 6. *Let* $f:[0,1]^3\to\mathbb{R}$ *be a continuous function which is differentiable at* $(0,0,0)$ *and* $u_n, v_n:[0,1]\to[0,1]$ *two sequences of continuous functions such that either* $\lim\limits_{n\to\infty} u_n = 0$ *uniformly on* $[0,1]$, *or* $\lim\limits_{n\to\infty} v_n = 0$ *uniformly on* $[0,1]$. *Then*

$$\iiint\limits_{[0,1]^3} f\left(u_n(x)v_n(y), u_n(y)v_n(z), u_n(z)v_n(x)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0)$$

$$+\left(\tfrac{\partial f}{\partial x}(0,0,0)+\tfrac{\partial f}{\partial y}(0,0,0)+\tfrac{\partial f}{\partial z}(0,0,0)\right)\left(\textstyle\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\textstyle\int_0^1 v_n(x)\,\mathrm{d}x\right)$$

$$+o\left(\left(\textstyle\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\textstyle\int_0^1 v_n(x)\,\mathrm{d}x\right)\right).$$

Proof. Let us take in Theorem 5, $A=[0,1]^3$, $a_n, b_n, c_n:[0,1]^3\to[0,1]$, $a_n(x,y,z)=u_n(x)v_n(y)$, $b_n(x,y,z)=u_n(y)v_n(z)$, $c_n(x,y,z)=u_n(z)v_n(x)$. Let us suppose, for example, that $\lim\limits_{n\to\infty} u_n = 0$ uniformly on $[0,1]$. From $0\le v_n(y)\le 1$, $\forall n\in\mathbb{N}$, $\forall y\in[0,1]$, we deduce that $0\le a_n(x,y,z)\le u_n(x)$, $\forall n\in\mathbb{N}$, $\forall (x,y,z)\in[0,1]^3$ and hence $\lim\limits_{n\to\infty} a_n=0$. Similarly $\lim\limits_{n\to\infty} b_n=0$, $\lim\limits_{n\to\infty} c_n=0$, all uniformly on $[0,1]^3$. Then

$$\iiint\limits_{[0,1]^3} f\left(a_n(x,y,z), b_n(x,y,z), c_n(x,y,z)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0)$$

$$+\frac{\partial f}{\partial x}(0,0,0)\iiint\limits_{[0,1]^3} a_n+\frac{\partial f}{\partial y}(0,0,0)\iiint\limits_{[0,1]^3} b_n+\frac{\partial f}{\partial z}(0,0,0)\iiint\limits_{[0,1]^3} c_n$$

$$+o\left(\iiint\limits_{[0,1]^3} a_n+\iiint\limits_{[0,1]^3} b_n+\iiint\limits_{[0,1]^3} c_n\right);$$

above and in the sequel of this proof we write simply $\iiint_{[0,1]^3} a_n$ instead of $\iiint_{[0,1]^3} a_n(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, etc. Since by Fubini's theorem

$$\iiint\limits_{[0,1]^3} a_n=\iiint\limits_{[0,1]^3} b_n=\iiint\limits_{[0,1]^3} c_n=\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n(x)\,\mathrm{d}x\right),$$

the evaluation from the statement follows. □

For $v_n(x)=1$ in Proposition 6 we get

Corollary 7. *Let* $f:[0,1]^3\to\mathbb{R}$ *be a continuous function which is differentiable at* $(0,0,0)$ *and* $u_n:[0,1]\to[0,1]$ *a sequence of continuous functions such that* $\lim\limits_{n\to\infty} u_n=0$ *uniformly on* $[0,1]$. *Then*

$$\iiint\limits_{[0,1]^3} f\left(u_n(x), u_n(y), u_n(z)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0)$$

$$+\left(\frac{\partial f}{\partial x}(0,0,0)+\frac{\partial f}{\partial y}(0,0,0)+\frac{\partial f}{\partial z}(0,0,0)\right)\int_0^1 u_n(x)\,\mathrm{d}x+o\left(\int_0^1 u_n(x)\,\mathrm{d}x\right).$$

Proposition 8. *Let $f:[0,1]^3\to\mathbb{R}$ be a continuous function which is differentiable at $(0,0,0)$ and $u_n,v_n:[0,1]\to[0,1]$ two sequences of continuous functions such that $\lim\limits_{n\to\infty}u_n=0$ uniformly on $[0,1]$. Then*

$$\iint\limits_{[0,1]^2} f\left(u_n(x),u_n(y),u_n(x)v_n(y)\right)\mathrm{d}x\mathrm{d}y=f(0,0,0)$$
$$+\left(\frac{\partial f}{\partial x}(0,0,0)+\frac{\partial f}{\partial y}(0,0,0)\right)\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)$$
$$+\frac{\partial f}{\partial z}(0,0,0)\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n(x)\,\mathrm{d}x\right)+o\left(\int_0^1 u_n(x)\,\mathrm{d}x\right).$$

Proof. Let us take in Theorem 5, $A=[0,1]^3$, $a_n,b_n,c_n:[0,1]^3\to[0,1]$, $a_n(x,y,z)=u_n(x)$, $b_n(x,y,z)=u_n(y)$, $c_n(x,y,z)=u_n(x)v_n(y)$. Let us note that from $\lim\limits_{n\to\infty}u_n=0$ uniformly on $[0,1]$ and $0\le v_n(x)\le 1$, $\forall x\in[0,1]$, $\forall n\in\mathbb{N}$ it follows that $\lim\limits_{n\to\infty}a_n=0$, $\lim\limits_{n\to\infty}b_n=0$, $\lim\limits_{n\to\infty}c_n=0$, all uniformly on $[0,1]^3$. Then

$$\iiint\limits_{[0,1]^3} f\left(u_n(x),u_n(y),u_n(x)v_n(y)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z=f(0,0,0)$$
$$+\frac{\partial f}{\partial x}(0,0,0)\iiint\limits_{[0,1]^3}a_n+\frac{\partial f}{\partial y}(0,0,0)\iiint\limits_{[0,1]^3}b_n+\frac{\partial f}{\partial z}(0,0,0)\iiint\limits_{[0,1]^3}c_n$$
$$+o\left(\iiint\limits_{[0,1]^3}a_n+\iiint\limits_{[0,1]^3}b_n+\iiint\limits_{[0,1]^3}c_n\right);$$

above and in the sequel of this proof we write simply $\iiint_{[0,1]^3}a_n$ instead of $\iiint_{[0,1]^3}a_n(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, etc. Since by Fubini's theorem

$$\iiint\limits_{[0,1]^3} f\left(u_n(x),u_n(y),u_n(x)v_n(y)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$=\iint\limits_{[0,1]^2} f\left(u_n(x),u_n(y),u_n(x)v_n(y)\right)\mathrm{d}x\mathrm{d}y,$$

$$\begin{aligned}\iiint\limits_{[0,1]^3}a_n &= \iiint\limits_{[0,1]^3}b_n=\int_0^1 u_n(x)\,\mathrm{d}x,\\ \iiint\limits_{[0,1]^3}c_n &= \left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n(x)\,\mathrm{d}x\right)\end{aligned}$$

we get

$$\iint\limits_{[0,1]^2} f\left(u_n(x), u_n(y), u_n(x) v_n(y)\right) \mathrm{d}x\mathrm{d}y = f(0,0,0)$$
$$+\left(\frac{\partial f}{\partial x}(0,0,0)+\frac{\partial f}{\partial y}(0,0,0)\right)\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)$$
$$+\frac{\partial f}{\partial z}(0,0,0)\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n(x)\,\mathrm{d}x\right)$$
$$+o\left(\int_0^1 u_n(x)\,\mathrm{d}x+\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n(x)\,\mathrm{d}x\right)\right). \tag{1}$$

From $0 \le v_n(x) \le 1$, $\forall x \in [0,1]$, $\forall n \in \mathbb{N}$, we deduce that $0 \le \int_0^1 v_n(x)\,\mathrm{d}x \le 1$, $\forall n \in \mathbb{N}$ and thus $\int_0^1 u_n(x)\,\mathrm{d}x + \left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n(x)\,\mathrm{d}x\right) \le 2\int_0^1 u_n(x)\,\mathrm{d}x$, $\forall n \in \mathbb{N}$. From Remark 3 and the relation (1) we deduce the evaluation from the statement. □

4. Some examples in the case of differentiable functions

In this section $f : [0,1]^3 \to \mathbb{R}$ is a continuous function which is differentiable at $(0,0,0)$ and $S = \frac{\partial f}{\partial x}(0,0,0) + \frac{\partial f}{\partial y}(0,0,0) + \frac{\partial f}{\partial z}(0,0,0)$. We begin with some applications of Corollary 7.

Corollary 9. *The following evaluation holds*

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n}{n}, \frac{y^n}{n}, \frac{z^n}{n}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0) + \frac{S}{n^2} + o\left(\frac{1}{n^2}\right).$$

Proof. Let $u_n : [0,1] \to [0,1]$, $u_n(x) = \frac{x^n}{n}$. We have $0 \le u_n(x) \le \frac{1}{n}$, $\forall n \in \mathbb{N}, \forall x \in [0,1]$, and thus $\lim\limits_{n\to\infty} u_n = 0$ uniformly on $[0,1]$, also $\int_0^1 u_n(x)\,\mathrm{d}x = \frac{1}{n(n+1)}$. We deduce that

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n}{n}, \frac{y^n}{n}, \frac{z^n}{n}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0) + \frac{S}{n(n+1)} + o\left(\frac{1}{n(n+1)}\right).$$

To finish the proof let us note that $\frac{1}{n+1} = \frac{1}{n} + o\left(\frac{1}{n}\right)$. □

Corollary 10. *The following evaluation holds*

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n}{n(x^n+1)}, \frac{y^n}{n(y^n+1)}, \frac{z^n}{n(z^n+1)}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$= f(0,0,0) + \frac{S\ln 2}{n^2} + o\left(\frac{1}{n^2}\right).$$

Proof. Let $u_n : [0,1] \to [0,1]$, $u_n(x) = \frac{x^n}{n(x^n+1)}$. We have $0 \le u_n(x) \le \frac{1}{n}$, $\forall n \in \mathbb{N}, \forall x \in [0,1]$, and thus $\lim_{n\to\infty} u_n = 0$ uniformly on $[0,1]$. We deduce that

$$\iiint_{[0,1]^3} f\left(\frac{x^n}{n(x^n+1)}, \frac{y^n}{n(y^n+1)}, \frac{z^n}{n(z^n+1)}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0)$$
$$+ \frac{S}{n}\int_0^1 \frac{x^n \mathrm{d}x}{x^n+1} + o\left(\frac{1}{n}\int_0^1 \frac{x^n \mathrm{d}x}{x^n+1}\right).$$

But, as is well-known, $\lim_{n\to\infty} n\int_0^1 \frac{x^n\mathrm{d}x}{x^n+1} = \ln 2$, that is, $\int_0^1 \frac{x^n\mathrm{d}x}{x^n+1} = \frac{\ln 2}{n} + o\left(\frac{1}{n}\right)$, see [6, problem 3.13]. The evaluation from the statement follows. □

Corollary 11. *The following evaluation holds*
$$\iiint_{[0,1]^3} f\left(\frac{1+x+x^2+\cdots+x^{n-1}}{n^2}, \frac{1+y+y^2+\cdots+y^{n-1}}{n^2}, \frac{1+z+z^2+\cdots+z^{n-1}}{n^2}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$= f(0,0,0) + \frac{S\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right).$$

Proof. Let $u_n : [0,1] \to [0,1]$, $u_n(x) = \frac{1+x+\cdots+x^{n-1}}{n^2}$. We have $0 \le u_n(x) \le \frac{1}{n}$, $\forall n \in \mathbb{N}, \forall x \in [0,1]$, and thus $\lim_{n\to\infty} u_n = 0$ uniformly on $[0,1]$. Also $\int_0^1 u_n(x)\,\mathrm{d}x = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = H_n$. We deduce that
$$\iiint_{[0,1]^3} f\left(\frac{1+x+x^2+\cdots+x^{n-1}}{n^2}, \frac{1+y+y^2+\cdots+y^{n-1}}{n^2}, \frac{1+z+z^2+\cdots+z^{n-1}}{n^2}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$= f(0,0,0) + \frac{SH_n}{n^2} + o\left(\frac{H_n}{n^2}\right).$$

To finish the proof, we recall that, as is well-known, $\lim_{n\to\infty} \frac{H_n}{\ln n} = 1$, that is $H_n = \ln n + o(\ln n)$. □

We continue with some applications of Proposition 6.

Corollary 12. *The following evaluation holds*
$$\iiint_{[0,1]^3} f\left(\frac{x^n y^n}{n}, \frac{y^n z^n}{n}, \frac{z^n x^n}{n}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0) + \frac{S}{n^3} + o\left(\frac{1}{n^3}\right).$$

Proof. Let $u_n, v_n : [0,1] \to [0,1]$, $u_n(x) = \frac{x^n}{n}$, $v_n(x) = x^n$. We have $0 \le u_n(x) \le \frac{1}{n}$, $0 \le v_n(x) \le 1$, $\forall n \in \mathbb{N}$, $\forall x \in [0,1]$, and thus $\lim_{n\to\infty} u_n = 0$ uniformly on $[0,1]$; also $\int_0^1 u_n(x)\,\mathrm{d}x = \frac{1}{n(n+1)}$, $\int_0^1 v_n(x)\,\mathrm{d}x = \frac{1}{n+1}$. We deduce that
$$\iiint_{[0,1]^3} f\left(\frac{x^n y^n}{n}, \frac{y^n z^n}{n}, \frac{z^n x^n}{n}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0) + \frac{S}{n(n+1)^2} + o\left(\frac{1}{n(n+1)^2}\right).$$

To finish the proof let us note that $\frac{1}{(n+1)^2} = \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$. □

Corollary 13. *The following evaluation holds*

$$\iiint_{[0,1]^3} f\left(\frac{x^n\left(1+y+\cdots+y^{n-1}\right)}{n^2}, \frac{y^n\left(1+z+\cdots+z^{n-1}\right)}{n^2}, \frac{z^n\left(1+x+\cdots+x^{n-1}\right)}{n^2}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= f(0,0,0) + \frac{S\ln n}{n^3} + o\left(\frac{\ln n}{n^3}\right).$$

Proof. Let $u_n, v_n : [0,1] \to [0,1]$, $u_n(x) = x^n$, $v_n(x) = \frac{1+x+\cdots+x^{n-1}}{n^2}$. We have $0 \le u_n(x) \le 1$, $0 \le v_n(x) \le \frac{1}{n}$, $\forall n \in \mathbb{N}$, $\forall x \in [0,1]$, and thus $\lim\limits_{n\to\infty} v_n = 0$ uniformly on $[0,1]$; also $\int_0^1 u_n(x)\,\mathrm{d}x = \frac{1}{n+1}$, $\int_0^1 v_n(x)\,\mathrm{d}x = \frac{H_n}{n^2}$, where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. We deduce that

$$\iiint\limits_{[0,1]^3} f\left(u_n(x)v_n(y), u_n(y)v_n(z), u_n(z)v_n(x)\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= f(0,0,0) + \frac{SH_n}{n^2(n+1)} + o\left(\frac{H_n}{n^2(n+1)}\right).$$

From the evaluation $H_n = \ln n + o(\ln n)$ we get the evaluation from the statement. □

We continue with an application of Proposition 8.

Corollary 14. *The following evaluation holds*

$$\iint\limits_{[0,1]^2} f\left(\frac{x^n}{n}, \frac{y^n}{n}, \frac{x^n y^n}{n}\right) \mathrm{d}x\mathrm{d}y = f(0,0,0) + \left(\frac{\partial f}{\partial x}(0,0,0) + \frac{\partial f}{\partial y}(0,0,0)\right)\frac{1}{n^2} + o\left(\frac{1}{n^2}\right).$$

Proof. Let us take in Proposition 8 $u_n, v_n : [0,1] \to [0,1]$, $u_n(x) = \frac{x^n}{n}$, $v_n(x) = y^n$. We get

$$\iint\limits_{[0,1]^2} f\left(\frac{x^n}{n}, \frac{y^n}{n}, \frac{x^n y^n}{n}\right) \mathrm{d}x\mathrm{d}y = f(0,0,0) + \left(\frac{\partial f}{\partial x}(0,0,0) + \frac{\partial f}{\partial y}(0,0,0)\right)\frac{1}{n(n+1)}$$

$$+ \frac{\partial f}{\partial z}(0,0,0)\frac{1}{n(n+1)^2} + o\left(\frac{1}{n(n+1)}\right).$$

By simple calculations we get the evaluation from the statement. □

We end this section with some applications of Theorem 5.

Corollary 15. *The following evaluation holds*

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n y^n z^n}{n}, \frac{x^{2n} y^{2n} z^{2n}}{n}, \frac{x^{3n} y^{3n} z^{3n}}{n}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0)$$

$$+\left(\frac{\partial f}{\partial x}(0,0,0) + \frac{\frac{\partial f}{\partial y}(0,0,0)}{8} + \frac{\frac{\partial f}{\partial z}(0,0,0)}{27}\right)\frac{1}{n^4} + o\left(\frac{1}{n^4}\right).$$

Proof. Let $A = [0,1]^3$, $a_n, b_n, c_n : [0,1]^3 \to [0,1]$, $a_n(x,y,z) = \frac{x^n y^n z^n}{n}$, $b_n(x,y,z) = \frac{x^{2n} y^{2n} z^{2n}}{n}$, $c_n(x,y,z) = \frac{x^{3n} y^{3n} z^{3n}}{n}$. From

$$0 \le a_n(x,y,z), b_n(x,y,z), c_n(x,y,z) \le \frac{1}{n}, \forall n \in \mathbb{N}, \forall (x,y,z) \in [0,1]^3$$

we deduce that $\lim\limits_{n\to\infty} a_n = 0$, $\lim\limits_{n\to\infty} b_n = 0$, $\lim\limits_{n\to\infty} c_n = 0$, all uniformly on $[0,1]^3$. We also have $\iiint_{[0,1]^3} a_n(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{n(n+1)^3}$, $\iiint_{[0,1]^3} b_n(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{n(2n+1)^3}$, $\iiint_{[0,1]^3} c_n(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{n(3n+1)^3}$. Then

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n y^n z^n}{n}, \frac{x^{2n} y^{2n} z^{2n}}{n}, \frac{x^{3n} y^{3n} z^{3n}}{n}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0)$$

$$+\frac{\frac{\partial f}{\partial x}(0,0,0)}{n(n+1)^3} + \frac{\frac{\partial f}{\partial y}(0,0,0)}{n(2n+1)^3} + \frac{\frac{\partial f}{\partial z}(0,0,0)}{n(3n+1)^3} + o\left(\frac{1}{n^4}\right).$$

Since $\frac{1}{(n+1)^3} = \frac{1}{n^3} + o\left(\frac{1}{n^3}\right)$, $\frac{1}{(2n+1)^3} = \frac{1}{8n^3} + o\left(\frac{1}{n^3}\right)$, $\frac{1}{(3n+1)^3} = \frac{1}{27n^3} + o\left(\frac{1}{n^3}\right)$, by simple calculations we get the evaluation from the statement. □

Corollary 16. *The following evaluation holds*

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n + y^n + z^n}{n}, \frac{x^{2n} + y^{2n} + z^{2n}}{n}, \frac{x^{3n} + y^{3n} + z^{3n}}{n}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= f(0,0,0) + 3\left(\frac{\partial f}{\partial x}(0,0,0) + \frac{\frac{\partial f}{\partial y}(0,0,0)}{2} + \frac{\frac{\partial f}{\partial z}(0,0,0)}{3}\right)\frac{1}{n^2} + o\left(\frac{1}{n^2}\right).$$

Proof. Let $A = [0,1]^3$ and $a_n, b_n, c_n : [0,1]^3 \to [0,1]$, $a_n(x,y,z) = \frac{x^n + y^n + z^n}{n}$, $b_n(x,y,z) = \frac{x^{2n} + y^{2n} + z^{2n}}{n}$, $c_n(x,y,z) = \frac{x^{3n} + y^{3n} + z^{3n}}{n}$. From

$$0 \le a_n(x,y,z), b_n(x,y,z), c_n(x,y,z) \le \frac{3}{n}, \forall n \in \mathbb{N}, \forall (x,y,z) \in [0,1]^3$$

we deduce that $\lim\limits_{n\to\infty} a_n = 0$, $\lim\limits_{n\to\infty} b_n = 0$, $\lim\limits_{n\to\infty} c_n = 0$, all uniformly on $[0,1]^3$. Also $\iiint_{[0,1]^3} a_n(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{3}{n(n+1)}$, $\iiint_{[0,1]^3} b_n(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z =$

$\frac{3}{n(2n+1)}$, $\iiint_{[0,1]^3} c_n(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{3}{n(3n+1)}$. Then

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n+y^n+z^n}{n}, \frac{x^{2n}+y^{2n}+z^{2n}}{n}, \frac{x^{3n}+y^{3n}+z^{3n}}{n}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= f(0,0,0) + \frac{3\frac{\partial f}{\partial x}(0,0,0)}{n(n+1)} + \frac{3\frac{\partial f}{\partial y}(0,0,0)}{n(2n+1)} + \frac{3\frac{\partial f}{\partial z}(0,0,0)}{n(3n+1)} + o\left(\frac{1}{n^2}\right).$$

Since $\frac{1}{n+1} = \frac{1}{n} + o\left(\frac{1}{n}\right)$, $\frac{1}{2n+1} = \frac{1}{2n} + o\left(\frac{1}{n}\right)$, $\frac{1}{3n+1} = \frac{1}{3n} + o\left(\frac{1}{n}\right)$, by simple calculations we get the evaluation from the statement. □

Corollary 17. *The following evaluation holds*

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n+y^n+z^n}{n}, \frac{x^ny^n+y^nz^n+z^nx^n}{n}, \frac{x^ny^nz^n}{n}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= f(0,0,0) + \frac{3\frac{\partial f}{\partial x}(0,0,0)}{n^2} + o\left(\frac{1}{n^2}\right).$$

Proof. Let $A = [0,1]^3$ and $a_n, b_n, c_n : [0,1]^3 \to [0,1]$, $a_n(x,y,z) = \frac{x^n+y^n+z^n}{n}$, $b_n(x,y,z) = \frac{x^ny^n+y^nz^n+z^nx^n}{n}$, $c_n(x,y,z) = \frac{x^ny^nz^n}{n}$. From

$$0 \le a_n(x,y,z), b_n(x,y,z) \le \frac{3}{n}, 0 \le c_n(x,y,z) \le \frac{1}{n}, \forall n \in \mathbb{N}, \forall (x,y,z) \in [0,1]^3$$

we deduce that $\lim\limits_{n\to\infty} a_n = 0$, $\lim\limits_{n\to\infty} b_n = 0$, $\lim\limits_{n\to\infty} c_n = 0$, all uniformly on $[0,1]^3$. Also $\iiint_{[0,1]^3} a_n(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{3}{n(n+1)}$, $\iiint_{[0,1]^3} b_n(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{3}{n(n+1)^2}$, $\iiint_{[0,1]^3} c_n(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{n(n+1)^3}$. Then

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n+y^n+z^n}{n}, \frac{x^ny^n+y^nz^n+z^nx^n}{n}, \frac{x^ny^nz^n}{n}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0)$$

$$+\frac{3\frac{\partial f}{\partial x}(0,0,0)}{n(n+1)} + \frac{3\frac{\partial f}{\partial y}(0,0,0)}{n(n+1)^2} + \frac{\frac{\partial f}{\partial z}(0,0,0)}{n(n+1)^3}$$

$$+o\left(\frac{1}{n(n+1)} + \frac{1}{n(n+1)^2} + \frac{1}{n(n+1)^3}\right).$$

From this evaluation by simple calculations we get the evaluation from the statement. □

5. THE CASE OF TWICE DIFFERENTIABLE FUNCTIONS

In this section we analyze the case of twice differentiable functions.

Theorem 18. *Let $f : [0,1]^3 \to \mathbb{R}$ be a continuous function which is twice differentiable at $(0,0,0)$. Let k be a natural number, $A \subset \mathbb{R}^k$ a compact Jordan measurable set, $a_n, b_n, c_n : A \to [0,1]$ be three sequences of continuous functions such that $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$, $\lim_{n\to\infty} c_n = 0$, all uniformly on A. Then*

$$\int_A f(a_n(x), b_n(x), c_n(x))\,\mathrm{d}x = f(0,0,0)\,\lambda_k(A)$$

$$+\frac{\partial f}{\partial x}(0,0,0)\int_A a_n(x)\,\mathrm{d}x + \frac{\partial f}{\partial y}(0,0,0)\int_A b_n(x)\,\mathrm{d}x + \frac{\partial f}{\partial z}(0,0,0)\int_A c_n(x)\,\mathrm{d}x$$

$$+\frac{\partial^2 f}{\partial x^2}(0,0,0)\int_A a_n^2(x)\,\mathrm{d}x + \frac{\partial^2 f}{\partial y^2}(0,0,0)\int_A b_n^2(x)\,\mathrm{d}x + \frac{\partial^2 f}{\partial z^2}(0,0,0)\int_A c_n^2(x)\,\mathrm{d}x$$

$$+2\frac{\partial^2 f}{\partial x\partial y}(0,0,0)\int_A a_n(x)\,b_n(x)\,\mathrm{d}x + 2\frac{\partial^2 f}{\partial x\partial z}(0,0,0)\int_A a_n(x)\,c_n(x)\,\mathrm{d}x$$

$$+2\frac{\partial^2 f}{\partial y\partial z}(0,0,0)\int_A b_n(x)\,c_n(x)\,\mathrm{d}x$$

$$+o\left(\int_A a_n^2(x)\,\mathrm{d}x + \int_A b_n^2(x)\,\mathrm{d}x + \int_A c_n^2(x)\,\mathrm{d}x\right).$$

Proof. Since f is twice differentiable at $(0,0,0)$, from the Maclaurin formula we have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{f(x,y,z) - f(0,0,0) - P_1(x,y,z) - P_2(x,y,z)}{x^2+y^2+z^2} = 0,$$

where $P_1(x,y,z) = \frac{\partial f}{\partial x}(0,0,0)\,x + \frac{\partial f}{\partial y}(0,0,0)\,y + \frac{\partial f}{\partial z}(0,0,0)\,z$,

$$P_2(x,y,z) = \frac{\partial^2 f}{\partial x^2}(0,0,0)\,x^2 + \frac{\partial^2 f}{\partial y^2}(0,0,0)\,y^2 + \frac{\partial^2 f}{\partial z^2}(0,0,0)\,z^2$$

$$+\,2\frac{\partial^2 f}{\partial x\partial y}(0,0,0)\,xy + 2\frac{\partial^2 f}{\partial x\partial z}(0,0,0)\,xz + 2\frac{\partial^2 f}{\partial y\partial z}(0,0,0)\,yz,$$

see [1]. From this limit we deduce that

$$f(x,y,z) = g(x,y,z) + o\left(x^2+y^2+z^2\right) \text{ for } (x,y,z)\to(0,0,0),$$

where $g(x,y,z) = f(0,0,0) + P_1(x,y,z) + P_2(x,y,z)$. We apply Theorem 4 in which $E = [0,1]^3$ and $h(x,y,z) = x^2+y^2+z^2$. □

We will prove in the sequel some applications of Theorem 18.

Proposition 19. *Let $f : [0,1]^3 \to \mathbb{R}$ be a continuous function which is twice differentiable at $(0,0,0)$ and $u_n, v_n : [0,1] \to [0,1]$ two sequences of* *continuous functions such that either* $\lim\limits_{n\to\infty} u_n = 0$ *uniformly on* $[0,1]$, *or* $\lim\limits_{n\to\infty} v_n = 0$ *uniformly on* $[0,1]$. *Then*

$$\iiint\limits_{[0,1]^3} f\left(u_n(x)v_n(y), u_n(y)v_n(z), u_n(z)v_n(x)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0)$$

$$+S\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n(x)\,\mathrm{d}x\right)$$

$$+(\Delta f)(0,0,0)\left(\int_0^1 u_n^2(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n^2(x)\,\mathrm{d}x\right)$$

$$+2T\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n(x)\,\mathrm{d}x\right)\left(\int_0^1 u_n(x)v_n(x)\,\mathrm{d}x\right)$$

$$+o\left(\left(\int_0^1 u_n^2(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n^2(x)\,\mathrm{d}x\right)\right),$$

where $S = \frac{\partial f}{\partial x}(0,0,0) + \frac{\partial f}{\partial y}(0,0,0) + \frac{\partial f}{\partial z}(0,0,0)$, $(\Delta f)(0,0,0) = \frac{\partial^2 f}{\partial x^2}(0,0,0) + \frac{\partial^2 f}{\partial y^2}(0,0,0) + \frac{\partial^2 f}{\partial z^2}(0,0,0)$, $T = \frac{\partial^2 f}{\partial x\partial y}(0,0,0) + \frac{\partial^2 f}{\partial x\partial z}(0,0,0) + \frac{\partial^2 f}{\partial y\partial z}(0,0,0)$.

Proof. Let us take in Theorem 18, $A = [0,1]^3$, $a_n, b_n, c_n : [0,1]^3 \to [0,1]$, $a_n(x,y,z) = u_n(x)v_n(y)$, $b_n(x,y,z) = u_n(y)v_n(z)$, $c_n(x,y,z) = u_n(z)v_n(x)$. We get

$$\iiint\limits_{[0,1]^3} f\left(a_n(x,y,z), b_n(x,y,z), c_n(x,y,z)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0)$$

$$+\frac{\partial f}{\partial x}(0,0,0)\iiint\limits_{[0,1]^3} a_n + \frac{\partial f}{\partial y}(0,0,0)\iiint\limits_{[0,1]^3} b_n + \frac{\partial f}{\partial z}(0,0,0)\iiint\limits_{[0,1]^3} c_n$$

$$+\frac{\partial^2 f}{\partial x^2}(0,0,0)\iiint\limits_{[0,1]^3} a_n^2 + \frac{\partial^2 f}{\partial y^2}(0,0,0)\iiint\limits_{[0,1]^3} b_n^2 + \frac{\partial^2 f}{\partial z^2}(0,0,0)\iiint\limits_{[0,1]^3} c_n^2$$

$$+2\frac{\partial^2 f}{\partial x\partial y}(0,0,0)\iiint\limits_{[0,1]^3} a_n b_n + 2\frac{\partial^2 f}{\partial x\partial z}(0,0,0)\iiint\limits_{[0,1]^3} a_n c_n$$

$$+2\frac{\partial^2 f}{\partial y\partial z}(0,0,0)\iiint\limits_{[0,1]^3} b_n c_n + o\left(\iiint\limits_{[0,1]^3} a_n^2 + \iiint\limits_{[0,1]^3} b_n^2 + \iiint\limits_{[0,1]^3} c_n^2\right);$$

above and in the sequel of this proof we simply write $\iiint_{[0,1]^3} a_n$ instead of $\iiint_{[0,1]^3} a_n(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, etc. By Fubini's theorem

$$\iiint\limits_{[0,1]^3} a_n = \iiint\limits_{[0,1]^3} b_n = \iiint\limits_{[0,1]^3} c_n = \left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n(x)\,\mathrm{d}x\right),$$

$$\iiint\limits_{[0,1]^3} a_n^2 = \iiint\limits_{[0,1]^3} b_n^2 = \iiint\limits_{[0,1]^3} c_n^2 = \left(\int_0^1 u_n^2\left(x\right)\mathrm{d}x\right)\left(\int_0^1 v_n^2\left(x\right)\mathrm{d}x\right),$$

and

$$\begin{aligned}\iiint\limits_{[0,1]^3} a_n b_n &= \iiint\limits_{[0,1]^3} a_n c_n = \iiint\limits_{[0,1]^3} b_n c_n \\ &= \left(\int_0^1 u_n\left(x\right)\mathrm{d}x\right)\left(\int_0^1 v_n\left(x\right)\mathrm{d}x\right)\left(\int_0^1 u_n\left(x\right)v_n\left(x\right)\mathrm{d}x\right),\end{aligned}$$

and thus we get the evaluation from the statement. □

Taking $v_n\left(x\right) = 1$ in Proposition 19 we get

Corollary 20. *Let $f : [0,1]^3 \to \mathbb{R}$ be a continuous function which is twice differentiable at $(0,0,0)$ and $u_n : [0,1] \to [0,1]$ a sequence of continuous functions such that $\lim\limits_{n\to\infty} u_n = 0$ uniformly on $[0,1]$. Then*

$$\iiint\limits_{[0,1]^3} f\left(u_n\left(x\right), u_n\left(y\right), u_n\left(z\right)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = f\left(0,0,0\right) +$$

$$+S\int_0^1 u_n\left(x\right)\mathrm{d}x + \left(\Delta f\right)\left(0,0,0\right)\int_0^1 u_n^2\left(x\right)\mathrm{d}x + 2T\left(\int_0^1 u_n\left(x\right)\mathrm{d}x\right)^2$$

$$+o\left(\int_0^1 u_n^2\left(x\right)\mathrm{d}x\right).$$

Proposition 21. *Let $f : [0,1]^3 \to \mathbb{R}$ be a continuous function which is twice differentiable at $(0,0,0)$ and $u_n, v_n : [0,1] \to [0,1]$ two sequences of continuous functions such that $\lim\limits_{n\to\infty} u_n = 0$ uniformly on $[0,1]$ and $\lim\limits_{n\to\infty} \int_0^1 v_n\left(x\right)\mathrm{d}x = 0$. Then*

$$\iint\limits_{[0,1]^2} f\left(u_n\left(x\right), u_n\left(y\right), u_n\left(x\right)v_n\left(y\right)\right)\mathrm{d}x\mathrm{d}y = f\left(0,0,0\right)$$

$$\begin{aligned}&+\left(\frac{\partial f}{\partial x}\left(0,0,0\right) + \frac{\partial f}{\partial y}\left(0,0,0\right)\right)\left(\int_0^1 u_n\left(x\right)\mathrm{d}x\right) \\ &+\frac{\partial f}{\partial z}\left(0,0,0\right)\left(\int_0^1 u_n\left(x\right)\mathrm{d}x\right)\left(\int_0^1 v_n\left(x\right)\mathrm{d}x\right) \\ &+\left(\frac{\partial^2 f}{\partial x^2}\left(0,0,0\right) + \frac{\partial^2 f}{\partial y^2}\left(0,0,0\right)\right)\int_0^1 u_n^2\left(x\right)\mathrm{d}x \\ &+2\frac{\partial^2 f}{\partial x \partial y}\left(0,0,0\right)\left(\int_0^1 u_n\left(x\right)\mathrm{d}x\right)^2 + o\left(\int_0^1 u_n^2\left(x\right)\mathrm{d}x\right).\end{aligned}$$

Proof. Let us take in Theorem 18, $A = [0,1]^3$, $a_n, b_n, c_n : [0,1]^3 \to [0,1]$, $a_n(x,y,z) = u_n(x)$, $b_n(x,y,z) = u_n(y)$, $c_n(x,y,z) = u_n(x)\, v_n(y)$. We get

$$\iiint_{[0,1]^3} f\left(a_n(x,y,z), b_n(x,y,z), c_n(x,y,z)\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0)$$

$$+\frac{\partial f}{\partial x}(0,0,0)\iiint_{[0,1]^3} a_n + \frac{\partial f}{\partial y}(0,0,0)\iiint_{[0,1]^3} b_n + \frac{\partial f}{\partial z}(0,0,0)\iiint_{[0,1]^3} c_n$$

$$+\frac{\partial^2 f}{\partial x^2}(0,0,0)\iiint_{[0,1]^3} a_n^2 + \frac{\partial^2 f}{\partial y^2}(0,0,0)\iiint_{[0,1]^3} b_n^2 + \frac{\partial^2 f}{\partial z^2}(0,0,0)\iiint_{[0,1]^3} c_n^2$$

$$+2\frac{\partial^2 f}{\partial x\partial y}(0,0,0)\iiint_{[0,1]^3} a_n b_n + 2\frac{\partial^2 f}{\partial x\partial z}(0,0,0)\iiint_{[0,1]^3} a_n c_n$$

$$+2\frac{\partial^2 f}{\partial y\partial z}(0,0,0)\iiint_{[0,1]^3} b_n c_n + o\left(\iiint_{[0,1]^3} a_n^2 + \iiint_{[0,1]^3} b_n^2 + \iiint_{[0,1]^3} c_n^2\right);$$

above and in the sequel of this proof we write simply $\iiint_{[0,1]^3} a_n$ instead of $\iiint_{[0,1]^3} a_n(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, etc. Since by Fubini's theorem

$$\begin{aligned}\iiint_{[0,1]^3} a_n^2 &= \iiint_{[0,1]^3} b_n^2 = \int_0^1 u_n^2(x)\,\mathrm{d}x,\\ \iiint_{[0,1]^3} c_n^2 &= \left(\int_0^1 u_n^2(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n^2(x)\,\mathrm{d}x\right)\end{aligned}$$

and from the hypothesis $0 \le v_n(x) \le 1$, $\forall x \in [0,1]$ we have

$$0 \le \left(\int_0^1 u_n^2(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n^2(x)\,\mathrm{d}x\right) \le \int_0^1 u_n^2(x)\,\mathrm{d}x,$$

we deduce that

$$0 \le \iiint_{[0,1]^3} a_n^2 + \iiint_{[0,1]^3} b_n^2 + \iiint_{[0,1]^3} c_n^2 \le 3\int_0^1 u_n^2(x)\,\mathrm{d}x.$$

Also

$$\begin{aligned}\iiint\limits_{[0,1]^3} a_n b_n &= \left(\int_0^1 u_n(x)\,\mathrm{d}x\right)^2,\\ \iiint\limits_{[0,1]^3} a_n c_n &= \left(\int_0^1 u_n^2(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n(x)\,\mathrm{d}x\right),\\ \iiint\limits_{[0,1]^3} b_n c_n &= \left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\int_0^1 u_n(x)\,v_n(x)\,\mathrm{d}x\right).\end{aligned}$$

Hence by Remark 3 we have

$$\begin{aligned}&\iint\limits_{[0,1]^2} f\left(u_n(x), u_n(y), u_n(x)\,v_n(y)\right)\mathrm{d}x\mathrm{d}y = f(0,0,0)\\ &\quad+\left(\frac{\partial f}{\partial x}(0,0,0)+\frac{\partial f}{\partial y}(0,0,0)\right)\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\\ &\quad+\frac{\partial f}{\partial z}(0,0,0)\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n(x)\,\mathrm{d}x\right)\\ &\quad+\left(\frac{\partial^2 f}{\partial x^2}(0,0,0)+\frac{\partial^2 f}{\partial y^2}(0,0,0)\right)\int_0^1 u_n^2(x)\,\mathrm{d}x\\ &+2\frac{\partial^2 f}{\partial x\partial y}(0,0,0)\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)^2\\ &+2\frac{\partial^2 f}{\partial x\partial z}(0,0,0)\left(\int_0^1 u_n^2(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n(x)\,\mathrm{d}x\right)\\ &+2\frac{\partial^2 f}{\partial y\partial z}(0,0,0)\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\int_0^1 u_n(x)\,v_n(x)\,\mathrm{d}x\right)\\ &+o\left(\int_0^1 u_n^2(x)\,\mathrm{d}x\right). \qquad (2)\end{aligned}$$

Since by hypothesis $\lim\limits_{n\to\infty}\int_0^1 v_n(x)\,\mathrm{d}x = 0$, we get

$$\left(\int_0^1 u_n^2(x)\,\mathrm{d}x\right)\left(\int_0^1 v_n(x)\,\mathrm{d}x\right) = o\left(\int_0^1 u_n^2(x)\,\mathrm{d}x\right).$$

From the Cauchy-Buniakovski-Schwarz inequality and $0 \le v_n(x) \le 1$, $\forall x \in [0,1]$, we have

$$\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\int_0^1 u_n(x)\,v_n(x)\,\mathrm{d}x\right) \le$$

$$\sqrt{\int_0^1 u_n^2(x)\,\mathrm{d}x}\sqrt{\int_0^1 u_n^2(x)\,\mathrm{d}x}\sqrt{\int_0^1 v_n^2(x)\,\mathrm{d}x} \le \left(\int_0^1 u_n^2(x)\,\mathrm{d}x\right)\sqrt{\int_0^1 v_n(x)\,\mathrm{d}x}$$

and again from the hypothesis $\lim\limits_{n\to\infty}\int_0^1 v_n(x)\,\mathrm{d}x = 0$ we deduce that

$$\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)\left(\int_0^1 u_n(x)\,v_n(x)\,\mathrm{d}x\right) = o\left(\int_0^1 u_n^2(x)\,\mathrm{d}x\right). \tag{3}$$

The evaluation from the statement follows from the relations (2) and (3). □

6. Some examples in the case of twice differentiable functions

To avoid the repetitions: in this section $f : [0,1]^3 \to \mathbb{R}$ is a continuous function which is twice differentiable at $(0,0,0)$, $S = \frac{\partial f}{\partial x}(0,0,0) + \frac{\partial f}{\partial y}(0,0,0) + \frac{\partial f}{\partial z}(0,0,0)$, $\Delta f(0,0,0) = \frac{\partial^2 f}{\partial x^2}(0,0,0) + \frac{\partial^2 f}{\partial y^2}(0,0,0) + \frac{\partial^2 f}{\partial z^2}(0,0,0)$, $T = \frac{\partial^2 f}{\partial x \partial y}(0,0,0) + \frac{\partial^2 f}{\partial x \partial z}(0,0,0) + \frac{\partial^2 f}{\partial y \partial z}(0,0,0)$.

We begin with two applications of Corollary 20.

Corollary 22. *The following evaluation holds*

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n}{n},\frac{y^n}{n},\frac{z^n}{n}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0) + \frac{S}{n^2} + \frac{\Delta f(0,0,0) - 2S}{2n^3} + o\left(\frac{1}{n^3}\right).$$

Proof. Let $u_n : [0,1] \to [0,1]$, $u_n(x) = \frac{x^n}{n}$. We have $\int_0^1 u_n(x)\,\mathrm{d}x = \frac{1}{n(n+1)}$, $\int_0^1 u_n^2(x)\,\mathrm{d}x = \frac{1}{n^2(2n+1)}$. We deduce that

$$\begin{aligned}\iiint\limits_{[0,1]^3} f\left(\frac{x^n}{n},\frac{y^n}{n},\frac{z^n}{n}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z &= f(0,0,0) + \frac{S}{n(n+1)} + \frac{\Delta f(0,0,0)}{n^2(2n+1)} \\ &\quad + \frac{2T}{n^2(n+1)^2} + o\left(\frac{1}{n^3}\right) \\ &= f(0,0,0) + \frac{S}{n(n+1)} + \frac{\Delta f(0,0,0)}{n^2(2n+1)} + o\left(\frac{1}{n^3}\right).\end{aligned}$$

Since $\frac{1}{n(n+1)} = \frac{1}{n^2} - \frac{1}{n^3} + o\left(\frac{1}{n^3}\right)$ and $\frac{1}{n^2(2n+1)} = \frac{1}{2n^3} + o\left(\frac{1}{n^3}\right)$, after some simple calculations we get the evaluation from the statement. □

Corollary 23. *The following evaluation holds*

$$\begin{aligned}&\iiint\limits_{[0,1]^3} f\left(\frac{x^n}{n(x^n+1)},\frac{y^n}{n(y^n+1)},\frac{z^n}{n(z^n+1)}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= f(0,0,0) + \frac{S\ln 2}{n^2} + \frac{(12\ln 2 - 6)\,\Delta f(0,0,0) - \pi^2 S}{12n^3} + o\left(\frac{1}{n^3}\right).\end{aligned}$$

Proof. Let $u_n : [0,1] \to [0,1]$, $u_n(x) = \frac{x^n}{n(x^n+1)}$. We get

$$\iiint\limits_{[0,1]^3} f(u_n(x), u_n(y), u_n(z))\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0)$$

$$+S\int_0^1 u_n(x)\,\mathrm{d}x + \Delta f(0,0,0)\int_0^1 u_n^2(x)\,\mathrm{d}x$$

$$+2T\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)^2 + o\left(\int_0^1 u_n^2(x)\,\mathrm{d}x\right).$$

We now use the well-known results that if $\varphi : [0,1] \to \mathbb{R}$ is continuous then $\int_0^1 nx^n\varphi(x^n)\,\mathrm{d}x = \int_0^1 \varphi(x)\,\mathrm{d}x + o(1)$ and

$$\lim_{n\to\infty} n\left(\int_0^1 nx^n\varphi(x^n)\,\mathrm{d}x - \int_0^1 \varphi(x)\,\mathrm{d}x\right) = -\int_0^1 \frac{1}{x}\left(\int_0^x \varphi(t)\,\mathrm{d}t\right)\mathrm{d}x$$

or equivalently

$$\int_0^1 nx^n\varphi(x^n)\,\mathrm{d}x = \int_0^1 \varphi(x)\,\mathrm{d}x - \frac{1}{n}\int_0^1 \frac{1}{x}\left(\int_0^x \varphi(t)\,\mathrm{d}t\right)\mathrm{d}x + o\left(\frac{1}{n}\right)$$

for $\varphi(x) = \frac{1}{x+1}$, see [6, problem 3.13]. We deduce that $\int_0^1 \frac{nx^n}{x^n+1}\mathrm{d}x = \ln 2 + o(1)$, $\int_0^1 \frac{nx^n}{x^n+1}\mathrm{d}x = \ln 2 - \frac{\pi^2}{12n} + o\left(\frac{1}{n}\right)$; we have used that $\int_0^1 \frac{\ln(x+1)}{x}\mathrm{d}x = \frac{\pi^2}{12}$, see [7, Proposition 11]. Also

$$\begin{aligned}\int_0^1 u_n^2(x)\,\mathrm{d}x &= \frac{1}{n^3}\int_0^1 \frac{nx^{2n}}{(x^n+1)^2}\mathrm{d}x = \frac{1}{n^3}\left(\int_0^1 \frac{t\mathrm{d}t}{(t+1)^2} + o(1)\right)\\ &= \frac{2\ln 2 - 1}{2n^3} + o\left(\frac{1}{n^3}\right)\end{aligned}$$

and

$$\int_0^1 u_n(x)\,\mathrm{d}x = \frac{1}{n^2}\int_0^1 \frac{nx^n\mathrm{d}x}{x^n+1} = \frac{\ln 2}{n^2} - \frac{\pi^2}{12n^3} + o\left(\frac{1}{n^3}\right).$$

Hence $\left(\int_0^1 u_n(x)\,\mathrm{d}x\right)^2 = o\left(\frac{1}{n^3}\right)$ and we get that

$$\begin{aligned}&\iiint\limits_{[0,1]^3} f(u_n(x), u_n(y), u_n(z))\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ =\ & f(0,0,0) + S\left(\frac{\ln 2}{n^2} - \frac{\pi^2}{12n^3}\right) + \frac{(2\ln 2 - 1)\Delta f(0,0,0)}{2n^3} + o\left(\frac{1}{n^3}\right).\end{aligned}$$

The stated evaluation is obtained from this by simple calculations. □

We continue with an application of Proposition 19.

Corollary 24. *The following evaluation holds*

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n y^n}{n}, \frac{y^n z^n}{n}, \frac{z^n x^n}{n}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0) + \frac{S}{n^3} + \frac{\Delta f(0,0,0) - 8S}{4n^4} + o\left(\frac{1}{n^4}\right).$$

Proof. With the choice $u_n, v_n : [0,1] \to [0,1]$, $u_n(x) = \frac{x^n}{n}$, $v_n(x) = x^n$, we have $\int_0^1 u_n(x)\,\mathrm{d}x = \frac{1}{n(n+1)}$, $\int_0^1 v_n(x)\,\mathrm{d}x = \frac{1}{n+1}$, $\int_0^1 u_n^2(x)\,\mathrm{d}x = \frac{1}{n^2(2n+1)}$, $\int_0^1 v_n^2(x)\,\mathrm{d}x = \frac{1}{2n+1}$, $\int_0^1 u_n(x)\,v_n(x)\,\mathrm{d}x = \frac{1}{n(2n+1)}$. We therefore get

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n y^n}{n}, \frac{y^n z^n}{n}, \frac{z^n x^n}{n}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0) + \frac{S}{n(n+1)^2} + \frac{\Delta f(0,0,0)}{n^2(2n+1)^2} + \frac{2T}{n^2(n+1)^2(2n+1)} + o\left(\frac{1}{n^4}\right)$$

and hence

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n y^n}{n}, \frac{y^n z^n}{n}, \frac{z^n x^n}{n}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0) + \frac{S}{n(n+1)^2} + \frac{\Delta f(0,0,0)}{n^2(2n+1)^2} + o\left(\frac{1}{n^4}\right).$$

From $\frac{1}{n(n+1)^2} = \frac{1}{n^3} - \frac{2}{n^4} + o\left(\frac{1}{n^4}\right)$ we deduce that

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n y^n}{n}, \frac{y^n z^n}{n}, \frac{z^n x^n}{n}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0) + S\left(\frac{1}{n^3} - \frac{2}{n^4}\right) + \frac{\Delta f(0,0,0)}{4n^4} + o\left(\frac{1}{n^4}\right),$$

that is, the stated evaluation. □

The next result is an application of Proposition 21.

Corollary 25. *The following evaluation holds*

$$\iint\limits_{[0,1]^2} f\left(\frac{x^n}{n}, \frac{y^n}{n}, \frac{x^n y^n}{n}\right) \mathrm{d}x\mathrm{d}y = f(0,0,0) + \left(\frac{\partial f}{\partial x}(0,0,0) + \frac{\partial f}{\partial y}(0,0,0)\right)\frac{1}{n^2}$$
$$+ \left[\frac{\partial f}{\partial z}(0,0,0) - \frac{\partial f}{\partial x}(0,0,0) - \frac{\partial f}{\partial y}(0,0,0) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0,0) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0,0)\right]\frac{1}{n^3} + o\left(\frac{1}{n^3}\right).$$

Proof. Let $u_n, v_n : [0,1] \to [0,1]$, $u_n(x) = \frac{x^n}{n}$, $v_n(x) = x^n$ in Proposition 21. We get

$$\iint\limits_{[0,1]^2} f\left(\frac{x^n}{n}, \frac{y^n}{n}, \frac{x^n y^n}{n}\right) \mathrm{d}x\mathrm{d}y = f(0,0,0)$$

$$+\left(\frac{\partial f}{\partial x}(0,0,0) + \frac{\partial f}{\partial y}(0,0,0)\right)\frac{1}{n(n+1)} + \frac{\partial f}{\partial z}(0,0,0)\frac{1}{n(n+1)^2}$$

$$+\left(\frac{\partial^2 f}{\partial x^2}(0,0,0) + \frac{\partial^2 f}{\partial y^2}(0,0,0)\right)\frac{1}{n^2(2n+1)} + 2\frac{\partial^2 f}{\partial x \partial y}(0,0,0)\frac{1}{n^2(n+1)^2}$$

$$+o\left(\frac{1}{n^2(2n+1)}\right)$$

and hence

$$\iint\limits_{[0,1]^2} f\left(\frac{x^n}{n}, \frac{y^n}{n}, \frac{x^n y^n}{n}\right) \mathrm{d}x\mathrm{d}y = f(0,0,0)$$

$$+\left(\frac{\partial f}{\partial x}(0,0,0) + \frac{\partial f}{\partial y}(0,0,0)\right)\frac{1}{n(n+1)} + \frac{\partial f}{\partial z}(0,0,0)\cdot\frac{1}{n^3}$$

$$+\left(\frac{\partial^2 f}{\partial x^2}(0,0,0) + \frac{\partial^2 f}{\partial y^2}(0,0,0)\right)\frac{1}{2n^3} + o\left(\frac{1}{n^3}\right).$$

From the evaluation $\frac{1}{n(n+1)} = \frac{1}{n^2} - \frac{1}{n^3} + o\left(\frac{1}{n^3}\right)$, by simple calculations we get the evaluation from the statement. □

We end this paper with an application of Theorem 18.

Corollary 26. *The following evaluation holds*

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n y^n z^n}{n}, \frac{x^{2n} y^{2n} z^{2n}}{n}, \frac{x^{3n} y^{3n} z^{3n}}{n}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0) + \frac{A}{n^4}$$

$$+\frac{B}{n^5} + o\left(\frac{1}{n^5}\right),$$

where $A = \frac{\partial f}{\partial x}(0,0,0) + \frac{1}{8}\frac{\partial f}{\partial y}(0,0,0) + \frac{1}{27}\frac{\partial f}{\partial z}(0,0,0)$, $B = -3\frac{\partial f}{\partial x}(0,0,0) - \frac{3}{16}\frac{\partial f}{\partial y}(0,0,0) - \frac{1}{27}\frac{\partial f}{\partial z}(0,0,0) + \frac{\partial^2 f}{\partial x^2}(0,0,0)\cdot\frac{1}{2^3} + \frac{\partial^2 f}{\partial y^2}(0,0,0)\cdot\frac{1}{4^3} + \frac{\partial^2 f}{\partial z^2}(0,0,0)\cdot\frac{1}{6^3} + \frac{\partial^2 f}{\partial x \partial y}(0,0,0)\cdot\frac{2}{3^3} + \frac{\partial^2 f}{\partial x \partial z}(0,0,0)\cdot\frac{2}{4^3} + \frac{\partial^2 f}{\partial y \partial z}(0,0,0)\cdot\frac{2}{5^3}$.

Proof. From Theorem 18 and obvious calculations we get

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n y^n z^n}{n}, \frac{x^{2n} y^{2n} z^{2n}}{n}, \frac{x^{3n} y^{3n} z^{3n}}{n}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(0,0,0)$$

$$+\frac{\partial f}{\partial x}(0,0,0)\cdot\frac{1}{n(n+1)^3} + \frac{\partial f}{\partial y}(0,0,0)\cdot\frac{1}{n(2n+1)^3} + \frac{\partial f}{\partial z}(0,0,0)\cdot\frac{1}{n(3n+1)^3}$$

$$+\frac{\partial^2 f}{\partial x^2}(0,0,0)\cdot\frac{1}{n^2(2n+1)^3}+\frac{\partial^2 f}{\partial y^2}(0,0,0)\cdot\frac{1}{n^2(4n+1)^3}$$
$$+\frac{\partial^2 f}{\partial z^2}(0,0,0)\cdot\frac{1}{n^2(6n+1)^3}+2\frac{\partial^2 f}{\partial x\partial y}(0,0,0)\cdot\frac{1}{n^2(3n+1)^3}$$
$$+2\frac{\partial^2 f}{\partial x\partial z}(0,0,0)\cdot\frac{1}{n^2(4n+1)^3}+2\frac{\partial^2 f}{\partial y\partial z}(0,0,0)\cdot\frac{1}{n^2(5n+1)^3}+o\left(\frac{1}{n^5}\right).$$

From the evaluations $\frac{1}{n(n+1)^3}=\frac{1}{n^4}-\frac{3}{n^5}+o\left(\frac{1}{n^5}\right)$; $\frac{1}{n(2n+1)^3}=\frac{1}{8n^4}-\frac{3}{16n^5}+o\left(\frac{1}{n^5}\right)$; $\frac{1}{n(3n+1)^3}=\frac{1}{27n^4}-\frac{1}{27n^5}+o\left(\frac{1}{n^5}\right)$ we deduce that

$$\iiint\limits_{[0,1]^3} f\left(\frac{x^n y^n z^n}{n},\frac{x^{2n}y^{2n}z^{2n}}{n},\frac{x^{3n}3y^n z^{3n}}{n}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z=f(0,0,0)$$
$$+\frac{\partial f}{\partial x}(0,0,0)\cdot\left(\frac{1}{n^4}-\frac{3}{n^5}\right)+\frac{\partial f}{\partial y}(0,0,0)\cdot\left(\frac{1}{8n^4}-\frac{3}{16n^5}\right)$$
$$+\frac{\partial f}{\partial z}(0,0,0)\cdot\left(\frac{1}{27n^4}-\frac{1}{27n^5}\right)+\frac{\partial^2 f}{\partial x^2}(0,0,0)\cdot\frac{1}{2^3n^5}+\frac{\partial^2 f}{\partial y^2}(0,0,0)\cdot\frac{1}{4^3n^5}$$
$$+\frac{\partial^2 f}{\partial z^2}(0,0,0)\cdot\frac{1}{6^3n^5}+2\frac{\partial^2 f}{\partial x\partial y}(0,0,0)\cdot\frac{1}{3^3n^5}+2\frac{\partial^2 f}{\partial x\partial z}(0,0,0)\cdot\frac{1}{4^3n^5}$$
$$+2\frac{\partial^2 f}{\partial y\partial z}(0,0,0)\cdot\frac{1}{5^3n^5}+o\left(\frac{1}{n^5}\right).$$

After some simple calculations we get the evaluation from the statement. □

References

[1] N. Boboc, *Analiză matematică*, partea a II-a, curs tipărit, Bucureşti, 1993.

[2] N. Bourbaki, *Functions of a Real Variable: Elementary Theory*, Transl. from the 1976 French original, Springer, Berlin, 2004.

[3] C. George, *Exercises in Integration*, Problem Books in Mathematics, Springer-Verlag, New York, 1984.

[4] B. M. Makarov, M. G. Goluzina, A. A. Lodkin, A. N. Podkorytov, *Selected Problems in Real Analysis*, Transl. Math. Monographs, 107, A.M.S., Providence, RI, 1992.

[5] B. Makarov, A. Podkorytov, *Real Analysis: Measures, Integrals and Applications.* Translated from the 2011 Russian original. Universitext, Springer, London, 2013.

[6] D. Popa, *Exerciţii de analiză matematică*, Biblioteca S. S. M. R., Editura Mira, Bucureşti, 2007.

[7] D. Popa, The Lebesgue dominated convergence theorem in action, *Gaz. Mat., Ser. A* **35** (2017), No. 3–4, 13–25.

MATHEMATICAL NOTES

A sequence of integrals on the multidimensional unit cube revisited

Ulrich Abel[1)], Vitaliy Kushnirevych[2)]

Abstract. We present a new proof of the limit of a sequence of integrals on the multidimensional unit cube. This limit was recently derived by Popa. The approach is based on a reduction of multivariate integrals and an application of a method of asymptotic analysis. Finally, we state a slight generalization.

Keywords: Integral formulas, Integrals of Riemann and Lebesgue type.
MSC: Primary 28-01; Secondary 26B20, 26A42.

1. Introduction

Let n,k be positive integers. Recently, Popa [2] considered the multivariate integrals

$$I_{n,k}\left[f\right] := \int_{[0,1]^k} \left(1-x_1x_2\cdots x_k\right)^n f\left(\left(1-x_1x_2\cdots x_k\right)^n\right) \mathrm{d}\mathbf{x},$$

where $\mathrm{d}\mathbf{x} = \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_k$ and derived, for continuous functions $f:[0,1]\to\mathbb{R}$, the limit

$$\lim_{n\to\infty} \frac{n}{\left(\ln n\right)^{k-1}} I_{n,k}\left[f\right] = \frac{1}{\Gamma\left(k\right)} \int_0^1 f\left(x\right) \mathrm{d}x := I_k\left[f\right] \qquad \left(k\in\mathbb{N}\right). \tag{1}$$

We start with some notation. The gamma function $\Gamma\left(z\right) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t$, for $\mathrm{Re}\left(z\right) > 0$, interpolates the factorials $(k-1)! = \Gamma\left(k\right)$ on the positive integers k. By convention, we set $0! = 1$.

In what follows, we use the Landau notation. Let g,h be two real-valued functions defined on an open interval containing $a\in\mathbb{R}\cup\{-\infty,+\infty\}$. We write $g\left(x\right) = O\left(h\left(x\right)\right)$ as $x\to a$, if there exists a positive constant M such that the inequality $\left|g\left(x\right)\right| \le M\cdot h\left(x\right)$ is valid in a certain neighborhood of a. For the one-sided limit $x\to a+0$, $a\in\mathbb{R}$, the definition of Landau's symbol O is obvious.

In the special case $k=1$, Eq. (1) takes the form

$$n\cdot I_{n,1}\left[f\right] = n\int_0^1 x^n f\left(x^n\right)\mathrm{d}x = \int_0^1 f\left(t\right) t^{\frac{1}{n}}\mathrm{d}t \to \int_0^1 f\left(t\right)\mathrm{d}t = I_1\left[f\right]$$

as $n\to\infty$ (see the references given in [2]).

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The purpose of this note is a short proof of Popa's formula (1). The last section presents a generalization.

2. An alternative proof

First, we take advantage of the reduction formula

$$\int_{[0,1]^k} f(x_1 x_2 \cdots x_k)\,\mathrm{d}\mathbf{x} = \frac{1}{\Gamma(k)} \int_0^1 (-\ln t)^{k-1} f(t)\,\mathrm{d}t \tag{2}$$

for integrals of Beukers's type (see [1, Theorem 1]) in order to obtain

$$I_{n,k}[f] = \frac{1}{\Gamma(k)} \int_0^1 (-\ln t)^{k-1} (1-t)^n f((1-t)^n)\,\mathrm{d}t. \tag{3}$$

As the next step we show Eq. (1) for monomials $e_r(x) = x^r$ $(r = 0, 1, 2, \ldots)$. We study $I_{n,k}[e_r]$ by using relation (3). A change of variable $t = 1 - e^{-s}$ leads to

$$I_{n,k}[e_r] = \frac{1}{\Gamma(k)} \int_0^\infty \left(-\ln\left(1-e^{-s}\right)\right)^{k-1} e^{-ms}\mathrm{d}s$$

with $m = (r+1)n + 1$. In the case $k = 1$, it immediately follows

$$\lim_{n\to\infty} n \cdot I_{n,1}[e_r] = \lim_{n\to\infty} \frac{n}{(r+1)n+1} = \frac{1}{r+1} = I_1[e_r].$$

Now we deal with $I_{n,k}[e_r]$, for $k \geq 2$. Differentiating

$$\int_0^\infty s^{\lambda-1} e^{-ms}\mathrm{d}s = m^\lambda \Gamma(\lambda)$$

k times with respect to λ, we obtain

$$\begin{aligned}\int_0^\infty s^{\lambda-1} (-\ln s)^k e^{-ms}\mathrm{d}s &= \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^k \left(m^{-\lambda}\Gamma(\lambda)\right) \\ &= m^{-\lambda} \sum_{j=0}^k (-1)^j \binom{k}{j} \Gamma^{(j)}(\lambda) (\ln m)^{k-j}.\end{aligned}$$

This implies (put $\lambda = 1$) the asymptotic formula

$$\int_0^\infty (-\ln s)^k e^{-ms}\mathrm{d}s = \frac{1}{m} (\ln m)^k + O\left(\frac{1}{m} (\ln m)^{k-1}\right) \qquad (m \to \infty).$$

Because

$$\left(-\ln\left(1-e^{-s}\right)\right)^k = \left(-\ln s - \ln\left(\frac{1-e^{-s}}{s}\right)\right)^k = (-\ln s)^k + O\left((-\ln s)^{k-1}\right)$$

as $s \to 0+$, since $\left(1-e^{-s}\right)/s = 1+O\left(s\right)$ as $s \to 0+$, we can choose a constant c with $0 < c < e^{-1}$, such that

$$\int_0^{\infty} \left(-\ln\left(1-e^{-s}\right)\right)^k e^{-ms}\mathrm{d}s = \int_0^c \left(-\ln s\right)^k e^{-ms}\mathrm{d}s + O\left(\int_0^c \left(-\ln s\right)^{k-1} e^{-ms}\mathrm{d}s\right)$$

as $m \to +\infty$. Here we tacitly used the fact that, for each $c > 0$,

$$\int_c^{\infty} \left(-\ln\left(1-e^{-s}\right)\right)^k e^{-ms}\mathrm{d}s = O\left(\exp\left(-cm/2\right)\right) \qquad \left(m \to +\infty\right)$$

(cf. [3, Eq. (2.21) on page 69]). This approach is valid also in a more general setting (see [3, Theorem 2 on page 70]). Combining the above formulas we obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{n}{\left(\ln n\right)^k} I_{n,k+1}\left[e_r\right] &= \frac{1}{k!}\lim_{n\to\infty}\left(\frac{\ln\left(\left(r+1\right)n+1\right)}{\ln n}\right)^k \frac{n}{\left(r+1\right)n+1} \\ &= \frac{1}{k!}\cdot\frac{1}{r+1} = I_{k+1}\left[e_r\right].\end{aligned}$$

By linearity, the limit (1) is valid for each polynomial f. As shown in [2] the desired result follows, for each continuous function f, by the Weierstraß approximation theorem. In order to keep the paper self-contained, we repeat the density argument. Put $J_{n,k}\left[f\right] := \left(n/\left(\ln n\right)^{k-1}\right) I_{n,k}\left[f\right]$. Given $\varepsilon > 0$, we can choose a polynomial p approximating $f \in C\left[0,1\right]$ such that $\left|f\left(x\right) - p\left(x\right)\right| < \varepsilon$ for all $x \in \left[0,1\right]$. Furthermore, choose an integer N, such that $\left|J_{n,k}\left[p\right] - I_k\left[p\right]\right| < \varepsilon$ and $\left|J_{n,k}\left[e_0\right] - I_k\left[e_0\right]\right| < 1$, for $n > N$. Noting that $I_k\left[e_0\right] = 1/\Gamma\left(k\right)$, we conclude that

$$\begin{aligned}\left|J_{n,k}\left[f\right] - I_k\left[f\right]\right| &\leq \left|J_{n,k}\left[f-p\right]\right| + \left|J_{n,k}\left[p\right] - I_k\left[p\right]\right| + \left|I_k\left[p-f\right]\right| \\ &< \varepsilon\cdot\left|J_{n,k}\left[e_0\right]\right| + \varepsilon + \varepsilon\cdot\left|I_k\left[e_0\right]\right| < 2\varepsilon + \varepsilon + \varepsilon,\end{aligned}$$

for $n > N$.

3. A generalization

Let $C\left[0,1\right]$ denote the linear space of continous functions $f : \left[0,1\right] \to \mathbb{R}$. Define the linear operator $L : C\left[0,1\right] \to C\left[0,1\right]$ by $\left(Lf\right)\left(0\right) = f\left(0\right)$ and

$$\left(Lf\right)\left(x\right) = \frac{1}{x}\int_0^x f\left(t\right)\mathrm{d}t \qquad \left(0 < x \leq 1\right).$$

Denote its iterates by $L^j = L \circ L^{j-1}$ $\left(j \in \mathbb{N}\right)$, where L^0 is the identity operator on $C\left[0,1\right]$. In particular, one has $L^j e_r = \left(r+1\right)^{-j} e_r$ $\left(r = 0,1,2,\ldots\right)$.

Furthermore, if we approximate a function $f \in C[0,1]$ by a polynomial p such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [0,1]$, we conclude that

$$|(L(f-p))(x)| \leq \frac{1}{x} \int_0^x \varepsilon \, \mathrm{d}t = \varepsilon \qquad (0 < x \leq 1),$$

and, by mathematical induction $\left|\left(L^j(f-p)\right)(x)\right| \leq \varepsilon$, for all $x \in [0,1]$ and $j \in \mathbb{N}$. Following the lines of Section 2 one can show the following generalization of Eq. (1).

Theorem 1. *Let n, k, j be positive integers. For continuous functions $f : [0,1] \to \mathbb{R}$, the equality*

$$\lim_{n\to\infty} \frac{n^j}{(\ln n)^{k-1}} \int_{[0,1]^k} (x_1 x_2 \cdots x_k)^{j-1} (1 - x_1 x_2 \cdots x_k)^n f((1 - x_1 x_2 \cdots x_k)^n) \, \mathrm{d}\mathbf{x}$$
$$= \frac{1}{\Gamma(k)} \int_0^1 \left(L^{j-1} f\right)(x) \, \mathrm{d}x$$

holds, where $\mathrm{d}\mathbf{x} = \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_k$.

Popa's result is the special case $j = 1$.

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References

[1] U. Abel, V. Kushnirevych, Reducing multiple integrals of Beukers's type, *Amer. Math. Monthly* **127** (2020), no. 10, 918–926, DOI: 10.1080/00029890.2020.1815477

[2] D. Popa, The limit of a sequence of integrals on the k-dimensional unit cube, *Gazeta Matematică seria A* **117** (2020), no. 1–2, 18–23.

[3] R. Wong, *Asymptotic Approximations of Integrals.* Classics in Applied Mathematics, 34. SIAM, Philadelphia, PA, 2001.

PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of November 2021**.

PROPOSED PROBLEMS

513. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ which verify the identity

$$xf'(x) + kf(-x) = x^2 \quad \text{for all } x \in \mathbb{R},$$

where $k \geq 1$ is an integer.

Poposed by Vasile Pop and Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

514. Evaluate the integral

$$\int_0^1 \frac{\log^n x}{\sqrt{x(1-x)}}\,\mathrm{d}x,$$

where n is a positive integer.

Proposed by Mircea Ivan, Technical University of Cluj-Napoca, Romania.

515. Let $S = \{(\alpha, \beta, \gamma) \in (0, \pi/2)^3 \; : \; \alpha + \beta + \gamma = \pi\}$. On S we define the real valued functions

$$a := a(\alpha, \beta, \gamma) = \sqrt{\sin^4 \beta + \sin^4 \gamma - 2\sin^2 \beta \sin^2 \gamma \cos 2\alpha},$$

$$b := b(\alpha, \beta, \gamma) = \sqrt{\sin^4 \alpha + \sin^4 \gamma - 2\sin^2 \alpha \sin^2 \gamma \cos 2\beta},$$

$$c := a(\alpha, \beta, \gamma) = \sqrt{\sin^4 \alpha + \sin^4 \beta - 2\sin^2 \alpha \sin^2 \beta \cos 2\gamma}.$$

Prove that the function $f : S \to S$,

$$f(\alpha, \beta, \gamma) = \left(\arccos\left(\frac{b^2 + c^2 - a^2}{2bc}\right), \arccos\left(\frac{a^2 + c^2 - b^2}{2ac}\right), \arccos\left(\frac{a^2 + b^2 - c^2}{2ab}\right)\right),$$

is well defined.

Is f injective? Is it surjective?

Proposed by Leonard Giugiuc, Colegiul Naţional Traian, Drobeta Turnu Severin, Romania, and Abdilkadîr Altıntaş, Emirdağ, Afyonkarahisar, Turkey.

516. Let $n \geq 2$ be an integer. Let $v_1, \dots, v_{n-1}$ be some orthonormal vectors and let v be a unit vector in $\mathbb{R}^n$. We regard $v_1, \dots, v_{n-1}, v$ as column vectors, i.e., as $n \times 1$ matrices.

We consider the $n \times n$ matrix

$$A = v_1 \cdot v_1^T + \cdots + v_{n-1} \cdot v_{n-1}^T - v \cdot v^T.$$

If A is not invertible, prove that $A^2 = A$ and determine its rank.

Proposed by Marian Panţiruc, Gheorghe Asachi Tehnical University of Iaşi, Romania.

517. Calculate the sum

$$S = \sum_{p,q,r=1}^{\infty} \frac{3p+r}{5^{p+q+r} r(p+q)(q+r)(r+p)}.$$

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

518. Calculate the integral:

$$\int_0^{\infty} \frac{x^2\sqrt{x}\ln x}{x^4+x^2+1} \mathrm{d}x.$$

Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.

519. Calculate

$$\sum_{n=1}^{\infty} (-1)^n n^2 \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} - \ln 2 + \frac{1}{4n} - \frac{1}{16n^2} \right).$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

520. Let $(x_n)_{n\geq 0}$ be a sequence with $x_0 \in (0, \pi/2)$ and

$$x_{n+1} = \begin{cases} \sin x_n & n \text{ is even,} \\ \cos x_n & n \text{ is odd.} \end{cases}$$

Prove that $x_{2n} \to a$ and $x_{2n+1} \to b$ when $n \to \infty$, where a and b are two constants that are independent of the choice of x_0.

Also determine if the series

$$\sum_{n=1}^{\infty} |x_{2n} - a|^{\alpha} \quad \text{and} \quad \sum_{n=1}^{\infty} |x_{2n+1} - b|^{\alpha}$$

are convergent for $\alpha > 0$.

Proposed by Radu Strugariu, Gheorghe Asachi Tehnical University of Iaşi, Romania.

SOLUTIONS

496. Calculate the integral:

$$\int_0^\infty \frac{\arctan x}{\sqrt{x^4+x^2+1}}\mathrm{d}x.$$

`Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.`

Solution by the author. Put $A=\displaystyle\int_0^\infty \frac{\arctan x}{\sqrt{x^4+x^2+1}}\mathrm{d}x.$

We also consider the integral: $B=\displaystyle\int_0^\infty \frac{\operatorname{arccot} x}{\sqrt{x^4+x^2+1}}\mathrm{d}x.$ We have

$$A+B=\int_0^\infty \frac{\arctan x+\operatorname{arccot} x}{\sqrt{x^4+x^2+1}}\mathrm{d}x=\frac{\pi}{2}\int_0^\infty \frac{1}{\sqrt{x^4+x^2+1}}\mathrm{d}x.$$

We are going to calculate the integral

$$C=\int_0^\infty \frac{1}{\sqrt{x^4+x^2+1}}\mathrm{d}x.$$

We can write

$$C=\int_0^\infty \frac{1}{\sqrt{x^4+x^2+1}}\mathrm{d}x=\int_0^1 \frac{1}{\sqrt{x^4+x^2+1}}\mathrm{d}x+\int_1^\infty \frac{1}{\sqrt{x^4+x^2+1}}\mathrm{d}x.$$

In the second integral we make the variable change $x=\dfrac{1}{t}$. We have

$$\frac{1}{\sqrt{x^4+x^2+1}}\mathrm{d}x=\frac{1}{\sqrt{\frac{1}{t^4}+\frac{1}{t^2}+1}}\left(-\frac{1}{t^2}\right)\mathrm{d}t=-\frac{1}{\sqrt{t^4+t^2+1}}\mathrm{d}t.$$

Hence $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^4+x^2+1}}\mathrm{d}x=-\int_1^0 \frac{1}{\sqrt{x^4+x^2+1}}\mathrm{d}x=\int_0^1 \frac{1}{\sqrt{t^4+t^2+1}}\mathrm{d}t.$

It follows that $C=2\displaystyle\int_0^1 \frac{1}{\sqrt{t^4+t^2+1}}\mathrm{d}t.$

We will show that the integral C can also be expressed using the complete elliptic integral of the first kind, which is defined by

$$K(k)=\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\theta}}\mathrm{d}\theta\,,\ \text{with} -1\le k\le 1.$$

More exactly, we will show that $C=K\left(\dfrac{1}{2}\right)$. To prove this, we write the right-hand side as

$$K\left(\frac{1}{2}\right)=\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\frac{1}{4}\sin^2\theta}}\mathrm{d}\theta.$$

Substitute $t = \tan\dfrac{\theta}{2}$. We have $\sin\theta = \dfrac{2t}{1+t^2}$ and $\mathrm{d}t = \dfrac{1}{2}\sec^2\dfrac{\theta}{2}\mathrm{d}\theta = \dfrac{1+t^2}{2}\mathrm{d}\theta$, so $\mathrm{d}\theta = \dfrac{2}{1+t^2}\mathrm{d}t$. It follows that

$$K\left(\frac{1}{2}\right) = \int_0^1 \frac{1}{\sqrt{1-\frac{1}{4}\frac{4t^2}{(1+t^2)^2}}}\cdot\frac{2}{1+t^2}\mathrm{d}t = 2\int_0^1 \frac{1}{\sqrt{t^4+^2+1}}\mathrm{d}t = C.$$

In the integral A we make the change of variable $x = \dfrac{1}{t}$. We have $\arctan x = \operatorname{arctan} t$, and as seen above, $\dfrac{1}{\sqrt{x^4+x^2+1}}\mathrm{d}x = -\dfrac{1}{\sqrt{t^4+t^2+1}}\mathrm{d}t$. We get

$$\int_0^\infty \frac{\arctan x}{\sqrt{x^4+x^2+1}}\mathrm{d}x = -\int_\infty^0 \frac{\arctan t}{\sqrt{t^4+t^2+1}}\mathrm{d}x,$$

i.e., $A = B$. Since $A+B = \dfrac{\pi}{2}C = \dfrac{\pi}{2}K\left(\dfrac{1}{2}\right)$, we get $A = \dfrac{\pi}{4}K\left(\dfrac{1}{2}\right)$. $\square$

Note from the Editor. We received similar proofs from Daniel Văcaru, from Piteşti, Romania and Sean Stewart, from Bomaderry, NSW, Australia.

Sean Stewart uses a different substitution to get $\int_0^1 \frac{1}{\sqrt{t^4+t^2+1}}\mathrm{d}t = \frac{1}{2}K(\frac{1}{2})$. Namely, he takes $x = \tan\theta$ instead of $\tan\frac{\theta}{2}$. He gets

$$\int_0^1 \frac{1}{\sqrt{t^4+t^2+1}}\mathrm{d}t = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sqrt{\tan^4\theta+\tan^2\theta+1}}\mathrm{d}\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sqrt{\sec^4\theta-\tan^2\theta}}\mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{\mathrm{d}\theta}{\sqrt{1-\sin^2\theta\cos^2\theta}} = \int_0^{\frac{\pi}{4}} \frac{\mathrm{d}\theta}{\sqrt{1-\frac{1}{4}\sin^2 2\theta}}.$$

Then, after the substitution $\theta \mapsto \theta/2$, he gets

$$\int_0^{\frac{\pi}{4}} \frac{\mathrm{d}\theta}{\sqrt{1-\frac{1}{4}\sin^2 2\theta}} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-\frac{1}{4}\sin^2\theta}} = \frac{1}{2}K\big(\frac{1}{2}\big).$$

497. Let $n \geq 4$ and let $a_1, \ldots, a_n$ be nonzero real numbers such that $\dfrac{1}{a_1} + \cdots + \dfrac{1}{a_n} = 0$. Prove that

$$\left(\frac{1}{a_1^2}+\cdots+\frac{1}{a_n^2}\right)\sum_{i<j}(a_i-a_j)^2 \geq n^3.$$

When do we have equality?

`Proposed by Leonard Giugiuc, Traian National College, Drobeta Turnu Severin, Romania.`

Solution by the author. We define the $3 \times n$ matrix

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_n \\ \frac{1}{a_1} & \cdots & \frac{1}{a_n} \end{pmatrix}.$$

From

$$AA^T = \begin{pmatrix} n & \sum_{i=1}^n a_i & \sum_{i=1}^n \frac{1}{a_i} \\ \sum_{i=1}^n a_i & \sum_{i=1}^n a_i^2 & n \\ \sum_{i=1}^n \frac{1}{a_i} & n & \sum_{i=1}^n \frac{1}{a_i^2} \end{pmatrix} = \begin{pmatrix} n & \sum_{i=1}^n a_i & 0 \\ \sum_{i=1}^n a_i & \sum_{i=1}^n a_i^2 & n \\ 0 & n & \sum_{i=1}^n \frac{1}{a_i^2} \end{pmatrix}$$

it follows that

$$\begin{aligned} \det(AA^T) &= \left(\sum_{i=1}^n \frac{1}{a_i^2}\right)\left(n\sum_{i=1}^n a_i^2 - \left(\sum_{i=1}^n a_i\right)^2\right) - n^3 \\ &= \left(\sum_{i=1}^n \frac{1}{a_i^2}\right)\sum_{i<j}(a_i - a_j)^2 - n^3. \end{aligned}$$

For every $i < j < k$ we denote by $A_{i,j,k}$ the 3×3 matrix

$$A_{i,j,k} = \begin{pmatrix} 1 & a & 1 \\ a_i & a_j & a_k \\ \frac{1}{a_i} & \frac{1}{a_j} & \frac{1}{a_k} \end{pmatrix}.$$

Then, by the Cauchy-Binet formula, we have

$$\begin{aligned} \left(\sum_{i=1}^n \frac{1}{a_i^2}\right)\sum_{i<j}(a_i - a_j)^2 - n^3 = \det(AA^T) &= \sum_{i<j<k} \det A_{i,j,k} \det A_{i,j,k}^T \\ &= \sum_{i<j<k} \det(A_{i,j,k})^2. \end{aligned}$$

It follows that $\left(\sum_{i=1}^n \frac{1}{a_i^2}\right)\sum_{i<j}(a_i - a_j)^2 \geq n^3$ with equality iff $\det A_{i,j,k} = 0$ $\forall i, j, k,\ i < j < k$.

But if we multiply the columns of $A_{i,j,k}$ by a_i, a_i and a_k, respectively, and permute the rows we get a Vandermonde matrix. So we have

$$\begin{aligned} a_i a_j a_k \det A_{i,j,k} = \det \begin{pmatrix} a_i & a_j & a_k \\ a_i^2 & a_j^2 & a_k^2 \\ 1 & 1 & 1 \end{pmatrix} &= \det \begin{pmatrix} 1 & 1 & 1 \\ a_i & a_j & a_k \\ a_i^2 & a_j^2 & a_k^2 \end{pmatrix} \\ &= (a_j - a_i)(a_k - a_i)(a_k - a_i). \end{aligned}$$

Hence the equality holds iff for every $i < j < k$ the numbers a_i, a_j, a_k are not mutually distinct, i.e., iff $|\{a_1, \ldots, a_n\}| \leq 2$. Since $\sum_{i=1}^n \frac{1}{a_n} = 0$, some of $a_1, \ldots, a_n$ are positive and some negative, so there are $a, b \in \mathbb{R}$, $a > 0 > b$, and some $1 \leq k \leq n-1$ such that the sequence $a_1, \ldots, a_n$ contains $n - k$

copies of a and k copies of b. Then $0 = \sum_{i=1}^{n} \frac{1}{a_i} = \frac{n-k}{a} + \frac{k}{b}$. It follows that $a = (n-k)c$ and $b = -kc$ for some $c > 0$.

In conclusion, the equality holds if and only if there is some $c > 0$ and some $1 \le k \le n-1$ such that the sequence $a_1, \ldots, a_n$ contains $n-k$ copies of $(n-k)c$ and k copies of $-kc$. $\square$

Solution by Marian Cucoaneş, Eremia Grigorescu Technological Highschool, Mărăşesti, Vrancea, Romania. We may assume that $a_1 \ge \cdots \ge a_n$. Since $\frac{1}{a_1} + \cdots + \frac{1}{a_n} = 0$, not all a_i have the same sign, so there is $1 \le k \le n-1$ such that $a_1 \ge \cdots \ge a_k > 0 > a_{k+1} \ge \cdots \ge a_n$.

Let $A = \{1, \ldots, k\}$ and $B = \{k+1, \ldots, n\}$. As $\sum_{1 \le i < j \le n} (a_i - a_j)^2 \ge \sum_{i \in A,\, j \in B} (a_i - a_j)^2$, it suffices to prove that

$$\left(\sum_{i=1}^{n} \frac{1}{a_i^2} \right) \sum_{i \in A,\, j \in B} (a_i - a_j)^2 \ge n^3.$$

For $j \in B$ we put $b_j = -a_j$, so that $b_j > 0$ for $j \in B$. Then the inequality from the hypothesis writes as $\frac{1}{a_1} + \cdots + \frac{1}{a_k} = \frac{1}{b_{k+1}} + \cdots + \frac{1}{b_n} =: X > 0$ and the inequality we want to prove writes as

$$\left(\frac{1}{a_1^2} + \cdots + \frac{1}{a_k^2} + \frac{1}{b_{k+1}^2} + \cdots + \frac{1}{b_n^2} \right) \sum_{i \in A,\, j \in B} (a_i + b_j)^2 \ge n^3. \tag{1}$$

We have

$$\begin{aligned} \sum_{i \in A,\, j \in B} (a_i + b_j)^2 =& (n-k)(a_1^2 + \cdots + a_k^2) + k(b_{k+1}^2 + \cdots + b_n^2) \\ &+ 2(a_1 + \cdots + a_k)(b_{k+1} + \cdots + b_n). \end{aligned} \tag{2}$$

Now the functions $f, g, h : (0, \infty) \to \mathbb{R}$, $f(x) = x^2$, $g(x) = x^{-2}$, and $h(x) = x^{-1}$ are convex. We apply Jensen's inequality for the functions f, g, h to the numbers $\frac{1}{a_1}, \ldots, \frac{1}{a_k} \in (0, \infty)$. Since $\frac{1}{a_1} + \cdots + \frac{1}{a_k} = X$, we have

$$\begin{aligned} \frac{X^2}{k} &= kf\left(\frac{X}{k}\right) \le f\left(\frac{1}{a_1}\right) + \cdots + f\left(\frac{1}{a_k}\right) = \frac{1}{a_1^2} + \cdots + \frac{1}{a_k^2}, \\ \frac{k^3}{X^2} &= kg\left(\frac{X}{k}\right) \le g\left(\frac{1}{a_1}\right) + \cdots + g\left(\frac{1}{a_k}\right) = a_1^2 + \cdots + a_k^2, \\ \frac{k^2}{X} &= kh\left(\frac{X}{k}\right) \le h\left(\frac{1}{a_1}\right) + \cdots + h\left(\frac{1}{a_k}\right) = a_1 + \cdots + a_k. \end{aligned}$$

Similarly, when we apply Jensen's inequality to $\frac{1}{b_{k+1}}, \ldots, \frac{1}{b_n}$, with $\frac{1}{b_{k+1}} + \cdots + \frac{1}{b_n} = X$, we get

$$\frac{1}{b_{k+1}^2} + \cdots + \frac{1}{b_n^2} \ge \frac{X^2}{n-k},\ b_{k+1}^2 + \cdots + b_n^2 \ge \frac{(n-k)^3}{X^2},\ b_{k+1} + \cdots + b_n \ge \frac{(n-k)^2}{X}.$$

It follows that

$$\frac{1}{a_1^2}+\cdots+\frac{1}{a_k^2}+\frac{1}{b_{k+1}^2}+\cdots+\frac{1}{b_n^2}\ge\frac{X^2}{k}+\frac{X^2}{n-k}=\frac{nX^2}{k(n-k)}$$

and, by (2), we also have

$$\sum_{i\in A,\, j\in B}(a_i+b_j)^2\ge(n-k)\cdot\frac{k^3}{X^2}+k\cdot\frac{(n-k)^3}{X^2}+2\cdot\frac{k^2}{X^2}\cdot\frac{(n-k)^2}{X^2}$$
$$=\frac{k(n-k)}{X^2}(k^2+(n-k)^2+2k(n-k))=\frac{k(n-k)}{X^2}\cdot n^2.$$

In conclusion, the right hand side of (1) is $\ge \frac{nX^2}{k(n-k)}\cdot\frac{k(n-k)}{X^2}\cdot n^2=n^3$, as claimed.

Since f,g,h are strictly convex, the equality in (2) holds iff $a_1=\cdots=a_k$ and $b_{k+1}=\cdots=b_n$, i.e., $a_{k+1}=\cdots=a_n$. But if $a_1=\cdots=a_k$ and $a_{k+1}=\cdots=a_n$, then $\sum_{1\le i<j\le n}(a_i-a_j)^2=\sum_{i\in A,\, j\in B}(a_i-a_j)^2$, so in fact the equality holds in the original inequality.

498. Let $A,B\in M_n(\mathbb{C})$ be two matrices such that

$$A^2-B^2-I_n=\frac{1}{3}(AB-BA).$$

Prove that:

(i) $\det(A^2-B^2)=\det(A-B)\det(A+B)=1$.

(ii) $(AB-BA)^n=0$.

Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

Solution by the author. Let $X=A+B$, $Y=A-B$, so that $A=\frac{1}{2}(X+Y)$ and $B=\frac{1}{2}(X-Y)$. Then the relation from the statement writes as

$$\frac{1}{4}((X+Y)^2-(X-Y)^2)-I_n=\frac{1}{12}((X+Y)(X-Y)-(X-Y)(X+Y)),$$

which implies that $2XY+YX=3I_n$.

Now XY and YX have the same characteristic polynomial, $P_{XY}(t)=P_{YX}(T)=:P(T)=(t-\lambda_1)\cdots(t-\lambda_n)$, with $\lambda_1,\dots,\lambda_n\in\mathbb{C}$. We have

$P(t) = \det(tI_n - XY)$ and also

$$\begin{aligned} P(t) &= \det(tI_n - YX) = \det(tI_n - 3I_n + 2XY) \\ &= \det\left((-2)\left(\frac{3-t}{2}I_n - XY\right)\right) = (-2)^n \det\left(\frac{3-t}{2}I_n - XY\right) \\ &= (-2)^n P\left(\frac{3-t}{2}\right) = (-2)^n \prod_{i=1}^n \left(\frac{3-t}{2} - \lambda_i\right) \\ &= \prod_{i=1}^n (-2)\left(\frac{3-t}{2} - \lambda_i\right) = \prod_{i=1}^n (t + 2\lambda_i - 3). \end{aligned}$$

Since $P(t) = \prod_{i=1}^n (t - \lambda_i) = \prod_{i=1}^n (t + 2\lambda_i - 3)$, we have that $\lambda_1, \ldots, \lambda_n$ are equal, in some order, to $3 - 2\lambda_1, \ldots, 3 - 2\lambda_n$, i.e., there is a permutation $\sigma \in S_n$ such that $\lambda_i = 3 - 2\lambda_{\sigma(i)}$, i.e., such that $\lambda_i - 1 = (-2)(\lambda_{\sigma(i)} - 1)$ $\forall 1 \leq i \leq n$.

Let $N > 0$ be an integer such that $\sigma^N = 1$. (Since $\sigma \in S_n$, we may take $N = |S_n| = n!$.) Then for every $1 \leq i \leq n$ we have

$$\begin{aligned} \lambda_i - 1 &= (-2)(\lambda_{\sigma(i)} - 1) = (-2)^2(\lambda_{\sigma^2(i)} - 1) = \cdots = (-2)^N(\lambda_{\sigma^N(i)} - 1) \\ &= (-2)^N(\lambda_i - 1). \end{aligned}$$

Hence $(1 - (-2)^N)(\lambda_i - 1) = 0$ and so $\lambda_i = 1$.

Thus we proved that $\lambda_1 = \cdots = \lambda_n = 1$ and so $P(t) = (t-1)^n$. We now prove our statements.

We clearly have

$$\det(A - B)\det(A + B) = \det(A + B)(A - B) = \det XY = \lambda_1 \cdots \lambda_n = 1.$$

From $A = \frac{1}{2}(X + Y)$, $B = \frac{1}{2}(X - Y)$, and $YX = 3I_n - 2XY$ we get

$$\begin{aligned} AB - BA &= \frac{1}{4}((X + Y)(X - Y) - (X - Y)(X + Y)) \\ &= \frac{1}{2}(YX - XY) = \frac{3}{2}(I_n - XY), \end{aligned}$$

which implies that

$$A^2 - B^2 = I_n + \frac{1}{3}(AB - BA) = I_n + \frac{1}{2}(I_n - XY) = \frac{1}{2}(3I_n - XY).$$

It follows that

$$\begin{aligned} \det(A^2 - B^2) &= \det\frac{1}{2}(3I_n - XY) = \frac{1}{2^n}\det(3I_n - XY) = \frac{1}{2^n}P_{XY}(3) \\ &= \frac{1}{2^n}(3-1)^n = 1, \end{aligned}$$

which concludes the proof of (i).

Next, we have

$$\begin{aligned}
P_{AB-BA}(t) &= \det(tI_n - (AB-BA)) = \det\left(tI_n - \frac{3}{2}(I_n - XY)\right) \\
&= \det\left(-\frac{3}{2}\right)\left(\left(-\frac{2}{3}t+1\right)I_n - XY\right) \\
&= \left(-\frac{3}{2}\right)^n \det\left(\left(-\frac{2}{3}t+1\right)I_n - XY\right) \\
&= \left(-\frac{3}{2}\right)^n P_{XY}\left(-\frac{2}{3}t+1\right) = \left(-\frac{3}{2}\right)^n\left(-\frac{2}{3}t+1-1\right)^n = t^n.
\end{aligned}$$

By the Cayley-Hamilton theorem, $(AB-BA)^n = 0$. □

Generalization by Cornel Băeţica. Let $A, B \in M_n(\mathbb{C})$ be two matrices such that

$$A^2 - B^2 + u(AB - BA) = vI_n,$$

where u, v are non-zero real numbers. Prove that:
(i) $\det(A^2 - B^2) = \det(A-B)\det(A+B) = v^n$.
(ii) $(AB - BA)^n = 0$.

By using the same substitutions, that is, $X = A + B$ and $Y = A - B$, we get

$$(1-u)XY + (1+u)YX = 2vI_n. \tag{1}$$

If $u = 1$, then $YX = vI_n$. It follows that $XY = vI_n$, and thus $XY - YX = 0$. The same conclusion holds for $u = -1$.

Suppose $u \neq \pm 1$. Then (1) is equivalent to

$$XY - \frac{u+1}{u-1}YX = \frac{2v}{1-u}I_n.$$

Now note that $\frac{u+1}{u-1} \neq -1, 0, 1$.

In the following we show that if $X, Y \in M_n(\mathbb{C})$ are such that

$$XY - aYX = bI_n, \tag{2}$$

with a, b non-zero real numbers, $a \neq \pm 1$, then $(XY - YX)^n = 0$.

Let us denote the characteristic polynomial of a matrix M by $P_M(T)$. We have

$$\begin{aligned}
P_{XY-YX}(T) &= \det\left(TI_n - (XY - YX)\right) = \det\left((T-b)I_n - (a-1)YX\right) \\
&= P_{(a-1)YX}(T-b) = P_{(a-1)XY}(T-b) \\
&= \det\left((T-b)I_n - (a-1)XY\right) = \det\left(TI_n - a(XY-YX)\right) \\
&= P_{a(XY-YX)}(T) = a^n P_{XY-YX}(a^{-1}T).
\end{aligned}$$

Writing

$$P_{XY-YX}(T) = a_0 + a_1T + \cdots + a_{n-1}T^{n-1} + T^n$$

we get

$$\begin{aligned}&a_0+a_1T+\cdots+a_{n-1}T^{n-1}+T^n\\&=a^n(a_0+a_1a^{-1}T+\cdots+a_{n-1}a^{-n+1}T^{n-1}+a^{-n}T^n).\end{aligned}$$

This leads to $a^{n-i}a_i=a_i$ for all $i=0,1,\dots,n-1$. Since $a^{n-i}\neq 1$ for all $i=0,1,\dots,n-1$ we obtain $a_i=0$ for all $i=0,1,\dots,n-1$, and thus $P_{XY-YX}(T)=T^n$. In particular, by Cayley-Hamilton Theorem we get $(XY-YX)^n=0$. Since $XY-YX=2(BA-AB)$ we get $(AB-BA)^n=0$ and (ii) is proved.

In order to show (i) notice that $XY=vI_n+\frac{u+1}{2}(XY-YX)$ and then $\det X\det Y=\det(XY)=\left(\frac{u+1}{2}\right)^nP_{XY-YX}\left(\frac{2v}{u+1}\right)=v^n$. On the other side, $\det(A^2-B^2)=\det\left(vI_n-u(AB-BA)\right)=u^nP_{AB-BA}\left(\frac{v}{u}\right)=v^n$.

Remark. Note that in (2) we may assume that a,b are non-zero complex numbers such that $a^j\neq 1$ for all $j=1,\dots,n$. Taking into account that $a=\frac{u+1}{u-1}$, it follows that the condition *u,v are non-zero complex numbers and u is not purely imaginary* can replace in the problem the condition *u,v are non-zero real numbers.*

499. Let $a,b\geq 0$. Calculate

$$\lim_{n\to\infty}\sqrt{n}\int_0^{\frac{\pi}{2}}\sqrt{a\sin^{2n}x+b\cos^{2n}x}\,\mathrm{d}x.$$

`Proposed by Ovidiu Furdui, Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

Solution by the authors. The limit equals $\left(\sqrt{a}+\sqrt{b}\right)\sqrt{\dfrac{\pi}{2}}$.

By making the substitution $t=\frac{\pi}{2}-x$ on the interval $[\frac{\pi}{4},\frac{\pi}{2}]$ we get

$$\begin{aligned}\sqrt{n}\int_0^{\frac{\pi}{2}}\sqrt{a\sin^{2n}x+b\cos^{2n}x}\mathrm{d}x=\sqrt{n}\int_0^{\frac{\pi}{4}}\sqrt{a\sin^{2n}x+b\cos^{2n}x}\mathrm{d}x\\+\sqrt{n}\int_0^{\frac{\pi}{4}}\sqrt{a\cos^{2n}x+b\sin^{2n}x}\mathrm{d}x.\end{aligned}\tag{1}$$

Let $I_n=\sqrt{n}\displaystyle\int_0^{\frac{\pi}{4}}\sqrt{a\sin^{2n}x+b\cos^{2n}x}\mathrm{d}x$. We have

$$\sqrt{b}\sqrt{n}\int_0^{\frac{\pi}{4}}\cos^nx\mathrm{d}x\leq I_n\leq\sqrt{a}\sqrt{n}\int_0^{\frac{\pi}{4}}\sin^nx\mathrm{d}x+\sqrt{b}\sqrt{n}\int_0^{\frac{\pi}{4}}\cos^nx\mathrm{d}x.\tag{2}$$

On the other hand

$$0\leq\sqrt{n}\int_0^{\frac{\pi}{4}}\sin^nx\mathrm{d}x\leq\sqrt{n}\cdot\frac{\pi}{4}\cdot\left(\frac{\sqrt{2}}{2}\right)^n$$

and it follows that

$$\lim_{n\to\infty} \sqrt{n} \int_0^{\frac{\pi}{4}} \sin^n x \,\mathrm{d}x = 0. \tag{3}$$

Also,

$$\sqrt{n} \int_0^{\frac{\pi}{4}} \cos^n x \,\mathrm{d}x = \sqrt{n} \int_0^{\frac{\pi}{2}} \cos^n x \,\mathrm{d}x - \sqrt{n} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^n x \,\mathrm{d}x.$$

We have

$$0 \le \sqrt{n} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^n x \,\mathrm{d}x \le \sqrt{n} \cdot \frac{\pi}{4} \cdot \left(\frac{\sqrt{2}}{2}\right)^n$$

and this implies that

$$\lim_{n\to\infty} \sqrt{n} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^n x \,\mathrm{d}x = 0.$$

On the other hand

$$\begin{aligned} \lim_{n\to\infty} \sqrt{n} \int_0^{\frac{\pi}{2}} \cos^n x \mathrm{d}x &= \lim_{n\to\infty} \sqrt{n} \cdot \frac{1}{2} \cdot \mathrm{B}\left(\frac{n+1}{2}, \frac{1}{2}\right) \\ &= \frac{1}{2} \lim_{n\to\infty} \sqrt{n} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \\ &= \frac{\sqrt{\pi}}{2} \cdot \lim_{n\to\infty} \sqrt{n} \cdot \frac{\left(\frac{n-1}{2}\right)!}{\left(\frac{n}{2}\right)!} = \sqrt{\frac{\pi}{2}}, \end{aligned}$$

where the last limit follows based on Stirling's formula. This implies that

$$\lim_{n\to\infty} \sqrt{n} \int_0^{\frac{\pi}{4}} \cos^n x \,\mathrm{d}x = \sqrt{\frac{\pi}{2}}. \tag{4}$$

Combining (2), (3) and (4) we get that

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} \sqrt{n} \int_0^{\frac{\pi}{4}} \sqrt{a \sin^{2n} x + b \cos^{2n} x} \mathrm{d}x = \sqrt{b}\sqrt{\frac{\pi}{2}}. \tag{5}$$

Similarly one can prove that

$$\lim_{n\to\infty} \sqrt{n} \int_0^{\frac{\pi}{4}} \sqrt{a \cos^{2n} x + b \sin^{2n} x} \mathrm{d}x = \sqrt{a}\sqrt{\frac{\pi}{2}}. \tag{6}$$

Combining (1), (5) and (6) one has that

$$\lim_{n\to\infty} \sqrt{n} \int_0^{\frac{\pi}{2}} \sqrt{a \sin^{2n} x + b \cos^{2n} x} \,\mathrm{d}x = \left(\sqrt{a} + \sqrt{b}\right) \sqrt{\frac{\pi}{2}}.$$

The problem is solved. □

Note from the Editor. We also received a solution from Seán Stewart, from Bomaderry, NSW, Australia.

500. Let C be a simplex in $\mathbb{R}^n$ with the vertices $A_1,\dots,A_{n+1}$ and let M be a point in the interior of C. For every $1\le i<j\le n+1$ we denote by $A_{i,j}$ the point where hyperplane generated by M and $A_1,\dots,\hat{A}_i,\dots,\hat{A}_j,\dots,A_n$ intersects the edge A_iA_j of C. We denote by D the convex hull of $\{A_{i,j} : 1\le i<j\le n+1\}$.

Prove that the volume of D is $\le (1-\frac{n+1}{2^n})V$, where V is the volume of C, and the equality is reached if and only if M is the centroid of C.

Proposed by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Costel Bălcău, University of Piteşti, and Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.

Solution by the authors. For convenience, for every $i<j$ we denote $A_{j,i}=A_{i,j}$ so that $A_{i,j}$ is defined for $i\ne j$.

Since M is in the interior of the simplex C, we have $M=c_1A_1+\cdots+c_{n+1}A_{n+1}$, where $c_1,\dots,c_{n+1}>0$ and $c_1+\cdots+c_{n+1}=1$. Moreover, M is the centroid of C iff $c_1=\cdots=c_{n+1}=\frac{1}{n+1}$.

We claim that $A_{i,j}=\frac{c_iA_i+c_jA_j}{c_i+c_j}$. Indeed, $A'_{i,j}:=\frac{c_iA_i+c_jA_j}{c_i+c_j}$ is a convex combination of A_i and A_j, so that $A'_{i,j}$ belongs to the edge A_iA_j. We also have $M=c_1A_1+\cdots+c_{n+1}A_{n+1}$ and $c_1+\cdots+c_{n+1}=1$, whence

$$A'_{i,j}=\frac{M-\sum_{k\ne i,j}c_kA_k}{c_i+c_j}\quad\text{and}\quad\frac{1-\sum_{k\ne i,j}c_k}{c_i+c_j}=\frac{c_i+c_j}{c_i+c_j}=1.$$

Thus $A'_{i,j}$ is an affine combination of M and A_k with $k\ne i,j$. Therefore $A'_{i,j}$ is on the hyperspace generated by M and A_k with $k\ne i,j$. By the definition of $A_{i,j}$, we have $A_{i,j}=A'_{i,j}$, as claimed.

The volume of D is equal to $V-\sum_{i=1}^{n+1}V_i$, where V_i is the volume of the simplex with the vertices A_i and $A_{i,j}$, with $j\ne i$. For very $j\ne i$ we have $A_{i,j}-A_i=\frac{c_iA_i+c_jA_j}{c_i+c_j}-A_i=\frac{c_j}{c_i+c_j}(A_j-A_i)$, so $A_{i,j}$ is on the edge A_iA_j with $|A_iA_{i,j}|=\frac{c_j}{c_i+c_j}|A_iA_j|$. It follows that $V_i=\prod_{j\ne i}\frac{c_j}{c_i+c_j}V$. Hence the volume of D is

$$V-\sum_{i=1}^{n+1}\prod_{j\ne i}\frac{c_j}{c_i+c_j}V=(1-f_n(c_1,\dots,c_{n+1}))V,$$

where $f_n(x_1,\dots,x_{n+1})=\sum_{i=1}^{n+1}\prod_{j\ne i}\frac{x_j}{x_i+x_j}$. We must prove that if

$$S=\{(x_1,\dots,x_{n+1})\in\mathbb{R}^n : x_1,\dots,x_{n+1}>0,\ x_1+\cdots+x_{n+1}=1\},$$

then $\min_{(x_1,\dots,x_{n+1})\in S}f_n(x_1,\dots,x_{n+1})=\frac{n+1}{2^n}$ and the minimum is reached only at $(x_1,\dots,x_{n+1})=(\frac{1}{n+1},\dots,\frac{1}{n+1})$.

Since the map f_n is homogeneous of degree 0, i.e., $f(tx_1,\dots,tx_{n+1})=f(x_1,\dots,x_{n+1})$ for every $t,x_1,\dots,x_{n+1}>0$, our statement is equivalent to $\min_{x_1,\dots,x_{n+1}>0}f_n(x_1,\dots,x_{n+1})=\frac{n+1}{2^n}$, with the minimum obtained only when $x_1=\cdots=x_{n+1}$. We have to prove this for $n\ge 2$, but we extend this result, minus the fact that the minimum is reached only when $x_1 = \cdots = x_{n+1}$, to $n = 0$ and 1. We will need these cases in an induction argument.

We have $f_0(x_1) = 1\ \forall x_1 > 0$ and $f_1(x_1, x_2) = \frac{x_1}{x_1+x_1} + \frac{x_2}{x_2+x_1} = 1$ $\forall x_1, x_2 > 0$. Since $\frac{2^0}{1} = \frac{2^1}{2} = 1$, our statement for $n = 0, 1$ is trivial.

Suppose now that $n \geq 2$. Note that if $x_1 = \ldots = x_{n+1} = x$ then each term $\prod_{j\neq i} \frac{x_j}{x_i+x_j}$ in the definition of $f_n(x_1, \ldots, x_{n+1})$ is equal to $(\frac{x}{x+x})^n = \frac{1}{2^n}$, so $f_n(x, \ldots, x) = \frac{n+1}{2^n}$.

First we prove that $\min_{(x_1,\ldots,x_{n+1})\in S} f(x_1, \ldots, x_{n+1})$ exists. Note that S is an open simplex of dimension n. For every $0 < \varepsilon < \frac{1}{n+1}$ we consider $S_\varepsilon = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^n : x_1, \ldots, x_{n+1} \geq \varepsilon,\ x_1 + \cdots + x_{n+1} = 1\}$. Then $S_\varepsilon \subset S$ is a closed simplex of dimension n containing $(\frac{1}{n+1}, \ldots, \frac{1}{n+1})$. Since S_ε is a compact set and f_n is continuous, $\min_{(x_1,\ldots,x_{n+1})\in S_\varepsilon} f_n(x_1, \ldots, x_{n+1})$ exists and it is at most $f_n(\frac{1}{n+1}, \ldots, \frac{1}{n+1}) = \frac{n+1}{2^n}$.

We prove that there is $\varepsilon > 0$ small enough such that $f_n(x_1, \ldots, x_{n+1}) > \frac{n+1}{2^n}$ for every $(x_1, \ldots, x_{n+1}) \in S \setminus S_\varepsilon$.

For every $0 \leq m < n$ we define

$$\delta_m = \left(\frac{(m+1)/2^m}{(n+1)/2^n}\right)^{\frac{1}{n-m}} - 1, \quad \text{so that} \quad (1 + \delta_m)^{n-m} = \frac{(m+1)/2^m}{(n+1)/2^n}.$$

Note that for $m \geq 1$ we have $2(m+1) > m+2$ and $\frac{m+1}{2^m} > \frac{m+2}{2^{m+1}}$. Hence

$$\frac{1}{2^0} = \frac{2}{2^1} > \frac{3}{2^2} > \cdots > \frac{n+1}{2^n}.$$

Consequently, $\frac{n+1}{2^n} < \frac{m+1}{2^m}$ for $0 \leq m < n$. Hence for every $0 \leq m < n$ we have $1 < \frac{(m+1)/2^m}{(n+1)/2^n} < 2^{n-m}$, i.e., $1 < (1 + \delta_m)^{n-m} < 2^{n-m}$, so $0 < \delta_m < 1$.

We now define $\varepsilon = \frac{\delta_0 \cdots \delta_n}{n+1}$. Since $0 < \delta_m < 1\ \forall m$, we have $0 < \varepsilon < \frac{1}{n+1}$.

Assume that $(x_1, \ldots, x_{n+1}) \in S \setminus S_\varepsilon$. Then $x_i < \varepsilon$ for some i. By reordering the variables, we may assume that $0 < x_1 \leq \cdots \leq x_{n+1}$ and so $x_1 < \varepsilon$. Since x_{n+1} is the largest term of the sum $x_1 + \cdots + x_{n+1} = 1$, we have

$$x_{n+1} \geq \frac{1}{n+1} = \frac{1}{\delta_0 \cdots \delta_n}\varepsilon > \frac{1}{\delta_0 \cdots \delta_n} x_1.$$

Hence

$$\delta_0 \cdots \delta_n > \frac{x_1}{x_{n+1}} = \frac{x_1}{x_2} \cdots \frac{x_n}{x_{n+1}}.$$

It follows that there is $0 \leq m \leq n-1$ with $\frac{x_{m+1}}{x_{m+2}} < \delta_m$.

If $i \leq m+1$ and $j \geq m+2$, then $x_i \leq x_{m+1}$ and $x_j \geq x_{m+2}$, so that $\frac{x_i}{x_j} \leq \frac{x_{m+1}}{x_{m+2}} < \delta_m$, which implies that $\frac{x_j}{x_i+x_j} = \frac{1}{1+x_i/x_j} > \frac{1}{1+\delta_m}$. Hence, if

$1 \le i \le m+1$, then

$$\prod_{1\le j\le n+1, j\ne i} \frac{x_j}{x_i+x_j} = \left(\prod_{1\le j\le m+1, j\ne i} \frac{x_j}{x_i+x_j}\right)\left(\prod_{j=m+2}^{n+1} \frac{x_j}{x_i+x_j}\right)$$
$$> \left(\prod_{1\le j\le m+1, j\ne i} \frac{x_j}{x_i+x_j}\right)\left(\frac{1}{1+\delta_m}\right)^{n-m}$$
$$= \frac{(n+1)/2^n}{(m+1)/2^m} \prod_{1\le j\le m+1, j\ne i} \frac{x_j}{x_i+x_j}.$$

It follows that

$$f_n(x_1,\dots,x_{n+1}) = \sum_{i=1}^{n+1} \prod_{1\le j\le n+1, j\ne i} \frac{x_j}{x_i+x_j} > \sum_{i=1}^{m+1} \prod_{1\le j\le n+1, j\ne i} \frac{x_j}{x_i+x_j}$$
$$> \frac{(n+1)/2^n}{(m+1)/2^m} \sum_{i=1}^{m+1} \prod_{1\le j\le m+1, j\ne i} \frac{x_j}{x_i+x_j}$$
$$= \frac{(n+1)/2^n}{(m+1)/2^m} f_m(x_1,\dots,x_{m+1})$$

But, by the induction hypothesis, we have $f_m(x_1,\dots,x_{m+1}) \ge \frac{m+1}{2^m}$, whence $f_n(x_1,\dots,x_{n+1}) > \frac{(n+1)/2^n}{(m+1)/2^m} f_m(x_1,\dots,x_{m+1}) \ge \frac{n+1}{2^n}$, as claimed.

Since $\min_{(x_1,\dots,x_{n+1})\in S_\varepsilon} f_n(x_1,\dots,x_{n+1})$ exists and it is at most $\frac{n+1}{2^n}$ and $f_n(x_1,\dots,x_{n+1}) > \frac{n+1}{2^n}$ $\forall (x_1,\dots,x_{n+1}) \in S\setminus S_\varepsilon$, we conclude that $\min_{(x_1,\dots,x_{n+1})\in S} f_n(x_1,\dots,x_{n+1})$ also exists and it is equal to the minimum of f_n on S_ε. As seen above, since f_n homogeneous of degree 0, this implies that $\min_{x_1,\dots,x_{n+1}>0} f_n(x_1,\dots,x_{n+1})$ also exists and is equal to the minimum of f_n on S.

Let $c_1,\dots,c_{n+1} > 0$ be such that

$$f(c_1,\dots,c_{n+1}) = \min_{(x_1,\dots,x_{n+1})\in S} f_n(x_1,\dots,x_{n+1}).$$

Then all partial derivatives of f_n at $(c_1,\dots,c_{n+1})$ are zero. We claim that $c_1 = \dots = c_{n+1}$. Assume the contrary. So, if l is an index for which it holds $c_l = \min\{c_1,\dots,c_{n+1}\}$, then at least one of the inequalities $c_i \ge c_l$ is strict. We show that $\frac{\mathrm{d}f_n}{\mathrm{d}x_l}(c_1,\dots,c_{n+1}) < 0$, which will result in a contradiction.

If $i \ne l$, then $\frac{\mathrm{d}}{\mathrm{d}x_l}\frac{x_l}{x_i+x_l} = \frac{x_i}{(x_i+x_l)^2}$ and all the other factors of the product $\prod_{j\ne i,l} \frac{x_j}{x_i+x_j}$ are independent of x_l. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x_l}\prod_{j\ne i}\frac{x_j}{x_i+x_j} = \frac{x_i}{(x_i+x_l)^2}\prod_{j\ne i,l}\frac{x_j}{x_i+x_j} = \frac{\prod_{j\ne l} x_j}{(x_i+x_l)^2\prod_{j\ne i,l}(x_i+x_j)}.$$

If $i \neq l$, then $\frac{\mathrm{d}}{\mathrm{d}x_l}\frac{x_i}{x_l+x_i} = -\frac{x_l}{(x_l+x_i)^2}$. It follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x_l}\prod_{j\neq l}\frac{x_j}{x_l+x_j} &= \sum_{i\neq l} -\frac{x_i}{(x_l+x_i)^2}\prod_{j\neq l,i}\frac{x_j}{x_l+x_j} \\ &= -\sum_{i\neq l}\frac{\prod_{j\neq l}x_j}{(x_i+x_l)^2\prod_{j\neq i,l}(x_l+x_j)}.\end{aligned}$$

By adding the formulas above, we get

$$\begin{aligned}&\frac{\mathrm{d}f_n}{\mathrm{d}x_l}(x_1,\dots,x_{n+1}) = \sum_{i\neq l}\frac{\mathrm{d}}{\mathrm{d}x_l}\prod_{j\neq i}\frac{x_j}{x_i+x_j} + \frac{\mathrm{d}}{\mathrm{d}x_l}\prod_{j\neq l}\frac{x_j}{x_l+x_j} \\ &= \sum_{i\neq l}\frac{\prod_{j\neq l}x_j}{(x_i+x_l)^2\prod_{j\neq i,l}(x_i+x_j)} - \sum_{i\neq l}\frac{\prod_{j\neq l}x_j}{(x_i+x_l)^2\prod_{j\neq i,l}(x_l+x_j)} \\ &= \prod_{j\neq l}x_j\sum_{i\neq l}\left(\frac{1}{(x_i+x_l)^2\prod_{j\neq i,l}(x_i+x_j)} - \frac{1}{(x_i+x_l)^2\prod_{j\neq i,l}(x_l+x_j)}\right).\end{aligned}$$

But $c_i \geq c_l$ for every $i \neq l$ and $c_i > c_l$ for at least one value of $i \neq l$. It follows that each term $\frac{1}{(c_i+c_l)^2\prod_{j\neq i,l}(c_i+c_j)} - \frac{1}{(c_i+c_l)^2\prod_{j\neq i,l}(c_l+c_j)}$ is ≤ 0 and at least one of them is < 0. Hence $\frac{\mathrm{d}f_n}{\mathrm{d}x_l}(c_1,\dots,c_{n+1}) < 0$, as claimed.

In conclusion, for every $c_1,\dots,c_{n+1} > 0$ with

$$f_n(c_1,\dots,c_{n+1}) = \min_{x_1,\dots,x_{n+1}>0} f_n(x_1,\dots,x_{n+1})$$

we have $c_1 = \cdots = c_{n+1} =: c$ and so
$\min_{x_1,\dots,x_{n+1}>0} f_n(x_1,\dots,x_{n+1}) = f_n(c,\dots,c) = \frac{n+1}{2^n}$. □

501. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then $f(x+y) - f(x) \geq yf'(x)$ $\forall x, y \in \mathbb{R}$ if and only if $n(f(x+1/n) - f(x)) \geq f'(x)$ $\forall x \in \mathbb{R}$ and for every positive integer n.

Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

Solution by the author. The "only if" part is trivial.

For the "if" part we prove that f is a convex function. Assuming the contrary, we infer that there are $a, b \in \mathbb{R}$ and $0 < t < 1$ such that $f(ta + (1-t)b) > tf(a) + (1-t)f(b)$. We define the function

$$\phi : [a,b] \to \mathbb{R}, \quad \phi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a).$$

Note that $\phi(a) = \phi(b) = 0$.

We have

$$\begin{aligned}\phi(ta+(1-t)b) &= f(ta+(1-t)b) - f(a) - \frac{f(b)-f(a)}{b-a}(ta+(1-t)b-a)\\ &= f(ta+(1-t)b) - (tf(a)+(1-t)f(b)) > 0.\end{aligned}$$

Let $M = \max_{x\in[a,b]} \phi(x)$, which exists because ϕ is continuous and $[a,b]$ is compact. We obviously have $M \geq \phi(ta+(1-t)b) > 0$. Define the set $A = \{x \in [a,b] : \phi(x) = M\}$. Then A is closed and $\emptyset \neq A \subseteq [a,b]$. Hence A is a non-empty compact set and so it has a maximum, say, $\max A = m$. From $\phi(m) = M > 0 = \phi(a) = \phi(b)$ it follows that $m \neq a, b$, i.e., $m \in (a,b)$. Then $\phi(m) = \max_{x\in[a,b]} f(x)$ implies $f'(m) = 0$.

Since $m < b$, there is an integer $n > 0$ such that $m + 1/n \leq b$ and so $m + 1/n \in [a,b]$. If we put $h = 1/n$, then $m + h \in [a,b]$ and the inequality $n(f(m+1/n) - f(m)) \geq f'(m)$, from the hypothesis, writes as $f(m+h) - f(m) \geq hf'(m)$.

Note that $f(x) = \phi(x) + f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$ entails $f'(x) = \phi'(x) + \frac{f(b)-f(a)}{b-a}$. Therefore the inequality $f(m+h) - f(m) \geq hf'(m)$ writes as

$$\begin{aligned}&\left(\phi(m+h) + f(a) + \frac{f(b)-f(a)}{b-a}(m+h-a)\right)\\ &\quad - \left(\phi(m) + f(a) + \frac{f(b)-f(a)}{b-a}(m-a)\right) \geq h\left(\phi'(m) + \frac{f(b)-f(a)}{b-a}\right),\end{aligned}$$

which is equivalent to $\phi(m+h) - \phi(m) \geq h\phi'(m) = 0$. In other words $\phi(m+h) \geq \phi(m) = M = \max_{x\in[a,b]} \phi(x)$, which implies that $\phi(m+h) = M$, that is $m + h \in A$. But this contradicts the fact that $m = \max A$.

Hence f is convex, which implies that one has $f(x+y) - f(x) \geq yf'(x)$ for all $x, y \in \mathbb{R}$. $\square$

502. Let $m \geq 0$ be an integer. Evaluate the series

$$\sum_{k=1}^{\infty} \frac{(x^m \log x)^{(k+m)}}{k!}, \qquad x > 1,$$

where $f^{(i)}$ is the derivative of order i of f.

`Proposed by Mircea Ivan, Technical University of Cluj-Napoca, Romania.`

Solution by the author. In order to calculate the required derivatives $(x^m \log x)^{(k+m)}$, $k = 1, 2, \ldots$, one can proceed with the Leibniz formula for the higher derivative of a product, which is quite laborious. We provide here a straightforward method.

Fix $k \geq 1$. For $m = 0$, we have

$$(\log x)^{(k)} = (-1)^{k-1}(k-1)! \cdot x^{-k}.$$

For $m \geq 1$, we obtain the recurrence relation:

$$\begin{aligned}(x^m \log x)^{(k+m)} &= \big((x^m \log x)'\big)^{(k+m-1)} = \big(m\, x^{m-1} \log x + x^{m-1}\big)^{(k+m-1)} \\ &= m \cdot \big(x^{m-1} \log x\big)^{(k+m-1)} + 0,\end{aligned}$$

hence,

$$(x^m \log x)^{(k+m)} = m! \cdot (\log x)^{(k)} = m!\, \cdot (-1)^{k-1}(k-1)!\, \cdot x^{-k}.$$

Now it is easy to find the answer:

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(x^m \log x)^{(k+m)}}{k!} &= m! \cdot \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^{-k} \overset{x^{-1} \in (0,1)}{=\!=\!=} m! \cdot \log\big(1 + x^{-1}\big) \\ &= m! \cdot \log \frac{1+x}{x}.\end{aligned}$$

Note. As pointed out by Mircea Rus, the derivative $(x^m \log x)^{(m+1)}$ can be calculated by using the Leibniz formula for $(fg)^{(n)}$. Then, with the help of some binomial formula, one gets $(x^m \log x)^{(m+1)} = m!x^{-1}$. From here we get $(x^m \log x)^{(k+m)} = m!(x^{-1})^{(k-1)} = m! \cdot (-1)^{k-1}(k-1)! \cdot x^{-k}$.

Solution by Mircea Rus, Technical University of Cluj-Napoca, Romania. Let $f(x) = x^m \log x$ $(x > 1)$ and fix x. Then

$$S := \sum_{k=1}^{\infty} \frac{(x^m \log x)^{(k+m)}}{k!} = \left(\sum_{k=1}^{\infty} \frac{f^{(k)}(x)}{k!}\right)^{(m)} = (f(x+1) - f(x))^{(m)}$$

where we used the power series expansion of the function f around x:

$$f(y) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}(y-x)^k, \quad y \in (0, 2x)$$

with $y := x + 1 \in (0, 2x)$ (it is an elementary task to show that the radius of convergence is x).

It remains to compute $(x^m \log x)^{(m)}$ for arbitrary $x > 1$ and $m \in \mathbb{N}$.

Denote $g_m(x) = (x^m \log x)^{(m)}$. Then $g_0(x) = \log x$ and

$$\begin{aligned}g_m(x) &= \big((x^m \log x)'\big)^{(m-1)} \\ &= \big(mx^{m-1} \log x + x^{m-1}\big)^{(m-1)} = m \cdot g_{m-1}(x) + (m-1)! \qquad (m \geq 1),\end{aligned}$$

hence

$$\frac{g_m(x)}{m!} = \frac{g_{m-1}(x)}{(m-1)!} + \frac{1}{m} \qquad (m \geq 1),$$

which leads to

$$\frac{g_m(x)}{m!} = \frac{g_0(x)}{0!} + \left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right) = \log x + H_m.$$

Concluding,

$$S = g_m(x+1) - g_m(x) = m! \cdot \log \frac{x+1}{x}.$$

Note from the Editor. We also received solutions from Seán Stewart from Bomaderry, NSW, Australia, Brian Bradie from Christopher Newport University Newport News, VA, USA, Ulrich Abel from Technische Hochschule Mittelhessen, Germany, and Daniel Văcaru from Piteşti, Romania. Their solutions are similar to the author's, in the sense that they first prove explicit formulas for the derivatives of $x^m \log x$. In the solutions by Bradie, Abel and Văcaru, they first calculate

$$\begin{aligned}(x^m \log x)^{(m)} &= \sum_{k=0}^{m} \binom{m}{k} (x^m)(m-k)(\log x)^{(k)} \\ &= m! \log x + \sum_{k=1}^{m} \binom{m}{k} \frac{m!}{k!} x^k \frac{(-1)^{k-1}(k-1)!}{x^k} = m! \log x + C,\end{aligned}$$

where C is a constant. From here one gets $(x^m \log x)^{(k+m)} = m! \frac{(-1)^{k-1}(k-1)!}{x^k}$ $\forall k \geq 1$ and so our sum writes as $m! \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{kx^k} = m! \log\left(1 + \frac{1}{x}\right)$.

Alternatively, as in Abel's proof, one may consider the function $f(z) = (z^m \log z)^{(m)} = m! \log z + C$, which is analytic in the open disk with center x and radius $x > 1$, and our sum writes as

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{f^{(k)}(x)}{k!} &= \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} 1^k - f(x) = f(x+1) - f(x) \\ &= m!(\log(x+1) - \log x) = m! \log\left(1 + \frac{1}{x}\right).\end{aligned}$$

503. The Poincaré half-space model of the non-Euclidean n-dimensional space is the upper half-space $\mathbb{H}_n = \{(x, y) : x \in \mathbb{R}^{n-1}, y > 0\}$. We regard the elements $x \in \mathbb{R}^{n-1}$ as a column vector, i.e., as an element of $\mathcal{M}_{n-1,1}(\mathbb{R})$.

Then the group of positively oriented isometries of $\mathbb{H}_n$ is made of the functions $f_{\alpha,A,a} : \mathbb{H}_n \to \mathbb{H}_n$, with $\alpha > 0$, $A \in O^+(n-1)$ and $a \in \mathbb{R}^{n-1}$, given by $(x, y) \mapsto \alpha(Ax + a, y)$ and the functions $g_{\alpha,A,r,a} : \mathbb{H}_n \to \mathbb{H}_n$, with $\alpha > 0$, $A \in O^-(n-1)$, $r, a \in \mathbb{R}^{n-1}$, given by $(x, y) \mapsto \alpha\left(\frac{A(x-r)}{|x-r|^2+y^2} + b, \frac{y}{|x-r|^2+y^2}\right)$.

Give a direct proof of the fact that if G is the set of all $f_{\alpha,A,a}$ and $g_{\alpha,A,r,a}$ then $(G, \circ)$ is a group.

Here $\circ$ denotes functional composition. Recall that the orthogonal group $O(n-1) = \{A \in \mathcal{M}_{n-1}(\mathbb{R}) : A^T A = I_{n-1}\}$ has a decomposition $O(n-1) = O^+(n-1) \cup O^-(n-1)$, where $O^\pm(n-1) = \{A \in O(n-1) : \det A = \pm 1\}$.

If $x = (x_1, \dots, x_{n-1})^T \in \mathbb{R}^{n-1}$, then $|x|$ denotes its Euclidean length, $|x|^2 = x_1^2 + \cdots + x_{n-1}^2$.

Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.

Solution by the author. We show that the composition of two functions $f, g \in G$ is always in G. We have four cases, corresponding to f and g being of the type $f_{\alpha,A,a}$ or $g_{\alpha,A,t,a}$.

I. $f = f_{\alpha,A,a}$, $g = f_{\beta,B,b}$. We have $f_{\beta,B,b}(x,y) = (\beta Bx + \beta b, \beta y)$ and

$$\begin{aligned} f_{\alpha,A,a} f_{\beta,B,b}(x,y) &= f_{\alpha,A,a}(\beta Bx + \beta b, \beta y) = \alpha\big(A(\beta Bx + \beta b) + a, \beta y\big) \\ &= \alpha\beta(ABx + Ab + \beta^{-1}a, y), \end{aligned}$$

so that $f_{\alpha,A,a} f_{\beta,B,b} = f_{\alpha\beta,AB,Ab+\beta^{-1}a} \in G$. (From $A, B \in O^+(n-1)$ we get $AB \in O^+(n-1)$.)

II. $f = f_{\alpha,A,a}$, $g = g_{\beta,B,s,b}$. With the help of the explicit formula $g_{\beta,B,s,b}(x,y) = (\beta\frac{B(x-s)}{|x-s|^2+y^2} + \beta b, \beta\frac{y}{|x-s|^2+y^2})$ we obtain

$$\begin{aligned} f_{\alpha,A,a} g_{\beta,B,s,b}(x,y) &= f_{\alpha,A,a}\left(\beta\frac{B(x-s)}{|x-s|^2+y^2} + \beta b, \beta\frac{y}{|x-s|^2+y^2}\right) \\ &= \alpha\left(A\big(\beta\frac{B(x-s)}{|x-s|^2+y^2} + \beta b\big) + a, \beta\frac{y}{|x-s|^2+y^2}\right) \\ &= \alpha\beta\left(\frac{AB(x-s)}{|x-s|^2+y^2} + Ab + \beta^{-1}a, \frac{y}{|x-s|^2+y^2}\right). \end{aligned}$$

Thus $f_{\alpha,A,a} g_{\beta,B,s,b}(x,y) = g_{\alpha\beta,AB,s,Ab+\beta^{-1}b} \in G$. (We have $A \in O^+(n-1)$ and $B \in O^-(n-1)$, so that $AB \in O^-(n-1)$.)

III. $f = g_{\alpha,A,r,a}$, $g = f_{\beta,B,b}$. We have $f_{\beta,B,b}(x,y) = (x', y')$, where we put $x' = \beta Bx + \beta b$, $y' = \beta y$, so

$$g_{\alpha,A,r,a} f_{\beta,B,b}(x,y) = g_{\alpha,A,r,a}(x',y') = \alpha\left(\frac{A(x'-r)}{|x'-r|^2 + y'^2} + a, \frac{y'}{|x'-r|^2 + y'^2}\right).$$

But $x' - r = \beta Bx + \beta b - r = \beta B(x-t)$, where $t = \beta^{-1}B^{-1}r - B^{-1}b$, so that $A(x'-r) = \beta AB(x-t)$ and

$$|x'-r|^2 + y'^2 = |\beta B(x-t)|^2 + (\beta y)^2 = \beta^2(|x-t|^2 + y^2).$$

(From $B \in O(n-1)$ we get $|\beta B(x-t)|^2 = \beta^2|B(x-t)|^2 = \beta^2|x-t|^2$.) In follows that

$$\begin{aligned} g_{\alpha,A,r,a} f_{\beta,B,b}(x,y) &= \alpha\left(\frac{\beta AB(x-t)}{\beta^2(|x-t|^2+y^2)} + a, \frac{\beta y}{\beta^2(|x-t|^2+y^2)}\right) \\ &= \frac{\alpha}{\beta}\left(\frac{AB(x-t)}{|x-t|^2+y^2} + \beta a, \frac{y}{|x-t|^2+y^2}\right). \end{aligned}$$

So $g_{\alpha,A,r,a} f_{\beta,B,b} = g_{\frac{\alpha}{\beta},AB,t,\beta a} = g_{\frac{\alpha}{\beta},AB,\beta^{-1}B^{-1}r - B^{-1}b,\beta a} \in G$. (We have used that $A \in O^-(n-1)$ and $B \in O^+(n-1)$ imply $AB \in O^-(n-1)$.)

IV. $f = g_{\alpha,A,r,a}$, $g = g_{\beta,B,s,b}$. We have

$$\begin{aligned} g_{\alpha,A,r,a}g_{\beta,B,s,b}(x,y) &= g_{\alpha,A,r,a}(x',y') \\ &= \alpha\left(A(x'-r)(|x'-r|^2+y'^2)^{-1}+a, y'(|x'-r|^2+y'^2)^{-1}\right), \end{aligned}$$

where

$$(x',y') = g_{\beta,B,s,b}(x,y) = \left(\beta\frac{B(x-s)}{|x-s|^2+y^2}+\beta b, \beta\frac{y}{|x-s|^2+y^2}\right).$$

We have $x'-r = \beta\frac{B(x-s)}{|x-s|^2+y^2}+\beta b-r = \beta\left(\frac{B(x-s)}{|x-s|^2+y^2}+c\right) = \beta\frac{B(x-s)+(|x-s|^2+y^2)c}{|x-s|^2+y^2}$, where $c = b-\beta^{-1}r$. We have two subcases:

(a) $c = 0$, i.e., $r = \beta b$. Then $x'-r = \beta\frac{B(x-s)}{|x-s|^2+y^2}$, so that

$$\begin{aligned} |x'-r|^2+y'^2 &= \left|\beta\frac{B(x-s)}{|x-s|^2+y^2}\right|^2+\left(\beta\frac{y}{|x-s|^2+y^2}\right)^2 \\ &= \frac{\beta^2}{\left(|x-s|^2+y^2\right)^2}\left(|B(x-s)|^2+y^2\right) = \frac{\beta^2}{|x-s|^2+y^2}. \end{aligned}$$

(We have $B\in O(n-1)$ and therefore $|B(x-s)|^2+y^2 = |x-s|^2+y^2$.)

It follows that $g_{\alpha,A,r,a}g_{\beta,B,s,b}(x,y) = \alpha\left(\frac{A(x'-r)}{|x'-r|^2+y'^2}+a, \frac{y'}{|x'-r|^2+y'^2}\right)$ writes as

$$\begin{aligned} &\alpha\left(A\beta\frac{B(x-s)}{|x-s|^2+y^2}\left(\frac{\beta^2}{|x-s|^2+y^2}\right)^{-1}+a, \beta\frac{y}{|x-s|^2+y^2}\left(\frac{\beta^2}{|x-s|^2+y^2}\right)^{-1}\right) \\ &\qquad = \alpha\left(\beta^{-1}AB(x-s)+a, \beta^{-1}y\right) = \frac{\alpha}{\beta}(ABx-ABs+\beta a, y). \end{aligned}$$

Thus $g_{\alpha,A,r,a}g_{\beta,B,s,b}(x,y) = f_{\frac{\alpha}{\beta},AB,\beta a-ABs}\in G$. (From $A,B\in O^-(n-1)$ it follows that $AB\in O^+(n-1)$.)

(b) $c\neq 0$, i.e., $r\neq\beta b$. Now we het

$$\begin{aligned} |x'-r|^2+y'^2 &= \left|\beta\frac{B(x-s)+(|x-s|^2+y^2)c}{|x-s|^2+y^2}\right|^2+\left(\beta\frac{y}{|x-s|^2+y^2}\right)^2 \\ &= \frac{\beta^2}{(|x-s|^2+y^2)^2}\left(\left|B(x-s)+(|x-s|^2+y^2)c\right|^2+y^2\right). \end{aligned}$$

We consider the following scalar product $\langle\cdot,\cdot\rangle : \mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\to\mathbb{R}$: if $u = (u_1,\dots,u_{n-1})^T$, $v = (v_1,\dots,v_{n-1})^T\in\mathbb{R}^{n-1}$ then $\langle u,v\rangle = u^Tv = v^Tu = u_1v_1+\cdots+u_{n-1}v_{n-1}$. We have $|u+v|^2 = |u|^2+|v|^2+2\langle u,v\rangle$. Moreover, if $C\in\mathcal{M}_{n-1}(\mathbb{R})$ then $\langle Cu,v\rangle = \langle u,C^Tv\rangle$. We also use the fact $B\in O(n-1)$, so $|B(x-s)| = |x-s|$. We get

$$\begin{aligned}
&\bigl|B(x-s) + \bigl(|x-s|^2+y^2\bigr)c\bigr|^2 + y^2 \\
&\quad = |B(x-s)|^2 + \bigl(|x-s|^2+y^2\bigr)^2|c|^2 + 2\bigl(|x-s|^2+y^2\bigr)\langle B(x-s), c\rangle + y^2 \\
&\quad = |x-s|^2 + \bigl(|x-s|^2+y^2\bigr)^2|c|^2 + 2\bigl(|x-s|^2+y^2\bigr)\langle x-s, B^Tc\rangle| + y^2 \\
&\quad = \bigl(|x-s|^2+y^2\bigr)\bigl(1 + \bigl(|x-s|^2+y^2\bigr)|c|^2 + 2\langle x-s, B^Tc\rangle\bigr) \\
&\quad = |c|^2\bigl(|x-s|^2+y^2\bigr)\left(\frac{1}{|c|^2} + |x-s|^2 + y^2 + 2\langle |x-s, \frac{B^Tc}{|c|^2}\rangle\right).
\end{aligned}$$

But $B^T \in O(n-1)$, so $|\frac{B^Tc}{|c|^2}|^2 = \frac{|B^Tc|^2}{|c|^4} = \frac{|c|^2}{|c|^4} = \frac{1}{|c|^2}$. Hence

$$\begin{aligned}
\frac{1}{|c|^2} + |x-s|^2 + y^2 + 2\langle x-s, \frac{B^Tc}{|c|^2}\rangle &= |x-s|^2 + |\frac{B^Tc}{|c|^2}|^2 + 2\langle x-s, \frac{B^Tc}{|c|^2}\rangle + y^2 \\
&= |x-s+\frac{B^Tc}{|c|^2}|^2 + y^2 = |x-t|^2 + y^2,
\end{aligned}$$

where $t = s - \frac{B^Tc}{|c|^2}$. In conclusion,

$$\begin{aligned}
|x'-r|^2 + y'^2 &= \frac{\beta^2}{(|x-s|^2+y^2)^2}(|B(x-s) + (|x-s|^2+y^2)c|^2 + y^2) \\
&= \frac{\beta^2}{(|x-s|^2+y^2)^2}|c|^2(|x-s|^2+y^2)(|x-t|^2+y^2) \\
&= \frac{\beta^2|c|^2(|x-t|^2+y^2)}{|x-s|^2+y^2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{x'-r}{|x'-r|^2+y^2} &= \beta\frac{B(x-s) + \bigl(|x-s|^2+y^2\bigr)c}{|x-s|^2+y^2}\left(\frac{\beta^2|c|^2(|x-t|^2+y^2)}{|x-s|^2+y^2}\right)^{-1} \\
&= \beta^{-1}|c|^{-2}\frac{B(x-s) + \bigl(|x-s|^2+y^2\bigr)c}{|x-t|^2+y^2}.
\end{aligned}$$

But $x-s = x-t-\frac{B^Tc}{|c|^2}$ and, again, $|\frac{B^Tc}{|c|^2}|^2 = \frac{1}{|c|^2}$. It follows that we have $|x-s|^2 = |x-t|^2 + \bigl|\frac{B^Tc}{|c|^2}\bigr|^2 - 2\langle x-t, \frac{B^Tc}{|c|^2}\rangle = |x-t|^2 + \frac{1}{|c|^2} - 2\langle B(x-t), \frac{c}{|c|^2}\rangle$.

Therefore

$$\begin{aligned}
&B(x-s)+\big(|x-s|^2+y^2\big)c \\
&= B\big(x-t-\frac{B^Tc}{|c|^2}\big)+\left(|x-t|^2+\frac{1}{|c|^2}-2\langle B(x-t),\frac{c}{|c|^2}\rangle+y^2\right)c \\
&= B(x-t)-\frac{BB^Tc}{|c|^2}+\big(|x-t|^2+y^2\big)c+\frac{c}{|c|^2}-2\langle B(x-t),\frac{c}{|c|^2}\rangle c \\
&= B(x-t)+(|x-t|^2+y^2)c-2\langle B(x-t),\frac{c}{|c|^2}\rangle c.
\end{aligned}$$

(We have $BB^T = I_{n-1}$, so $-\frac{BB^Tc}{|c|^2}$ and $\frac{c}{|c|^2}$ cancel each other.)

We regard $\langle B(x-t),\frac{c}{|c|^2}\rangle = \big(\frac{c}{|c|^2}\big)^T B(x-t) \in \mathbb{R}$ as a 1×1 matrix, which can be multiplied to the right by the $(n-1)\times 1$ matrix c. With $\langle B(x-t),\frac{c}{|c|^2}\rangle c = c(\frac{c}{|c|^2})^T B(x-t) = \frac{c^Tc}{|c|^2}B(x-t)$ we can rewrite the expression $B(x-s)+(|x-s|^2+y^2)c$ as

$$B(x-t)+(|x-t|^2+y^2)c-2\frac{c^Tc}{|c|^2}B(x-t) = (I_{n-1}-2\frac{c^Tc}{|c|^2})B(x-t)+(|x-t|^2+y^2)c.$$

Hence

$$\begin{aligned}
\frac{x'-r}{|x'-r|^2+y^2} &= \beta^{-1}|c|^{-2}\frac{\big(I_{n-1}-2\frac{c^Tc}{|c|^2}\big)B(x-t)+\big(|x-t|^2+y^2\big)c}{|x-t|^2+y^2} \\
&= \beta^{-1}|c|^{-2}\frac{\big(I_{n-1}-2\frac{c^Tc}{|c|^2}\big)B(x-t)}{|x-t|^2+y^2}+\beta^{-1}|c|^{-2}c.
\end{aligned}$$

We also have

$$\begin{aligned}
y'\big(|x'-r|^2+y^2\big)^{-1} &= \frac{\beta y}{|x-s|^2+y^2}\left(\frac{\beta^2|c|^2\big(|x-t|^2+y^2\big)}{|x-s|^2+y^2}\right)^{-1} \\
&= \beta^{-1}|c|^{-2}\frac{y}{|x-t|^2+y^2}.
\end{aligned}$$

Then $\alpha^{-1}g_{\alpha,A,r,a}g_{\beta,B,s,b}(x,y) = \Big(\frac{A(x'-r)}{|x'-r|^2+y'^2}+a,\frac{y'}{|x'-r|^2+y'^2}\Big)$ writes as

$$\begin{aligned}
&\beta^{-1}|c|^{-2}\left(A\left(\frac{\big(I_{n-1}-2\frac{c^Tc}{|c|^2}\big)B(x-t)}{|x-t|^2+y^2}+c\right)+a,\frac{y}{|x-t|^2+y^2}\right) \\
&= \beta^{-1}|c|^{-2}\left(\frac{A\big(I_{n-1}-2\frac{c^Tc}{|c|^2}\big)B(x-t)}{|x-t|^2+y^2}+c+\beta|c|^2a,\frac{y}{|x-t|^2+y^2}\right).
\end{aligned}$$

Thus $g_{\alpha,A,r,a}g_{\beta,B,s,b}(x,y) = g_{\alpha\beta^{-1}|c|^{-2},A(I_{n-1}-2\frac{c^Tc}{|c|^2})B,t,c+\beta|c|^2a}$. To prove that this function belongs to G, one needs to show that $A(I_{n-1} - 2\frac{c^Tc}{|c|^2})B \in O^-(n-1)$. As $A, B \in O^-(n-1)$, this is equivalent to $I_{n-1} - 2\frac{c^Tc}{|c|^2} \in O^-(n-1)$. If $C := \frac{c^Tc}{|c|^2}$, then $C = C^T$ and $C^2 = \frac{c^T(cc^T)c}{|c|^4}$. With $cc^T = |c|^2$, we obtain $C^2 = \frac{c^Tc}{|c|^2} = C$. It follows that $(I_{n-1} - 2C)^T(I_{n-1} - 2C) = (I_{n-1} - 2C)^2 = I_{n-1} - 4C + 4C^2 = I_{n-1}$ and $I_{n-1} - 2\frac{c^Tc}{|c|^2} = I_{n-1} - 2C \in O(n-1)$.

To compute the determinant of $I_{n-1} - 2\frac{c^Tc}{|c|^2}$ we use a well known result which states that if C is a $k \times l$ matrix and D is an $l \times k$ matrix then $X^l P_{CD}(X) = X^k P_{DC}(X)$. We take $X = 1$ and we get $P_{CD}(1) = P_{DC}(1)$, i.e., $\det(I_k - CD) = \det(I_l - DC)$. We take the matrices $C = \frac{c}{|c|^2}$ and $D = 2c^T$ of sizes $(n-1) \times 1$ and $1 \times (n-1)$, respectively, and get $\det(I_{n-1} - 2\frac{cc^T}{|c|^2}) = \det(I_1 - 2\frac{c^Tc}{|c|^2})$. But the right side is a 1×1 matrix, i.e., a number, so its determinant is itself. Hence $\det(I_{n-1} - 2\frac{cc^T}{|c|^2}) = 1 - 2\frac{c^Tc}{|c|^2} = 1 - 2\frac{|c|^2}{|c|^2} = -1$, which shows that $I_{n-1} - 2\frac{cc^T}{|c|^2} \in O^-(n-1)$.

Alternatively, we note that $\left(I_{n-1} - 2\frac{cc^T}{|c|^2}\right)c = c - 2\frac{cc^Tc}{|c|^2} = c - 2c = -c$. (We have $c^Tc = |c|^2$.) On the other hand, if $u \in \mathbb{R}^{n-1} = \mathcal{M}_{n-1,1}(\mathbb{R})$ is orthogonal on c, then $\langle c, u\rangle = c^Tu = 0$, so $(I_{n-1} - 2\frac{cc^T}{|c|^2})u = u - 2\frac{cc^Tu}{|c|^2} = u - 0 = u$. Thus $I_{n-1} - 2\frac{cc^T}{|c|^2}$ coincides with the symmetry with respect to c, which is known to belong to $O^-(n-1)$.

The identity map can be written as $f_{1,I_{n-1},0} \in G$. To conclude the proof we must prove that for every $f \in G$ there is $g \in G$ with $fg = gf = f_{1,I_{n-1},0}$. We consider the two possible cases for f.

If $f = f_{\alpha,A,a}$, then we look for an inverse of the type $g = f_{\beta,B,b}$. By case **I**, we have $f_{\alpha,A,a}f_{\beta,B,b} = f_{\alpha\beta,AB,Ab+\beta^{-1}a}$. So in order that $f_{\beta,B,b}$ is a right inverse of $f_{\alpha,A,a}$ we need that $\alpha\beta = 1$, $AB = I_{n-1}$ and $Ab + \beta^{-1}a = 0$, i.e., that $(\beta, B, b) = (\alpha^{-1}, A^{-1}, -\alpha A^{-1}a)$. (The third coordinate follows from $Ab = -\beta^{-1}a = -\alpha a$.) But if $(\beta, B, b) = (\alpha^{-1}, A^{-1}, -\alpha A^{-1}a)$, then also $(\alpha, A, a) = (\beta^{-1}, B^{-1}, -\beta B^{-1}b)$ and so $f_{\alpha,A,a}$ is a right inverse of $f_{\beta,B,b}$. Thus $f^{-1}_{\alpha,A,a} = f_{\alpha^{-1},A^{-1},-\alpha A^{-1}a}$.

If $f = g_{\alpha,A,r,a}$, then we are looking for an inverse $g = g_{\beta,B,s,b}$. In order that we are in case **IV**, we need that $r = \beta b$. Then $g_{\alpha,A,r,a}g_{\beta,B,s,b} = g_{\frac{\alpha}{\beta},AB,\beta a-ABs}$. Hence in order that $g_{\beta,B,s,b}$ is a right inverse of $g_{\alpha,A,r,a}$ one needs that $r = \beta b$ and $(\frac{\alpha}{\beta}, AB, \beta a - ABs) = (1, I_{n-1}, 0)$. This is equivalent to $(\beta, B, s, b) = (\alpha, A^{-1}, \alpha a, \alpha^{-1}r)$. (The formulas for s and b follow from $\alpha a = \beta a = ABs = s$ and $r = \beta b = \alpha b$.) But if $(\beta, B, s, b) = (\alpha, A^{-1}, \alpha a, \alpha^{-1}r)$, then we also have $(\alpha, A, r, a) = (\beta, B^{-1}, \beta b, \beta^{-1}s)$, so that $g_{\alpha,A,r,a}$ is a right inverse of $g_{\beta,B,s,b}$. Thus $g^{-1}_{\alpha,A,r,a} = g_{\alpha,A^{-1},\alpha a,\alpha^{-1}r}$. $\square$