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ARTICLES

Power series expansion for the pendulum angle function

Nicolae Anghel[1)]

Abstract. A complete power series expansion for the simple pendulum angle function is given, about the initial time $t = 0$. Its coefficients are expressed as trigonometric polynomials of the initial angle θ_0.

Keywords: Simple pendulum, Pendulum period, Power series expansion, Trigonometric polynomials.

MSC: Primary 34A34; Secondary 34C15.

The pendulum motion is a delight for the Grandfather clock and Physics enthusiasts [2, 3, 4, 5], but not so much for the Math ones. We set out to rectify this. The simple gravity pendulum is a weight attached at one end of a massless rod of length l and swinging freely from the other end, which is kept fixed. The idealized motion, free of air resistance, is planar and perpetual, a result of a uniform gravitational field g on the weight (identified with a mass point).

We are interested in the angle function $\theta(t)$ measuring (in radians) the signed angle formed by the rod and the vertical direction at time t. For definiteness, we assume that at time $t = 0$ the pendulum reaches its angular amplitude θ_0, $0 < \theta_0 < \pi$, at which instance its velocity vanishes, $\theta'(0) = 0$. The motion is periodic with period $T > 0$, and clearly $T/4$ is the first instance when θ vanishes.

The well-known initial value non-linear second order differential equation governing the angle function $\theta(t)$,

$$\theta''(t) + \frac{g}{l}\sin\theta(t) = 0, \qquad \theta(0) = \theta_0,\ \theta'(0) = 0, \tag{1}$$

[1)]Department of Mathematics, University of North Texas, Denton, TX 76203, anghel@unt.edu

is easily deduced by applying Newton's second law of dynamics to the angular length $l\theta(t)$, which is subject to only one force, the tangential component $g\sin\theta(t)$ of gravity, acting against the motion.

Multiplying Equation (1) by $2\theta'(t)$ leads to the first integral of motion,

$$\left(\theta'(t)\right)^2 = \frac{2g}{l}(\cos\theta(t) - \cos\theta_0). \tag{2}$$

Even Equation (2) can be derived by physical considerations, pertaining to the conservation of the gravitational potential energy and the kinetic energy associated to an infinitesimal vertical drop of the pendulum weight [5].

The immediate conclusion of Equation (2) is that the exact expression of $\theta(t)$ is an elliptic integral, and as such not expressible in terms of elementary functions. Also, by locally inverting θ, Equation (2) yields the value for the period T of the motion as the improper elliptic integral

$$T = 4\sqrt{\frac{l}{2g}}\int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\,\mathrm{d}\theta. \tag{3}$$

A noteworthy and fast converging equivalent expression of T [3] is given via Gauss' arithmetic-geometric mean of 1 and $\cos(\theta_0/2)$, by

$$T = 2\pi\sqrt{\frac{l}{g}}\frac{1}{\operatorname{agm}(1, \cos(\theta_0/2))}. \tag{4}$$

As a consequence of well-known existence and uniqueness results for the solutions of initial value differential equations, $\theta(t)$ is analytic in a neighborhood of $t = 0$. What is then its power series expansion about $t = 0$?

We mention in passing a Fourier series expansion for $\theta(t)$ [7],

$$\theta(t) = 8\sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1}\frac{q^{l+1/2}}{1+q^{2l+1}}\cos\left((2l+1)\frac{2\pi}{T}t\right), \tag{5}$$

where $q = \exp\left(-\frac{K'(\sin(\theta_0/2))}{K(\sin(\theta_0/2))}\right)$, and K is the elliptic integral function

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2 u}}\,\mathrm{d}u, \quad -1 < k < 1.$$

Also, in [1] a hybrid power series expansion is presented for t, as a function of $\dfrac{\sin(\theta/2)}{\sin(\theta_0/2)}$.

The power series expansion for $\theta(t)$ about $t = 0$ is necessarily complicated. Just look at the Faà di Bruno's recursive expression for the derivatives

$\theta^{(n)}(0)$, given by formulas [6]

$$\theta^{(n+2)}(0) = -\frac{g}{l}\sum \frac{n!}{m_1!m_2!\cdots m_n!}\cdot \sin^{(m_1+m_2+\cdots+m_n)}(\theta_0)\cdot \prod_{j=1}^{n}\left(\frac{\theta^{(j)}(0)}{j!}\right)^{m_j},$$

where the sum is to be performed over all n-tuples of non-negative integers satisfying $1\cdot m_1+2\cdot m_2+\cdots+n\cdot m_n = n$. Fortunately, there is a way around this difficulty provided by the following lemma.

Lemma 1. *For any positive integer n and any real number t,*

$$\theta^{(2n)}(t) = \left(\frac{g}{l}\right)^n \sum_{j=1}^{n} a_{nj}\sin(j\theta(t)), \tag{6}$$

where a_{nj}, $j = 1, 2, \dots n$, are constants depending on θ_0, given by the recurrence relations

$$\begin{aligned} a_{n+1,j} &= -\frac{(j-1)(2j-1)}{2}a_{n,j-1} + 2j^2\cos\theta_0\cdot a_{nj} - \frac{(j+1)(2j+1)}{2}a_{n,j+1}\\ a_{11} &= -1, \quad j = 1, \dots, n+1. \end{aligned} \tag{7}$$

In (7) it is understood that $a_{nj} = 0$, if $j \notin \{1, 2, \dots, n\}$.

Proof. The verification of (6) goes obviously by induction on n, the case $n = 1$ reducing to the starting point of our discussion, provided by the differential equation (1).

Assuming now (6) true for some fixed value of n and any t, two differentiations of it yield

$$\begin{aligned} \theta^{(2n+1)}(t) &= \left(\frac{g}{l}\right)^n \sum_{j=1}^{n} ja_{nj}\cos(j\theta(t))\cdot\theta'(t),\\ \theta^{(2n+2)}(t) &= -\left(\frac{g}{l}\right)^n \sum_{j=1}^{n} j^2a_{nj}\sin(j\theta(t))\cdot\left(\theta'(t)\right)^2\\ &\quad+\left(\frac{g}{l}\right)^n \sum_{j=1}^{n} ja_{nj}\cos(j\theta(t))\cdot\theta''(t). \end{aligned}$$

Replacing $(\theta'(t))^2$ and $\theta''(t)$ in the expression of $\theta^{(2n+2)}(t)$ above by their values (2) and (1) respectively, and leaving out t for an easier read, further gives

$$
\begin{aligned}
\theta^{2(n+1)} &= \left(\frac{g}{l}\right)^{n+1} \sum_{j=1}^{n} (-2j^2) a_{nj} \sin(j\theta) \cdot (\cos\theta - \cos\theta_0) \\
&\quad + \left(\frac{g}{l}\right)^{n+1} \sum_{j=1}^{n} (-j) a_{nj} \cos(j\theta) \cdot \sin\theta \\
&= \left(\frac{g}{l}\right)^{n+1} \sum_{j=1}^{n} (-j^2) a_{nj} (\sin(j+1)\theta + \sin(j-1)\theta) \\
&\quad + \left(\frac{g}{l}\right)^{n+1} \sum_{j=1}^{n} (-j) a_{nj} \frac{\sin(j+1)\theta - \sin(j-1)\theta}{2} \\
&\quad + \left(\frac{g}{l}\right)^{n+1} \sum_{j=1}^{n} (2j^2) \cos\theta_0 \cdot a_{nj} \sin(j\theta) \\
&= \left(\frac{g}{l}\right)^{n+1} \sum_{j=2}^{n+1} - \left((j-1)^2 + \frac{j-1}{2} \right) a_{n,j-1} \sin(j\theta) \\
&\quad + \left(\frac{g}{l}\right)^{n+1} \sum_{j=1}^{n-1} - \left((j+1)^2 - \frac{j+1}{2} \right) a_{n,j+1} \sin(j\theta) \\
&\quad + \left(\frac{g}{l}\right)^{n+1} \sum_{j=1}^{n} (2j^2) \cos\theta_0 \cdot a_{nj} \sin(j\theta) \\
&= \left(\frac{g}{l}\right)^{n+1} \sum_{j=1}^{n+1} a_{n+1,j} \sin(j\theta),
\end{aligned}
$$

as claimed. □

The recurrence relations (7) appear to be difficult to put in closed form, in general, however two particular cases are manageable, that is

$$
\begin{aligned}
a_{nn} &= (-1)^n \frac{(n-1)!\,(2n-1)!!}{2^{n-1}}, \\
a_{n,n-1} &= (-1)^{n-1} \frac{n!\,(2n-1)!!}{3 \cdot 2^{n-2}} \cos\theta_0, \quad n = 2, 3, \dots,
\end{aligned}
\tag{8}
$$

where the double factorial stands for the product of all odd positive integers up to the indicated one. Equations (8) can be derived by iterating the recursions

$$
\begin{aligned}
a_{nn} &= -\frac{(n-1)(2n-1)}{2} a_{n-1,n-1} \quad \text{and} \\
a_{n,n-1} &= -\frac{(n-2)(2n-3)}{2} a_{n-1,n-2} + 2(n-1)^2 \cos\theta_0 \cdot a_{n-1,n-1},
\end{aligned}
$$

respectively.

The values of a_{nj} for $n = 1, 2, 3$ and 4 are as follows:

$$\begin{aligned} &a_{11} = -1, \quad a_{21} = -2\cos\theta_0, \quad a_{22} = \frac{3}{2}, \quad a_{31} = -2\cos(2\theta_0) - \frac{13}{2}, \\ &a_{32} = 15\cos\theta_0, \quad a_{33} = -\frac{15}{2}, \quad a_{41} = -2\cos(3\theta_0) - 60\cos\theta_0, \\ &a_{42} = 63\cos(2\theta_0) + 126, \quad a_{43} = -210\cos\theta_0, \; a_{44} = \frac{315}{4}. \end{aligned} \tag{9}$$

The finer structure of the constants a_{nj} is fully revealed by the following

Proposition 2. *For any $j = 1, 2, \ldots, n$ and $p = 0, 1, \ldots, \left[\frac{n-j}{2}\right]$, there are constants $c_{n,j,n-j-2p}$ independent of θ_0, such that*

$$a_{nj} = \sum_{p=0}^{\left[\frac{n-j}{2}\right]} c_{n,j,n-j-2p} \cdot \cos(n-j-2p)\theta_0, \tag{10}$$

where $[\cdot]$ denotes integer part.

Moreover, for $j = 1, 2, \ldots, n+1$ and $p = 0, 1, 2, \ldots, \left[\frac{n+1-j}{2}\right]$, we have recursively, starting with $c_{1,1,0} = -1$,

$$c_{n+1,j,n+1-j-2p} = \begin{cases} e_{n,j,p}, & \text{if} \quad p \neq \frac{n-j}{2} \\ e_{n,j,p} + j^2 c_{n,j,0}, & \text{if} \quad p = \frac{n-j}{2}, \end{cases} \tag{11}$$

where $e_{n,j,p}$ is short-hand for

$$\begin{aligned} e_{n,j,p} = &-\frac{(j-1)(2j-1)}{2} c_{n,j-1,n+1-j-2p} + j^2 c_{n,j,n-j-2p} \\ &+ j^2 c_{n,j,n-j-2(p-1)} - \frac{(j+1)(2j+1)}{2} c_{n,j+1,n+1-j-2p}. \end{aligned} \tag{12}$$

In (12), for $j \notin \{1, 2, \ldots, n\}$ or $k \notin \left\{n-j, n-j-2, \ldots, n-j-2\left[\frac{n-j}{2}\right]\right\}$, $c_{n,j,k}$ is to be taken 0.

Proof. The proof, again by induction on n, rests on a careful implementation of Equation (7). Indeed, if Equation (10) is assumed to hold true for some n

and every $j \in \{1, 2, \ldots, n\}$, then Equation (7) becomes

$$\begin{aligned} a_{n+1,j} = &- \frac{(j-1)(2j-1)}{2} \sum_{p=0}^{\left[\frac{n-j+1}{2}\right]} c_{n,j-1,n-j+1-2p} \cdot \cos(n-j+1-2p)\theta_0 \\ &+ 2j^2 \sum_{p=0}^{\left[\frac{n-j}{2}\right]} c_{n,j,n-j-2p} \cdot \cos(n-j-2p)\theta_0 \cdot \cos\theta_0 \\ &- \frac{(j+1)(2j+1)}{2} \sum_{p=0}^{\left[\frac{n-j-1}{2}\right]} c_{n,j+1,n-j-1-2p} \cdot \cos(n-j-1-2p)\theta_0, \end{aligned}$$

for $j = 1, 2, \ldots, n+1$. Keeping in mind when $c_{n,j,k}$ vanishes automatically, we further have

$$\begin{aligned} a_{n+1,j} = &\sum_{p=0}^{\left[\frac{n+1-j}{2}\right]} -\frac{(j-1)(2j-1)}{2} c_{n,j-1,n+1-j-2p} \cdot \cos(n+1-j-2p)\theta_0 \\ &+ \sum_{p=0}^{\left[\frac{n+1-j}{2}\right]} j^2 c_{n,j,n-j-2p} \cdot \big(\cos(n+1-j-2p)\theta_0 \\ &\qquad\qquad + \cos(n+1-j-2(p+1)\big)\theta_0 \\ &- \sum_{p=0}^{\left[\frac{n-j-1}{2}\right]} \frac{(j+1)(2j+1)}{2} c_{n,j+1,n+1-j-2(p+1)} \cos(n+1-j-2(p+1))\theta_0, \end{aligned}$$

or equivalently

$$
\begin{aligned}
a_{n+1,j} = & \sum_{p=0}^{\left[\frac{n+1-j}{2}\right]} -\frac{(j-1)(2j-1)}{2} c_{n,j-1,n+1-j-2p} \cdot \cos(n+1-j-2p)\theta_0 \\
& + \sum_{p=0}^{\left[\frac{n+1-j}{2}\right]} j^2 c_{n,j,n-j-2p} \cdot \cos(n+1-j-2p)\theta_0 \\
& + \sum_{p=0}^{\left[\frac{n+1-j}{2}\right]+1} j^2 c_{n,j,n-j-2(p-1)} \cdot \cos(n+1-j-2p)\theta_0 \\
& + \sum_{p=0}^{\left[\frac{n+1-j}{2}\right]} -\frac{(j+1)(2j+1)}{2} c_{n,j+1,n+1-j-2p} \cdot \cos(n+1-j-2p)\theta_0 \\
= & \sum_{p=0}^{\left[\frac{n+1-j}{2}\right]} c_{n+1,j,n+1-j-2p} \cdot \cos(n+1-j-2p)\theta_0.
\end{aligned}
$$

□

We list here for further use all the values of $c_{n,j,n-j-2p}$ for $n = 1, 2, 3$ and 4, which can be easily read from Equations (9):

$$
\begin{aligned}
& c_{1,1,0} = -1,\ c_{2,1,1} = -2,\quad c_{2,2,0} = \frac{3}{2},\qquad c_{3,1,2} = -2,\ \ c_{3,1,0} = -\frac{13}{2}, \\
& c_{3,2,1} = 15,\ \ c_{3,3,0} = -\frac{15}{2},\ c_{4,1,3} = -2,\quad c_{4,1,1} = -60, \\
& c_{4,2,2} = 63,\ \ c_{4,2,0} = 126,\quad c_{4,3,1} = -210,\ c_{4,4,0} = \frac{315}{4}.
\end{aligned}
\tag{13}
$$

We are now in a position to state and prove the main result of our paper, from which the power series expansion for the simple pendulum angle function about $t = 0$ follows immediately.

Theorem 3. *All the derivatives at $t = 0$ of the simple pendulum angle function $\theta(t)$ are trigonometric polynomials of the pendulum amplitude θ_0. Precisely, for $n = 1, 2, \ldots$,*

$$
\begin{aligned}
\theta^{(2n)}(0) = & \left(\frac{g}{l}\right)^n \sum_{p=0}^{\left[\frac{n-1}{2}\right]} \frac{1}{2} \left(\sum_{j=1}^{n-2p} c_{n,j,n-j-2p} - \sum_{j=1}^{p} c_{n,j,n+j-2p} \right. \\
& \left. + \sum_{j=n-2p}^{n-p} c_{n,j,-n+j+2p} \right) \cdot \sin(n-2p)\theta_0,
\end{aligned}
\tag{14}
$$

$$\theta^{(2n-1)}(0) = 0.$$

Consequently, Equations (14), (11), *and* (12) *provide a recursive way of generating the power series expansion of the simple pendulum function* $\theta(t)$ *about* $t = 0$, *given by*

$$\theta(t) = \theta_0 + \sum_{n=1}^{\infty} \frac{\theta^{(2n)}(0)}{(2n)!} t^{2n}. \tag{15}$$

In particular,

$$\begin{aligned}\theta(t) = \theta_0 &+ \frac{1}{2!}\frac{g}{l}\left(-\sin\theta_0\right)t^2 + \frac{1}{4!}\left(\frac{g}{l}\right)^2\left(\frac{1}{2}\sin(2\theta_0)\right)t^4 \\ &+ \frac{1}{6!}\left(\frac{g}{l}\right)^3\left(-\sin(3\theta_0) + 2\sin\theta_0\right)t^6 \\ &+ \frac{1}{8!}\left(\frac{g}{l}\right)^4\left(\frac{17}{4}\sin(4\theta_0) - 8\sin(2\theta_0)\right)t^8 + O(t^{10}).\end{aligned} \tag{16}$$

Proof. The vanishing of $\theta^{(2n-1)}(0)$ follows immediately from the initial condition $\theta'(0) = 0$ and the expression of $\theta^{(2n+1)}(t)$ appearing in the proof of Lemma 1. As for $\theta^{(2n)}(0)$, from Equations (6) and (10) we have

$$\begin{aligned}\theta^{(2n)}(0) &= \left(\frac{g}{l}\right)^n \sum_{j=1}^{n}\sum_{p=0}^{\left[\frac{n-j}{2}\right]} c_{n,j,n-j-2p}\cdot\cos(n-j-2p)\theta_0\cdot\sin(j\theta_0) \\ &= \left(\frac{g}{l}\right)^n \sum_{j=1}^{n}\sum_{p=0}^{\left[\frac{n-j}{2}\right]} c_{n,j,n-j-2p}\frac{\sin(n-2p)\theta_0 + \sin(2j+2p-n)\theta_0}{2} \\ &= \left(\frac{g}{l}\right)^n \sum_{j=1}^{n}\sum_{p=0}^{\left[\frac{n-j}{2}\right]} \frac{1}{2}c_{n,j,n-j-2p}\cdot\sin(n-2p)\theta_0 \\ &\quad + \left(\frac{g}{l}\right)^n \sum_{j=1}^{n}\sum_{p=0}^{\left[\frac{n-j}{2}\right]} \frac{1}{2}c_{n,j,n-j-2p}\cdot\sin(2j+2p-n)\theta_0.\end{aligned}$$

Now, a careful reverse in order of summation gives

$$\begin{aligned}&\left(\frac{g}{l}\right)^n \sum_{j=1}^{n}\sum_{p=0}^{\left[\frac{n-j}{2}\right]} \frac{1}{2}c_{n,j,n-j-2p}\cdot\sin(n-2p)\theta_0 \\ &\qquad = \left(\frac{g}{l}\right)^n \sum_{p=0}^{\left[\frac{n-1}{2}\right]}\sum_{j=1}^{n-2p} \frac{1}{2}c_{n,j,n-j-2p}\cdot\sin(n-2p)\theta_0.\end{aligned}$$

Further,

$$\begin{aligned}\left(\frac{g}{l}\right)^n \sum_{j=1}^{n} \sum_{p=0}^{\left[\frac{n-j}{2}\right]} \frac{1}{2} c_{n,j,n-j-2p} \cdot \sin(2j+2p-n)\theta_0 \\ = \left(\frac{g}{l}\right)^n \sum_{\substack{1\le j\le n, 0\le p\le\left[\frac{n-j}{2}\right] \\ 2(j+p)<n}} \frac{1}{2} c_{n,j,n-j-2p} \cdot \sin(2j+2p-n)\theta_0 \\ + \left(\frac{g}{l}\right)^n \sum_{\substack{1\le j\le n, 0\le p\le\left[\frac{n-j}{2}\right] \\ 2(j+p)>n}} \frac{1}{2} c_{n,j,n-j-2p} \cdot \sin(2j+2p-n)\theta_0.\end{aligned}$$

In order to reverse the order of summation in the last two double sums above we need to perform suitable changes of variables. Namely $(j,p) \mapsto (j',p')$, $j' = j, p' = j+p$, yields

$$\begin{aligned}\left(\frac{g}{l}\right)^n \sum_{\substack{1\le j\le n, 0\le p\le\left[\frac{n-j}{2}\right] \\ 2(j+p)<n}} \frac{1}{2} c_{n,j,n-j-2p} \cdot \sin(2j+2p-n)\theta_0 \\ = \left(\frac{g}{l}\right)^n \sum_{p=0}^{\left[\frac{n-1}{2}\right]} \sum_{j=1}^{p} \frac{1}{2} c_{n,j,n+j-2p} \cdot \sin(n-2p)\theta_0,\end{aligned}$$

while $(j,p) \mapsto (j',p')$, $j' = j, p' = n-j-p$, yields

$$\begin{aligned}\left(\frac{g}{l}\right)^n \sum_{\substack{1\le j\le n, 0\le p\le\left[\frac{n-j}{2}\right] \\ 2(j+p)>n}} \frac{1}{2} c_{n,j,n-j-2p} \cdot \sin(2j+2p-n)\theta_0 \\ = \left(\frac{g}{l}\right)^n \sum_{p=0}^{\left[\frac{n-1}{2}\right]} \sum_{j=n-2p}^{n-p} \frac{1}{2} c_{n,j,-n+j+2p} \cdot \sin(n-2p)\theta_0.\end{aligned}$$

Putting all these results together provides the desired expression (14) of $\theta^{(2n)}(0)$.

Lastly the order 9 power series expansion of $\theta(t)$ about $t=0$ given by (16) follows easily from Equations (15), (14), and (13). □

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The classical SIR model in epidemiology

Aurelian Gheondea[1)]

Abstract. This is a survey note in which we describe the classical SIR model in mathematical epidemiology, a bit of qualitative analysis, its Euler discretisation, and some simulations.

Keywords: Dynamical systems, SIR model, bifurcation, mathematical epidemiolgoy.

MSC: Primary 92B05; Secondary 34A30, 39A05.

1. Introduction

The interest for mathematical models that describe the dynamical evolution of infectious diseases is rather old. The first mathematical model was obtained by Daniel Bernoulli [1] in 1760 and it describes the evolution of an epidemic of small pox. One of the most important steps in establishing mathematical epidemiology was made by the physician R. Ross [7], the recipient of the Nobel Prize for medicine in 1902 for his contributions for the study and understanding malaria. Ross pushed forward his investigations on malaria by mathematically formalising his research. There are, of course, many empirical models obtained by collecting statistical data on epidemics over the time and for different geographical regions. The most reliable mathematical model, that is still in use, was obtained by W.O. Kermack and A.G. McKendrick [6] in 1927. For a comprehensive and pertinent presentation of the mathematical and statistical models in epidemiology we recommend the survey article of H.W. Hethcote [3]

There are, basically, two types of models that are used in epidemiology, the mathematical models, mainly dynamical models, and statistical models. These two types of models are complementary one to each other: the mathematical models make the skeletal structure of any scientific approach to the study of contagious diseases while the statistical models make the muscular structure that make the connections to the real data. Since these models are

[1)] Department of Mathematics, Bilkent University, 06800 Bilkent, Ankara, Turkey, and Institutul de Matematică al Academiei Române, C.P. 1-764, 014700 Bucureşti, România, `aurelian@fen.bilkent.edu.tr` and `A.Gheondea@imar.ro`

rather general, they apply not only to contagious diseases of human beings but also to animals and they are of remarkable efficiency in the study of the dynamics of various ecosystems.

The SIR model, that we describe in this note, is usually presented in most of the dynamical systems courses, e.g. see the textbook of M. Hirsh, S. Smale, and R.L. Devaney [4]. However, although these mathematical models look rather simple, they pose difficult problems to mathematicians and hence they are subjects of active research, e.g. see T. Harko, F.S.N. Lobo, and M.K. Mak [2]. A final satisfactory answer on explicit solutions is not yet available.

2. The Classical SIR Model

2.1. **Compartments Models.** The dynamical system models that are in use in epidemiology for the evolution of infectious diseases are based on the idea of transfer between compartments. We consider a population that lives in a clearly specified area (a village or a city, a metropolitan area, a county, a country, etc.) with a stable number of individuals N (we ignore births, deaths and migrations), relatively homogeneous, in which the interactions between the individuals are happening continuously. This population is split in compartments denoted by M, E, S, I, and R, with the following definitions:

- M is the number of babies that have passive immunity, inherited naturally from their mother, but that lasts a limited number of months.
- S is the number of individuals that are susceptible to be infected, having no immunity.
- E is the number of individuals that are in the latent period of infection, that are infected but not yet contagious.
- I is the number of infected individuals that are contagious as well.
- R is the number of those individuals that have been infected but have been cured, are no more infectious and got immunity, or died.

The transfer between compartments can be done only in the following ways:

- From the compartment M an individual can pass only to the compartment S.
- From the compartment S an individual can pass only to the compartment E where stays for a relatively constant interval of time.
- From the compartment E an individual can pass only to the compartment I of infected people.
- From compartment I and individual can pass only to the compartment R either cured and getting immunity or dies.

$$M \quad \Rightarrow \quad S \quad \Rightarrow \quad E \quad \Rightarrow \quad I \quad \Rightarrow \quad R$$

This model is called MSEIR and is considered the most comprehensive.

2.2. **The SIR Model.** W.O. Kermack and A.G. McKendrick [6] published a simplified dynamical system model called SIR, with only three compartments, S, I, and R, and it is considered as the classical mathematical model in studying epidemics. In this model we have only two possible transfers:

- From compartment S to the compartment I;
- From compartment I to the compartment R.

$$S \quad \Rightarrow I \quad \Rightarrow \quad R$$

The SIR model is obtained by the following assumptions:

(sir1) The dynamical system evolves in continuous time and the functions S, I, and R are functions of time t indefinitely differentiable on the interval $[0, \infty)$;

(sir2) β, the *infection rate*, is the parameter that controls the transfer between the compartments S and I and has the significance of the average number of contacts that are infected, in the time unit;

(sir3) γ, the *recovery rate*, is the transfer rate between the compartments I and R, that represents both the rate of cure and of death, without making a difference between these, with the significance that $1/\gamma$ is the interval of time in which an individual remains infected.

Both β and γ have interpretation as probability rates and hence take values in $[0, 1]$ only. From the assumption that the number of individuals in the designated area remains constant N we have

$$S(t) + I(t) + R(t) = N, \quad t \in [0, \infty), \tag{1}$$

which, by differentiation yields

$$S'(t) + I'(t) + R'(t) = 0, \quad t \in [0, \infty). \tag{2}$$

Then, from the definitions of the parameters β in (sir2), of γ in (sir3), and (2) we get the following system of ordinary differential equations (ODE) of order 1

$$\begin{cases} \frac{\mathrm{d}\, S(t)}{\mathrm{d}\, t} & = -\beta \frac{S(t)}{N} I(t), \\ \frac{I(t)}{\mathrm{d}\, t} & = \beta \frac{S(t)}{N} I(t) - \gamma(t), \\ \frac{R(t)}{\mathrm{d}\, t} & = \gamma I(t). \end{cases} \tag{3}$$

It is more convenient to normalise the functions S, I, and R in the following way:

$$s(t) = \frac{S(t)}{N}, \quad i(t) = \frac{I(t)}{N}, \quad r(t) = \frac{R(t)}{N}, \tag{4}$$

where N is the total number of individuals, and then, from (1) we get

$$s(t) + i(t) + r(t) = 1, \quad t \in [0, \infty), \tag{5}$$

and hence, by taking derivatives, we get

$$s'(t) + i'(t) + r'(t) = 0, \quad t \in [0, \infty). \tag{6}$$

Then the system (3) becomes

$$\begin{cases} \frac{\mathrm{d}\, s(t)}{\mathrm{d}\, t} &= -\beta s(t) i(t), \\ \frac{\mathrm{d}\, i(t)}{\mathrm{d}\, t} &= \beta s(t) i(t) - \gamma i(t), \\ \frac{\mathrm{d}\, r(t)}{\mathrm{d}\, t} &= \gamma i(t). \end{cases} \tag{7}$$

We usually associate to the system of first order differential equations (7) the initial conditions $s(0) = s_0$, $i(0) = i_0$, and $r(0) = r_0$ and in this case we talk about an *Initial Value Problem* (IVP). The three equations from (7) are not independent, for example the third is obtained from the first and the second, by using (5) and its consequence $s'(t) + i'(t) + r'(t) = 0$.

2.3. **Bifurcation.** The SIR model is not linear. As simple as it may look at the first glance, explicit analytical solutions are not known yet: some parametric solutions as well as other equivalent representations have been recently obtained by T. Harko, F.S.N. Lobo, and M.K. Mak [2]. For the moment, we can draw some qualitative conclusions. The functions s, i, and r take only nonnegative values and hence, from the first and the third equation in (7) it follows that $s'(t)$ is always nonpositive, hence s is nonincreasing, while $r'(t)$ is always nonnegative, hence r is nondecreasing. The behaviour of the function i is different. Since the second equation from (7) can be written as

$$\frac{\mathrm{d}\, i(t)}{\mathrm{d}\, t} = \big(\beta s(t) - \gamma\big) i(t), \quad t \in [0, \infty), \tag{8}$$

it follows that the function i increases as long as the function $\beta s - \gamma$ is positive and decreases as long as the same function $\beta s - \gamma$ is negative. In particular, we first have to look at the initial condition $t = 0$, that is, at the number $\beta s_0 - \gamma$, and see whether it is positive or negative. Equivalently, letting

$$\rho = \frac{s_0 \beta}{\gamma}, \tag{9}$$

called the *reproduction number*, we see that the monotonicity of the function i depends on how ρ stays with respect to the value 1. The reproduction number ρ plays one of the most important role in understanding the evolution of a contagious disease in a population.

Mathematically, a dynamical system has a *bifurcation* if, for a small change of a certain parameter, that is called the *bifurcation parameter*, we may have a change of the qualitative behaviour of the evolution of the system.

In the case of the SIR model (7), it is the reproduction number ρ that makes the bifurcation parameter, in the following sense:

- If $\rho > 1$ then the function i, that describes the normalised function of infected people, increases from the initial value i_0 to a certain maximal value and then decreases, and we say that we have an *epidemic*.
- If $\rho < 1$ then the function i decreases from the initial value i_0 to values close to 0 and, in this case, we do not have an epidemic.

2.4. **The Susceptibles and Recovered Equations.** In this subsection, we follow closely the article [2]. Firstly, let us observe that, following the brief qualitative analysis in the previous subsection, we see that the solution function s does not vanish on $(0, \infty)$ hence we can perform divisions with this function. We can interpret the first ODE in (7) as

$$i(t) = -\frac{1}{\beta}\frac{s'(t)}{s(t)}, \tag{10}$$

and, by differentiation the first ODE in (7) with respect to time we get

$$s''(t) = -\beta s'(t)\, i(t) - \beta s(t)\, i'(t),$$

from which, solving for $i'(t)$ and taking into account of (10) we get

$$i'(t) = -\frac{1}{\beta}\left[\frac{s''(t)}{s(t)} - \left(\frac{s'(t)}{s(t)}\right)^2\right]. \tag{11}$$

From the second ODE in (7) and (11), taking into account once more of (10) we get the second order nonlinear ODE of the function s, counting the susceptible individuals

$$s''(t) = s'^2(t) - \gamma s(t)s'(t) + \beta s^2(t)s'(t). \tag{12}$$

In a similar way, one obtains the second order nonlinear ODE of the function r, counting the recovered individuals,

$$r''(t) = \beta s_0 \mathrm{e}^{\frac{\beta}{\gamma}(r_0 - r(t))} r'(t) - \gamma r'(t). \tag{13}$$

Both second order ODE (12) and (13) are highly nonlinear and explicit solutions for them are not known.

2.5. **Euler Discretisation.** Since explicit compact solutions for the functions s, i, and r are not available, in order to get both qualitative and quantitative information from the SIR model one usually performs an *Euler discretisation* of the system (7). The Euler method of discretisation means that we make a sampling of the interval $[0, \infty)$ in intervals of equal length ΔT and, instead of functions, we work with sequences $(s_n)_n$, $(i_n)_n$, and $(r_n)_n$, with initial values s_0, i_0, and r_0.

Let us briefly describe the Euler's method of discretisation: one considers a differentiable function $y\colon [0,\infty) \to \mathbb{R}$ and the length of the sampling interval ΔT and let slope_{n-1} have the meaning of a discretised derivative

$$\text{slope}_{n-1} = \frac{y_n - y_{n-1}}{\Delta T}, \quad n \geq 1,$$

hence, we get the sequence $(y_n)_n$, with initial value y_0, and first order recurrence relation:

$$y_n = y_{n-1} + \text{slope}_{n-1}\Delta T, \quad n \geq 1. \tag{14}$$

Applying the Euler's discretisation method to the SIR model, from (7) and (14) we get the following system, valid for all integer $n \geq 1$,

$$\begin{cases} s_n &= s_{n-1} - \beta\, s_{n-1}\, i_{n-1}\, \Delta T, \\ i_n &= i_{n-1} + \beta s_{n-1}\, i_{n-1}\, \Delta T - \gamma\, i_{n-1}\, \Delta T, \\ r_n &= r_{n-1} + \gamma i_{n-1}\, \Delta T, \end{cases} \tag{15}$$

which is called a *system with difference equations* and to which we associate the initial values s_0, i_0, and r_0.

From the quantitative point of view, numerical calculations, simulations, and real data, there is a significant advantage of replacing the original system of differential equations (7) with the system of difference equations (15).

2.6. **Simulations.** Based on the system with finite differences (15) we made a few simulation with MATLAB for $\Delta T = 1$ and for an interval of time of 100 days, for the functions s of susceptible individuals, i of infected invividuals, and r of those recovered, normalised by number of total individuals.

We first have a simulation of the discrete SIR model (15) for the simple case with fixed parameters β and γ, and hence for a fixed reproduction number $\rho > 1$. This is the classical evolution of an epidemic, in which the function i, of infected people, increases up to a maximal value and then decreases down to zero. For this simulation, we have to keep in mind that no exterior intervention is performed and the epidemic is free to evolve, see Figure 1 for $\beta = 0.3$, $\gamma = 1/23$, $i(0) = 1/27000$, and $r(0) = 0$.

Epidemiologists call this scenario *herd immunity* in which case individuals get infected and then get recovered either by surviving and becoming immune or by dying. In this scenario, the rate of death is not considered and this raises discussions related to public policy, ethics, etc.

A second simulation points out the different scenarios of evolution of the function i, representing the normalised number of infected individuals, by varying the parameter β. We observe how the reproduction number ρ controls the shape of the curve i and points out the bifurcation in the neighbourhood of $\rho = 1$, see Figure 2, for parameter $\gamma = 1/23$, initial values $i(0) = 0.1$, $r(0) = 0.03$, for an interval of time of 100 days, with values reported daily. The

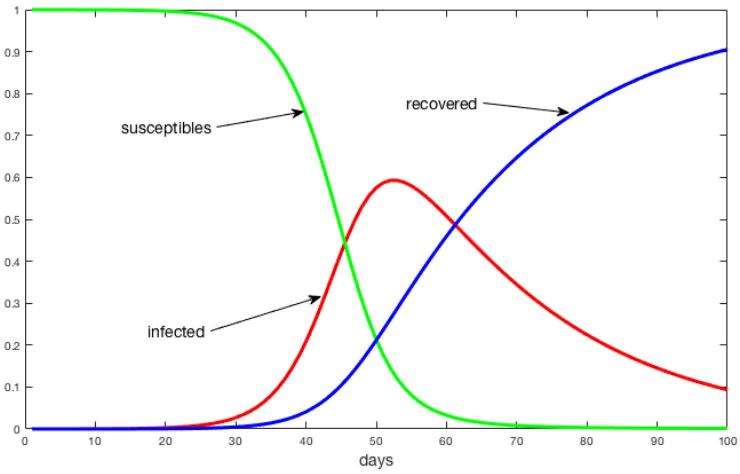

FIGURE 1. The evolution of a contagious disease as described by the classical discretised SIR model in cae of an epidemic with reproduction number $\rho > 1$. The function s is depicted by the green line, the function r is depicted by the blue line, and the function i is depicted by the red line.

bifurcation appears for the critical value $\beta = 0.04349$, when the reproduction number ρ passes the value 1.

A third simulation was performed with the following scenario: an epidemic with reproduction number $\rho > 1$ outbursts and then, after an interval of time, by taking measures of social distancing, for example, the reproduction number is brought to a value $\rho < 1$, then the measures of social distancing are relaxed and the reproduction number becomes $\rho > 1$ again. In such a scenario, the number of infected individuals oscillates, with intervals of time of increase followed by intervals of time of decrease, see Figure 3.

2.7. **Some Conclusions.** The dynamical system model SIR offers the possibility to understand what can happen with the evolution of a contagious disease and what we should expect for. The number of reproduction number ρ tells us wether we will have and epidemic or not, and we have two parameters β and γ on which we have to work on in order to control the epidemic. The parameter β can be controlled by social organising and public institutions while the parameter γ is controlled by medical aspects. When medical remedies of type antiviral medication and vaccine are missing, the only leverage that we can have on controlling the reproduction number ρ is to work on the parameter β. Otherwise, the propagation of the disease in a given population follows the mathematical model and the social and individual consequences are very difficult to estimate.

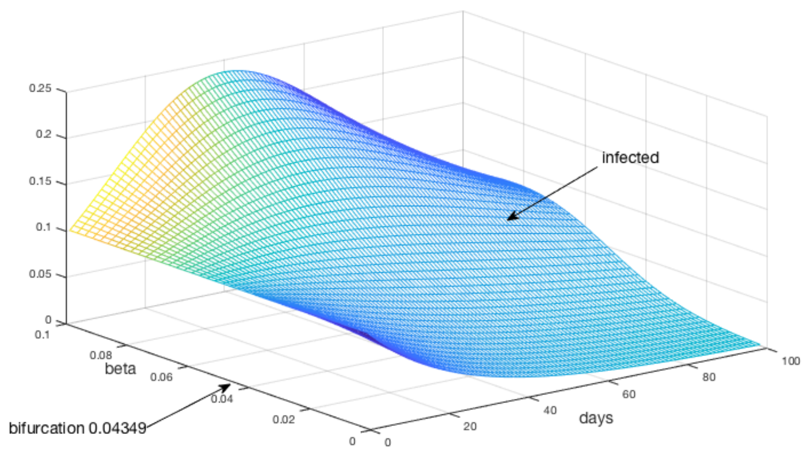

Figure 2. The dependence of the type of evolution of the function i of normalised infected people by a contagious disease, for a varying parameter β taking values between 0.0025 and 0.1, with sampling interval of 0.0025.

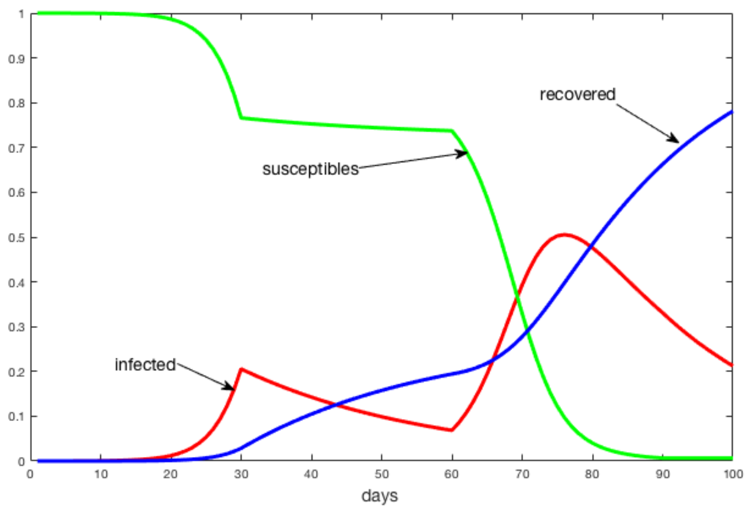

Figure 3. We observe the scenario for variations of the parameter β either 0.4 or 0.1 for intervals of time of 30 days.

In reality, neither β nor γ is constant in time, but the general evolution of the function i of infected people can be obtained from the discrete version of the SIR model, by making $\beta = (\beta_n)_n$ and $\gamma = (\gamma_n)$ sequences. In particular, when the reproduction number ρ oscillates, talking about "reaching the maximal level" does not make too much sense because there may be more maximal levels and more intervals of time of increase and decrease, a kind of "waves". The reader may combine our qualitative analysis with the current pandemic of COVID-19 using the data provided by the Imperial College COVID-19 Response Team [5].

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The limit of a sequence of integrals on the k-dimensional unit cube

DUMITRU POPA[1]

Abstract. We prove that for every natural number k and every continuous function $f : [0,1] \to \mathbb{R}$ the following equality holds

$$\lim_{n\to\infty} \frac{n}{(\ln n)^{k-1}} \int \cdots \int_{[0,1]^k} (1 - x_1 \cdots x_k)^n f\left((1 - x_1 \cdots x_k)^n\right) \mathrm{d}x_1 \cdots \mathrm{d}x_k$$
$$= \frac{1}{(k-1)!} \int_0^1 f(x)\,\mathrm{d}x.$$

Keywords: Riemann integral, Fubini theorem, limit of sequences of integrals, k-dimensional unit cube.

MSC: Primary 26B15; Secondary 28A35.

[1] Department of Mathematics, Ovidius University, Constanţa, Romania, dpopa@univ-ovidius.ro

1. Introduction

In this paper we find the limit of a sequence of integrals on the k-dimensional unit cube. This limit can be viewed as a natural extension of the well-known limit in the case of the Riemann integral on $[0,1]$: if $f:[0,1]\to\mathbb{R}$ is a continuous function then, $\lim\limits_{n\to\infty} n\int_0^1 x^n f(x^n)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x$, see [2], [3], [4]. The notation and notions used and not defined in this paper are standard. For two sequences of real numbers $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, $(y_n\neq 0,\ \forall n\geq n_0)$ the notation $x_n \backsim y_n$ means $\lim\limits_{n\to\infty}\frac{x_n}{y_n}=1$. For details regarding the multiple Riemann integral we recommend the reader the excellent treatment of this concept in the book of N. Boboc, see [1].

2. The main results

Proposition 1. *For every $n\in\mathbb{N}$ we define*

$$A_n^{(2)} = \frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right)$$

and $A_n^{(k+1)} = \frac{1}{n+1}\left(1+A_1^{(k)}+\cdots+A_n^{(k)}\right)$, for $k\geq 2$. Then:
(i) For every $k\geq 2$ and every $n\in\mathbb{N}$ we have

$$\int\cdots\int_{[0,1]^k}(1-x_1\cdots x_k)^n\,\mathrm{d}x_1\cdots\mathrm{d}x_k = A_n^{(k)}.$$

(ii) For every $k\geq 2$, we have $A_n^{(k)} \backsim \frac{1}{(k-1)!}\cdot\frac{(\ln n)^{k-1}}{n}$.

Proof. First we prove that for every $a\in\mathbb{R}$

$$\int_0^1 (1-at)^n\,\mathrm{d}t = \frac{1}{n+1}\Big(1+(1-a)+(1-a)^2+\cdots+(1-a)^n\Big). \qquad (1)$$

If $a=0$ the equality is obvious. If $a\neq 0$ then

$$\begin{aligned}\int_0^1 (1-at)^n\,\mathrm{d}t &= \left.\frac{-(1-at)^{n+1}}{(n+1)\,a}\right|_0^1 = \frac{1}{n+1}\cdot\frac{1-(1-a)^{n+1}}{a}\\ &= \frac{1}{n+1}\Big(1+(1-a)+(1-a)^2+\cdots+(1-a)^n\Big).\end{aligned}$$

(i) The proof is by induction on k. For $k=2$ by Fubini's theorem and (1), for every $n\in\mathbb{N}$, we have

$$\begin{aligned}\iint\limits_{[0,1]^2}(1-xy)^n\,\mathrm{d}x\mathrm{d}y &= \int_0^1\left(\int_0^1(1-xy)^n\,\mathrm{d}y\right)\mathrm{d}x\\ &= \frac{1}{n+1}\int_0^1\left(1+(1-x)+(1-x)^2+\cdots+(1-x)^n\right)\mathrm{d}x\\ &= \frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right)=A_n^{(2)}.\end{aligned}$$

Let us suppose that $\int\cdots\int_{[0,1]^k}(1-x_1\cdots x_k)^n\,\mathrm{d}x_1\cdots\mathrm{d}x_k=A_n^{(k)}$, for some integer $k\geq 2$ and every $n\in\mathbb{N}$.
Then by Fubini's theorem, (1) and the inductive hypothesis we get

$$\begin{aligned}&\int\cdots\int_{[0,1]^{k+1}}(1-x_1\cdots x_kx_{k+1})^n\,\mathrm{d}x_1\cdots\mathrm{d}x_k\mathrm{d}x_{k+1}\\ =\;&\int\cdots\int_{[0,1]^k}\left(\int_0^1(1-x_1\cdots x_kx_{k+1})^n\,\mathrm{d}x_{k+1}\right)\mathrm{d}x_1\cdots\mathrm{d}x_k\\ =\;&\frac{1}{n+1}\int_{[0,1]^k}\left(1+(1-x_1\cdots x_k)+\cdots+(1-x_1\cdots x_k)^n\right)\mathrm{d}x_1\cdots\mathrm{d}x_k\\ =\;&\frac{1}{n+1}\left(1+A_1^{(k)}+\cdots+A_n^{(k)}\right)=A_n^{(k+1)}.\end{aligned}$$

(ii) The proof is by induction on k. As it is well-known, $1+\frac{1}{2}+\cdots+\frac{1}{n+1}\backsim\ln n$ and thus, $A_n^{(2)}=\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right)\backsim\frac{\ln n}{n}$. Let $k\geq 2$ and suppose that $A_n^{(k)}\backsim\frac{(\ln n)^{k-1}}{n}\cdot\frac{1}{(k-1)!}$. Since the series $\sum\limits_{n=1}^{\infty}\frac{(\ln n)^{k-1}}{n}$ diverges, by the Stolz-Cesàro Lemma, the case $\left[\frac{\cdot}{\infty}\right]$, $1+A_1^{(k)}+\cdots+A_n^{(k)}\backsim\frac{1}{(k-1)!}\sum\limits_{i=1}^{n}\frac{(\ln i)^{k-1}}{i}$. By the Stolz-Cesàro Lemma we deduce that $\sum\limits_{i=1}^{n}\frac{(\ln i)^{k-1}}{i}\backsim\frac{(\ln n)^k}{k}$ and hence $A_n^{(k+1)}\backsim\frac{1}{k!}\cdot\frac{(\ln n)^k}{n}$. □

Theorem 2. *Let $k\in\mathbb{N}$ and $f:[0,1]\to\mathbb{R}$ be a continuous function. Then*

$$\begin{aligned}\lim_{n\to\infty}\frac{n}{(\ln n)^{k-1}}&\int\cdots\int_{[0,1]^k}(1-x_1\cdots x_k)^n f\left((1-x_1\cdots x_k)^n\right)\mathrm{d}x_1\cdots\mathrm{d}x_k\\ &=\frac{1}{(k-1)!}\int_0^1 f(x)\,\mathrm{d}x.\end{aligned}$$

Proof. The case $k=1$ is well-known, see [2] or [3]. Let $k\geq 2$.

The case of polynomials. First note that by Proposition 1 for every $i \in \mathbb{Z}$, $i \geq 0$, we have

$$\int \cdots \int_{[0,1]^k} (1 - x_1 \cdots x_k)^{n(i+1)} \,\mathrm{d}x_1 \cdots \mathrm{d}x_k \backsim \frac{1}{(k-1)!} \cdot \frac{(\ln(n(i+1)))^{k-1}}{n(i+1)}$$

and hence

$$\int \cdots \int_{[0,1]^k} (1 - x_1 \cdots x_k)^{n(i+1)} \,\mathrm{d}x_1 \cdots \mathrm{d}x_k \backsim \frac{1}{(k-1)!} \cdot \frac{1}{i+1} \frac{(\ln n)^{k-1}}{n}. \quad (2)$$

Let $P(t) = a_0 + a_1 t + \cdots + a_k t^k$ be a polynomial. From (2) we deduce that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{n}{(\ln n)^{k-1}} \int \cdots \int_{[0,1]^k} (1 - x_1 \cdots x_k)^n P((1 - x_1 \cdots x_k)^n) \,\mathrm{d}x_1 \cdots \mathrm{d}x_k \\
&\quad = \sum_{i=0}^{k} a_i \lim_{n\to\infty} \frac{n}{(\ln n)^{k-1}} \int \cdots \int_{[0,1]^k} (1 - x_1 \cdots x_k)^{n(i+1)} \,\mathrm{d}x_1 \cdots \mathrm{d}x_k \\
&\quad = \frac{1}{(k-1)!} \sum_{i=0}^{k} \frac{a_i}{i+1},
\end{aligned}$$

and so

$$\begin{aligned}
&\lim_{n\to\infty} \frac{n}{(\ln n)^{k-1}} \int \cdots \int_{[0,1]^k} (1 - x_1 \cdots x_k)^n P((1 - x_1 \cdots x_k)^n) \,\mathrm{d}x_1 \cdots \mathrm{d}x_k \\
&\quad = \frac{1}{(k-1)!} \int_0^1 P(x) \,\mathrm{d}x.
\end{aligned}$$

The case of continuous functions. Let $f : [0,1] \to \mathbb{R}$ be a continuous function and let $\varepsilon > 0$. By Bernstein's theorem, there exists a polynomial $P_\varepsilon : [0,1] \to \mathbb{R}$ such that $|f(x) - P_\varepsilon(x)| < \varepsilon$, $\forall x \in [0,1]$, that is,

$$-\varepsilon + P_\varepsilon(x) \leq f(x) \leq \varepsilon + P_\varepsilon(x), \forall x \in [0,1]. \quad (3)$$

From (3) we deduce that for every natural number n we have

$$\begin{aligned}
&\frac{n}{(\ln n)^{k-1}} \int \cdots \int_{[0,1]^k} (1 - x_1 \cdots x_k)^n P_\varepsilon((1 - x_1 \cdots x_k)^n) \,\mathrm{d}x_1 \cdots \mathrm{d}x_k \\
&\qquad - \frac{\varepsilon n}{(\ln n)^{k-1}} \int \cdots \int_{[0,1]^k} (1 - x_1 \cdots x_k)^n \,\mathrm{d}x_1 \cdots \mathrm{d}x_k \\
&\leq \frac{n}{(\ln n)^{k-1}} \int \cdots \int_{[0,1]^k} (1 - x_1 \cdots x_k)^n f((1 - x_1 \cdots x_k)^n) \,\mathrm{d}x_1 \cdots \mathrm{d}x_k \\
&\leq \frac{n}{(\ln n)^{k-1}} \int \cdots \int_{[0,1]^k} (1 - x_1 \cdots x_k)^n P_\varepsilon((1 - x_1 \cdots x_k)^n) \,\mathrm{d}x_1 \cdots \mathrm{d}x_k \\
&\qquad + \frac{\varepsilon n}{(\ln n)^{k-1}} \int \cdots \int_{[0,1]^k} (1 - x_1 \cdots x_k)^n \,\mathrm{d}x_1 \cdots \mathrm{d}x_k. \quad (4)
\end{aligned}$$

Then, from (3) and (4) we get

$$\begin{aligned}
&\frac{1}{(k-1)!}\int_0^1 P_\varepsilon(x)\,\mathrm{d}x-\frac{\varepsilon}{(k-1)!}\\
\le\ &\liminf_{n\to\infty}\frac{n}{(\ln n)^{k-1}}\int\cdots\int_{[0,1]^k}(1-x_1\cdots x_k)^n f\left((1-x_1\cdots x_k)^n\right)\mathrm{d}x_1\cdots\mathrm{d}x_k\\
\le\ &\limsup_{n\to\infty}\frac{n}{(\ln n)^{k-1}}\int\cdots\int_{[0,1]^k}(1-x_1\cdots x_k)^n f\left((1-x_1\cdots x_k)^n\right)\mathrm{d}x_1\cdots\mathrm{d}x_k\\
\le\ &\frac{1}{(k-1)!}\int_0^1 P_\varepsilon(x)\,\mathrm{d}x+\frac{\varepsilon}{(k-1)!}. \qquad (5)
\end{aligned}$$

But, from (3)

$$\int_0^1 f(x)\,\mathrm{d}x-\varepsilon\le\int_0^1 P_\varepsilon(x)\,\mathrm{d}x\le\int_0^1 f(x)\,\mathrm{d}x+\varepsilon \qquad (6)$$

and hence from (5) and (6) we get

$$\begin{aligned}
&\frac{1}{(k-1)!}\int_0^1 f(x)\,\mathrm{d}x-\frac{2\varepsilon}{(k-1)!}\\
\le\ &\liminf_{n\to\infty}\frac{n}{(\ln n)^{k-1}}\int\cdots\int_{[0,1]^k}(1-x_1\cdots x_k)^n f\left((1-x_1\cdots x_k)^n\right)\mathrm{d}x_1\cdots\mathrm{d}x_k\\
\le\ &\limsup_{n\to\infty}\frac{n}{(\ln n)^{k-1}}\int\cdots\int_{[0,1]^k}(1-x_1\cdots x_k)^n f\left((1-x_1\cdots x_k)^n\right)\mathrm{d}x_1\cdots\mathrm{d}x_k\\
\le\ &\frac{1}{(k-1)!}\int_0^1 f(x)\,\mathrm{d}x+\frac{2\varepsilon}{(k-1)!}.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary we obtain

$$\begin{aligned}
&\frac{1}{(k-1)!}\int_0^1 f(x)\,\mathrm{d}x\\
\le\ &\liminf_{n\to\infty}\frac{n}{(\ln n)^{k-1}}\int\cdots\int_{[0,1]^k}(1-x_1\cdots x_k)^n f\left((1-x_1\cdots x_k)^n\right)\mathrm{d}x_1\cdots\mathrm{d}x_k\\
\le\ &\limsup_{n\to\infty}\frac{n}{(\ln n)^{k-1}}\int\cdots\int_{[0,1]^k}(1-x_1\cdots x_k)^n f\left((1-x_1\cdots x_k)^n\right)\mathrm{d}x_1\cdots\mathrm{d}x_k\\
\le\ &\frac{1}{(k-1)!}\int_0^1 f(x)\,\mathrm{d}x.
\end{aligned}$$

Thus

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{n}{(\ln n)^{k-1}} \int\cdots\int_{[0,1]^k} (1-x_1\cdots x_k)^n f\left((1-x_1\cdots x_k)^n\right) \mathrm{d}x_1\cdots \mathrm{d}x_k \\
= &\limsup_{n\to\infty} \frac{n}{(\ln n)^{k-1}} \int\cdots\int_{[0,1]^k} (1-x_1\cdots x_k)^n f\left((1-x_1\cdots x_k)^n\right) \mathrm{d}x_1\cdots \mathrm{d}x_k \\
= &\frac{1}{(k-1)!}\int_0^1 f(x)\,\mathrm{d}x,
\end{aligned}$$

and the theorem is proved □

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The 14th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2020

MARIAN PANŢIRUC[1)], VASILE POP[2)], MIRCEA RUS[3)], RADU STRUGARIU[4)], TIBERIU TRIF[5)]

Abstract. The 14th South Eastern European Mathematical Olympiad for University Students (SEEMOUS 2020) took place on March 3–8, 2020, in Thessaloniki, Greece. We present the competition problems and their solutions, as given by the corresponding authors. Alternative solutions provided by members of the jury or by the contestants are also included.

Keywords: Adjugate matrix, similar matrices, minimal polynomial, diagonalizable matrix, rank, trace, eigenvalues, eigenvectors, Jordan form, Riemann integral, integral convergence, periodic function, differentiable function, sequences of real numbers, series of real numbers.

MSC: Primary 15A03; Secondary 15A21, 26D15.

The Mathematical Society of South-Eastern Europe launched this year the 14th South Eastern European Mathematical Competition for University

[1)] Gheorghe Asachi Technical University of Iaşi, Romania, `marian.pantiruc@tuiasi.ro`

[2)] Technical University of Cluj-Napoca, Romania, `Vasile.Pop@math.utcluj.ro`

[3)] Technical University of Cluj-Napoca, Romania, `rus.mircea@math.utcluj.ro`

[4)] Gheorghe Asachi Technical University of Iaşi, Romania, `rstrugariu@tuiasi.ro`

[5)] Babeş-Bolyai University, Cluj-Napoca, Romania, `ttrif@math.ubbcluj.ro`

Students with International Participation (SEEMOUS 2020). The competition was hosted by the School of Mathematics of the Aristotle University of Thessaloniki, Greece, between March 3 and March 8, 2020. Due to the evolution of the COVID-19 epidemic in Europe, some of the teams were not able to participate, which significantly reduced the number of contestants to 58, representing 16 universities from Bulgaria, Romania, Greece and Republic of North Macedonia. The jury awarded 7 gold medals, 14 silver medals and 18 bronze medals. The student Andrei Robert Bâra from University of Bucharest was the absolute winner of the competition, being the only contestant that obtained full marks to all four problems.

We present the problems from the contest and their solutions as given by the corresponding authors, together with alternative solutions provided by members of the jury or by the contestants.

Problem 1. Consider $A \in \mathcal{M}_{2020}(\mathbb{C})$ such that

$$A + A^* = I_{2020}, \qquad A \cdot A^* = I_{2020}, \tag{1}$$

where A^* is the adjugate of A, i.e., the matrix whose elements are $a^*_{ij} = (-1)^{i+j} d_{ji}$, where d_{ji} is the determinant obtained from A by eliminating the row j and the column i. Find the maximum number of matrices A verifying (1) such that any two of them are not similar.

Marian Panţiruc, Gheorghe Asachi Technical University of Iaşi, Romania

The jury considered the problem to be easy. With almost 15% *of the contestants providing a full solution and another* 10% *getting close to a complete one, the jury's evaluation was correct.*

Author's solution. It is known that $A \cdot A^* = \det A \cdot I_{2020}$, hence, from the second relation in (1) we obtain that $\det A = 1$ and A is invertible. Next, multiplying in the first relation of (1) by A, we get $A^2 - A + I_{2020} = O_{2020}$. It follows that the minimal polynomial of A divides $X^2 - X + 1 = (X - \omega)(X - \bar{\omega})$, where $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3}$. Because the factors of the minimal polynomial of A are of degree 1, it follows that A is diagonalizable, so A is similar to a matrix of the form

$$A_k = \begin{pmatrix} \omega I_k & O_{k,2020-k} \\ O_{2020-k,k} & \bar{\omega} I_{2020-k} \end{pmatrix}, \quad k \in \{0, 1, \dots, 2020\}.$$

Since $\det A = 1$, it follows that

$$\begin{aligned} \omega^k \bar{\omega}^{2020-k} = 1 \Leftrightarrow \omega^{2k-2020} = 1 \Leftrightarrow \cos\frac{(2k-2020)\pi}{3} + i \sin\frac{(2k-2020)\pi}{3} = 1 \\ \Leftrightarrow \cos\frac{(2k+2)\pi}{3} + i \sin\frac{(2k+2)\pi}{3} = 1 \\ \Leftrightarrow k = 3n - 1 \in \{0, \dots, 2020\} \Leftrightarrow k \in \{2, 5, 8, \dots, 2018\}. \end{aligned}$$

Two matrices A_{k_1} and A_{k_2} that verify the given relations are not similar if and only if $k_1 \neq k_2$, so the required maximum number of matrices is 673.

Remark 1. The notation A^* is used to denote different objects in different countries, so a more clear statement of the problem is:

Find the maximum number of non similar matrices in $\mathcal{M}_{2020}(\mathbb{C})$ of determinant 1 such that the sum between a matrix and its inverse is the unit matrix I_{2020}.

Remark 2. The interested reader can verify that if $A \in \mathcal{M}_n(\mathbb{C})$ is singular, then the relation $A + A^* = I_n$ holds if and only if $\operatorname{rank} A = n-1$ and $A^2 = A$.

Problem 2. Let $k > 1$ be a real number. Calculate:

(a) $L = \lim\limits_{n\to\infty} \displaystyle\int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x$;

(b) $\lim\limits_{n\to\infty} n \left[L - \displaystyle\int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x\right]$.

Ovidiu Furdui, Alina Sîntămărian, Technical University of Cluj-Napoca, Romania

The jury considered the problem to be of low to medium difficulty. Yet, only one contestant obtained the maximum number of points for this problem, while about 40% of the contestants got a blank score, proving it to be the most difficult problem.

Authors' solution. (a) Using the substitution $x = y^n$, we have that

$$I_n := \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x = nk^n \int_0^1 \left(\frac{y}{y+k-1}\right)^{n-1} \frac{\mathrm{d}y}{y+k-1}.$$

By the second substitution $t = \dfrac{y}{y+k-1}$, we obtain after some calculations that $I_n = nk^n \displaystyle\int_0^{\frac{1}{k}} \frac{t^{n-1}}{1-t}\,\mathrm{d}t$. Next, the integration by parts leads to

$$I_n = \frac{k}{k-1} - k^n \int_0^{\frac{1}{k}} \frac{t^n}{(1-t)^2}\,\mathrm{d}t.$$

It follows that $L := \lim\limits_{n\to\infty} I_n = \dfrac{k}{k-1}$, since

$$0 < k^n \int_0^{\frac{1}{k}} \frac{t^n}{(1-t)^2}\,\mathrm{d}t < k^n \cdot \frac{k^2}{(k-1)^2} \int_0^{\frac{1}{k}} t^n\,\mathrm{d}t = \frac{k}{(n+1)(k-1)^2} \to 0$$

as $n \to \infty$.

(b) Continuing from $L - I_n = k^n \int_0^{\frac{1}{k}} \frac{t^n}{(1-t)^2}\,\mathrm{d}t$, the integration by parts gives

$$L - I_n = \frac{1}{n+1} \cdot \frac{k}{(k-1)^2} - \frac{2k^n}{n+1} \int_0^{\frac{1}{k}} \frac{t^{n+1}}{(1-t)^3}\,\mathrm{d}t.$$

This implies that

$$\begin{aligned}\lim_{n\to\infty} n\,(L - I_n) &= \lim_{n\to\infty} \left[\frac{n}{n+1} \cdot \frac{k}{(k-1)^2} - \frac{2k^n n}{n+1} \int_0^{\frac{1}{k}} \frac{t^{n+1}}{(1-t)^3}\,\mathrm{d}t\right] \\ &= \frac{k}{(k-1)^2} - 2 \lim_{n\to\infty} k^n \int_0^{\frac{1}{k}} \frac{t^{n+1}}{(1-t)^3}\,\mathrm{d}t = \frac{k}{(k-1)^2},\end{aligned}$$

since

$$0 < k^n \int_0^{\frac{1}{k}} \frac{t^{n+1}}{(1-t)^3}\,\mathrm{d}t < \frac{k^{n+3}}{(k-1)^3} \int_0^{\frac{1}{k}} t^{n+1}\,\mathrm{d}t = \frac{k}{(k-1)^3(n+2)} \to 0$$

as $n \to \infty$.

Solution proposed by Tiberiu Trif. (a) Similarly to the authors' solution, after the first substitution $x = y^n$ we get

$$I_n := \int_0^1 \left(\frac{k}{\sqrt[n]{x} + k - 1}\right)^n \mathrm{d}x = \int_0^1 kn \left(\frac{ky}{y+k-1}\right)^{n-1} \frac{\mathrm{d}y}{y+k-1}\,.$$

Further, a second substitution $t = \left(\frac{ky}{y+k-1}\right)^n$ yields

$$\mathrm{d}t = k(k-1)n\left(\frac{ky}{y+k-1}\right)^{n-1} \frac{\mathrm{d}y}{(y+k-1)^2}, \qquad y + k - 1 = \frac{k(k-1)}{k - t^{1/n}}\,,$$

whence

$$I_n = \int_0^1 \frac{y+k-1}{k-1}\, k(k-1)n \left(\frac{ky}{y+k-1}\right)^{n-1} \frac{\mathrm{d}y}{(y+k-1)^2} = k \int_0^1 \frac{\mathrm{d}t}{k - t^{1/n}}\,.$$

By virtue of the Arzelà bounded convergence theorem, we deduce that

$$\lim_{n\to\infty} I_n = k \lim_{n\to\infty} \int_0^1 \frac{\mathrm{d}t}{k - t^{1/n}} = k \int_0^1 \frac{\mathrm{d}t}{k-1} = \frac{k}{k-1}\,.$$

(b) Taking into account the representation of I_n obtained in part (a), we have

$$\begin{aligned}
n\left(\frac{k}{k-1}-I_n\right) &= n\left(\frac{k}{k-1}-k\int_0^1 \frac{\mathrm{d}t}{k-t^{1/n}}\right)\\
&= kn\int_0^1\left(\frac{1}{k-1}-\frac{1}{k-t^{1/n}}\right)\mathrm{d}t\\
&= \frac{k}{k-1}\int_{0+0}^1 \frac{n\left(1-t^{1/n}\right)}{k-t^{1/n}}\mathrm{d}t.
\end{aligned}$$

Consider the sequence of functions $f_n:(0,1]\to(0,\infty)$ $(n\geq 1)$, defined by

$$f_n(t):=\frac{n\left(1-t^{1/n}\right)}{k-t^{1/n}}\quad \text{for all } t\in(0,1].$$

It is immediately seen that

$$\lim_{n\to\infty} f_n(t) = -\frac{\ln t}{k-1} =: f(t)\quad \text{for all } t\in(0,1]$$

and that

$$f_n(t)\leq f(t)\quad \text{for all } n\geq 1 \text{ and all } t\in(0,1]$$

because

$$n\left(1-t^{1/n}\right)\leq -\ln t\quad\text{and}\quad \frac{1}{k-t^{1/n}}\leq\frac{1}{k-1}\,.$$

Since

$$\int_{0+0}^1 f(t)\,\mathrm{d}t = -\frac{1}{k-1}\int_{0+0}^1 \ln t\,\mathrm{d}t=\frac{1}{k-1}\,,$$

by applying the dominated convergence theorem we deduce that

$$\begin{aligned}
\lim_{n\to\infty} n\left(\frac{k}{k-1}-I_n\right) &= \frac{k}{k-1}\lim_{n\to\infty}\int_{0+0}^1 f_n(t)\,\mathrm{d}t\\
&= \frac{k}{k-1}\int_{0+0}^1 f(t)\,\mathrm{d}t=\frac{k}{(k-1)^2}\,.
\end{aligned}$$

Solution to part (a), proposed by Tiberiu Trif. Consider the sequence of functions $f_n:(0,1]\to(0,\infty)$ $(n\geq 1)$, defined by

$$f_n(x):=\left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n\quad \text{for all } x\in(0,1].$$

A standard computation shows that

$$\lim_{n\to\infty} f_n(x) = x^{-1/k} =: f(x)\quad \text{for all } x\in(0,1].$$

We claim that

$$f_n(x)\leq f(x)\quad \text{for all } n\geq 1 \text{ and all } x\in(0,1].$$

Indeed, given any $x \in (0,1]$, we have

$$f_n(x) \le f(x) \iff \frac{k}{\sqrt[n]{x}+k-1} \le x^{-\frac{1}{nk}} \iff x^{\frac{1}{n}} + k - 1 \ge kx^{\frac{1}{nk}}.$$

Letting $t := x^{\frac{1}{nk}}$, the last inequality becomes

$$t^k + k - 1 \ge kt. \tag{2}$$

To prove (2), note that $t \in (0,1]$ and by virtue of Bernoulli's inequality, we have

$$t^k + k - 1 = (1 + (t-1))^k + k - 1 \ge 1 + k(t-1) + k - 1 = kt,$$

hence (2) holds. Since

$$\int_{0+0}^{1} f(x)\,\mathrm{d}x = \int_{0+0}^{1} x^{-1/k}\,\mathrm{d}x = \frac{k}{k-1}\,,$$

by applying the dominated convergence theorem we conclude that

$$\lim_{n\to\infty}\int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x = \lim_{n\to\infty}\int_{0+0}^1 f_n(x)\mathrm{d}x = \int_{0+0}^1 f(x)\mathrm{d}x = \frac{k}{k-1}\,.$$

Solution proposed by Mircea Rus. The problem can be seen as a particular case of the following set of limits:

$$(i)\quad L = \lim_{n\to\infty}\int_0^1 f^n\left(\sqrt[n]{x}\right)\,\mathrm{d}x,$$

$$(ii)\quad \lim_{n\to\infty}\left(L - \int_0^1 f^n\left(\sqrt[n]{x}\right)\,\mathrm{d}x\right),$$

where $f : [0,1] \to \mathbb{R}$ is of class C^2 such that $f(1) = 1$ and the mapping $g : [0,1] \to \mathbb{R}$, $g(t) = tf(t)$, has no critical points (i.e., g' has no roots). It is easy to check that $f(t) = \dfrac{k}{t+k-1}$ verifies these conditions.

By the substitution $t = \sqrt[n]{x}$, we can write

$$I_n = \int_0^1 f^n\left(\sqrt[n]{x}\right)\,\mathrm{d}x = n\int_0^1 t^{n-1} f^n(t)\,\mathrm{d}t = n\int_0^1 f(t)\cdot g^{n-1}(t)\,\mathrm{d}t.$$

Since $g(0) = 0$, $g(1) = f(1) = 1$ and g' has no roots, the function g has a differentiable inverse $h : [0,1] \to [0,1]$. Substituting $y = g(t)$ (i.e., $t = h(y)$), it follows that

$$I_n = n\int_0^1 y^{n-1}\cdot f(h(y))h'(y)\,\mathrm{d}y = n\int_0^1 y^{n-1}F(y)\,\mathrm{d}y, \quad \text{where } F = (f\circ h)\cdot h'.$$

Finally, a last change of variable $x = y^n$ leads to

$$I_n = \int_0^1 F\left(\sqrt[n]{x}\right)\,\mathrm{d}x.$$

Note that F is of class C^1, since f (hence g and h) is of class C^2. Since the sequence $(|F(\sqrt[n]{x})|)_n$ is uniformly bounded by $\max\limits_{y\in[0,1]} |F(y)|$ and

$$\lim_{n\to\infty} F\left(\sqrt[n]{x}\right) = F(1) \quad \text{for all } x \in (0,1],$$

we conclude by Arzelà bounded convergence theorem that

$$\begin{aligned} L &= \lim_{n\to\infty} \int_0^1 f^n\left(\sqrt[n]{x}\right) \mathrm{d}x = \lim_{n\to\infty} \int_0^1 F\left(\sqrt[n]{x}\right) \mathrm{d}x = \int_0^1 F(1)\, \mathrm{d}x = F(1) \\ &= f(h(1)) \cdot h'(1) = \frac{1}{g'(1)} = \frac{1}{1+f'(1)}, \end{aligned}$$

which for $f(t) = \dfrac{k}{t+k-1}$ yields the result $\dfrac{k}{k-1}$.

For the second limit, we rewrite

$$n\,(L - I_n) = n \int_0^1 \left(F(1) - F\left(\sqrt[n]{x}\right)\right) \mathrm{d}x = \int_0^{1-0} \frac{F\left(\sqrt[n]{x}\right) - F(1)}{\sqrt[n]{x}-1} \cdot \frac{1-\sqrt[n]{x}}{\frac{1}{n}}\, \mathrm{d}x. \tag{3}$$

We have that

$$\left| \frac{F\left(\sqrt[n]{x}\right) - F(1)}{\sqrt[n]{x}-1} \right| \le \max_{t\in[0,1]} \left|F'(t)\right| \quad \text{for all } n \ge 1 \text{ and } x \in [0,1).$$

Also, it is easy to check that

$$\left| \frac{1-\sqrt[n]{x}}{\frac{1}{n}} \right| = \frac{1-\sqrt[n]{x}}{\frac{1}{n}} \le -\ln x \quad \text{for all } n \ge 1 \text{ and } x \in (0,1].$$

Because

$$\lim_{n\to\infty} \left(\frac{F\left(\sqrt[n]{x}\right) - F(1)}{\sqrt[n]{x}-1} \cdot \frac{1-\sqrt[n]{x}}{\frac{1}{n}} \right) = -F'(1) \cdot \ln x \quad \text{for all } x \in (0,1),$$

we can now apply the dominated convergence theorem and conclude from (3) that

$$\lim_{n\to\infty} n\,(L - I_n) = -F'(1) \int_{0+0}^{1-0} \ln x \, \mathrm{d}x = F'(1).$$

An elementary computation gives

$$\begin{aligned} F'(1) &= f'(h(1)) \cdot \left(h'(1)\right)^2 + f(h(1)) \cdot h''(1) = \frac{f'(1)}{\left(g'(1)\right)^2} - \frac{g''(1)}{\left(g'(1)\right)^3} \\ &= \frac{f'(1)}{\left(1+f'(1)\right)^2} - \frac{2f'(1)+f''(1)}{\left(1+f'(1)\right)^3} = \frac{\left(f'(1)\right)^2 - f'(1) - f''(1)}{\left(1+f'(1)\right)^3}, \end{aligned}$$

which, in the case of $f(t) = \dfrac{k}{t+k-1}$, provides the result $\dfrac{k}{(k-1)^2}$.

Problem 3. Let n be a positive integer, $k \in \mathbb{C}$ and $A \in \mathcal{M}_n(\mathbb{C})$ such that $\operatorname{Tr} A \neq 0$ and

$$\operatorname{rank} A + \operatorname{rank}\left((\operatorname{Tr} A) \cdot I_n - kA\right) = n. \tag{4}$$

Find $\operatorname{rank} A$.

Vasile Pop, Technical University of Cluj-Napoca, Romania
Mihai Opincariu, Avram Iancu National College, Brad, Romania

The jury considered this problem to be also of low to medium difficulty. This assessment was right, since about 15% *of the contestants obtained a maximum score, while other* 10% *got close to a complete solution.*

Authors' solution. For simplicity, denote $\alpha = \operatorname{Tr} A$. Consider the block matrix

$$M = \left[\begin{array}{c|c} A & O_n \\ \hline O_n & \alpha I_n - kA \end{array}\right].$$

We perform on M a sequence of elementary transformations on rows and columns (that do not change the rank) as follows:

$$\begin{aligned} M &\xrightarrow{R_1} \left[\begin{array}{c|c} A & O_n \\ \hline A & \alpha I_n - kA \end{array}\right] \xrightarrow{C_1} \left[\begin{array}{c|c} A & kA \\ \hline A & \alpha I_n \end{array}\right] \\ &\xrightarrow{R_2} \left[\begin{array}{c|c} A - \frac{k}{\alpha}A^2 & O_n \\ \hline A & \alpha I_n \end{array}\right] \xrightarrow{C_2} \left[\begin{array}{c|c} A - \frac{k}{\alpha}A^2 & O_n \\ \hline O_n & \alpha I_n \end{array}\right] = N, \end{aligned}$$

where R_1 is the left multiplication by $\left[\begin{array}{c|c} I_n & O_n \\ \hline I_n & I_n \end{array}\right]$; C_1 is the right multiplication by $\left[\begin{array}{c|c} I_n & kI_n \\ \hline O_n & I_n \end{array}\right]$; R_2 is the left multiplication by $\left[\begin{array}{c|c} I_n & -\frac{k}{\alpha}A \\ \hline O_n & I_n \end{array}\right]$; C_2 is the right multiplication by $\left[\begin{array}{c|c} I_n & O_n \\ \hline -\frac{1}{\alpha}A & I_n \end{array}\right]$.

It follows that

$$n = \operatorname{rank}\, A + \operatorname{rank}\left(\alpha I_n - kA\right) = \operatorname{rank}\, M = \operatorname{rank}\, N = \operatorname{rank}\left(A - \frac{k}{\alpha}A^2\right) + n,$$

hence $\operatorname{rank}\left(A - \dfrac{k}{\alpha}A^2\right) = 0$, which leads to $A - \frac{k}{\alpha}A^2 = O_n$. Letting $B := \frac{k}{\alpha}A$, it follows that $B^2 = B$, hence

$$\operatorname{rank}\, B = \operatorname{Tr} B = \operatorname{Tr}\left(\frac{k}{\alpha}A\right) = \frac{k}{\alpha}\operatorname{Tr} A = k = \operatorname{rank} A.$$

Solution proposed by Cornel Băeţica. It is known (see, e.g., [1, Theorem 2.6]) that the equality in Sylvester's rank inequality

$$\operatorname{rank} AB \geq \operatorname{rank} A + \operatorname{rank} B - n$$

holds if and only if $\operatorname{Ker} A \subseteq \operatorname{Im} B$, i.e., the null space of A is a subset of the image space of B. Let $B = (\operatorname{Tr} A) \cdot I_n - kA$. Then for all $x \in \operatorname{Ker} A$,

$$x = \frac{1}{\operatorname{Tr} A}((\operatorname{Tr} A) \cdot I_n - kA)x = \frac{1}{\operatorname{Tr} A} \cdot Bx \in \frac{1}{\operatorname{Tr} A} \cdot \operatorname{Im} B = \operatorname{Im} B,$$

hence $\operatorname{Ker} A \subseteq \operatorname{Im} B$, which means that

$$\operatorname{rank}(A \cdot ((\operatorname{Tr} A) \cdot I_n - kA)) = \operatorname{rank} A + \operatorname{rank}((\operatorname{Tr} A) \cdot I_n - kA) - n \overset{(4)}{=} 0.$$

This shows that $A \cdot ((\operatorname{Tr} A) \cdot I_n - kA) = O_n$, which implies that $k \neq 0$ and the minimal polynomial of A divides $X\left(\frac{\operatorname{Tr} A}{k} - X\right)$, hence A is diagonalizable and the possible eigenvalues are 0 and $\frac{\operatorname{Tr} A}{k} \neq 0$. It follows that $\operatorname{rank} A$ equals the multiplicity of $\frac{\operatorname{Tr} A}{k}$ and taking into account that $\operatorname{Tr} A$ is the sum of all eigenvalues of A, we obtain $\operatorname{Tr} A = \frac{\operatorname{Tr} A}{k} \cdot \operatorname{rank} A$, so $\operatorname{rank} A = k$.

Solution proposed by Marian Panţiruc. We see, as above, that $k \neq 0$. Then, obviously, $\operatorname{rank}((\operatorname{Tr} A) \cdot I_n - kA) = \operatorname{rank}\left(\frac{\operatorname{Tr} A}{k} I_n - A\right)$ and for convenience, denote $B = \frac{\operatorname{Tr} A}{k} \cdot I_n - A$. Then

$$\begin{aligned}
n = \operatorname{rank}\left(\frac{\operatorname{Tr} A}{k} \cdot I_n\right) &= \operatorname{rank}(A + B) = \dim(\operatorname{Im}(A + B)) \\
&\leq \dim(\operatorname{Im} A + \operatorname{Im} B) = \dim(\operatorname{Im} A) + \dim(\operatorname{Im} B) - \dim(\operatorname{Im} A \cap \operatorname{Im} B) \\
&= \operatorname{rank} A + \operatorname{rank} B - \dim(\operatorname{Im} A \cap \operatorname{Im} B) \leq \operatorname{rank} A + \operatorname{rank} B \overset{(4)}{=} n.
\end{aligned}$$

Since there is equality in the above inequalities, it follows that $\operatorname{Im} A \cap \operatorname{Im} B = \{0\}$. Now, let $x \in \mathbb{C}^n$. Note that $AB = BA$. Then $ABx = BAx \in \operatorname{Im} A \cap \operatorname{Im} B = \{0\}$, hence $ABx = 0$ for all $x \in \mathbb{C}^n$, so $AB = A \cdot \left(\frac{\operatorname{Tr} A}{k} \cdot I_n - A\right) = O_n$. The conclusion follows, repeating the final argument from the first proof.

Solution proposed by Cristian Grecu, contestant. Observe first that $k \neq 0$, otherwise (4) would imply that $\operatorname{rank} A = 0$, so $\operatorname{Tr} A = 0$, which is false. Using the rank theorem, (4) gives $\operatorname{def} A + \operatorname{def}((\operatorname{Tr} A) \cdot I_n - kA) = n$, hence

$$\operatorname{def} A + \operatorname{def}\left(\frac{\operatorname{Tr} A}{k} \cdot I_n - A\right) = n.$$

This means that we can choose a basis $\mathcal{B} = \{u_1, \ldots, u_p\}$ in $\ker A$ and a basis $\mathcal{B}' = \{v_1, \ldots, v_{n-p}\}$ in $\ker\left(\frac{\operatorname{Tr} A}{k} \cdot I_n - A\right)$. Observe that for every

$j = 1, \ldots, n-p$

$$Av_j = \left(\frac{\operatorname{Tr} A}{k} \cdot I_n - \left(\frac{\operatorname{Tr} A}{k} \cdot I_n - A\right)\right) \cdot v_j = \frac{\operatorname{Tr} A}{k} \cdot v_j,$$

so each of the $n-p$ linearly independent eigenvectors v_j of $\frac{\operatorname{Tr} A}{k} \cdot I_n - A$ (corresponding to the eigenvalue 0) is an eigenvector for A, corresponding to the non-zero eigenvalue $\frac{\operatorname{Tr} A}{k}$. Since A already has p linearly independent eigenvectors corresponding to 0 (the elements of $\mathcal{B}$), it follows that $\{u_1, \ldots, u_p, v_1, \ldots, v_{n-p}\}$ is, in fact, a basis in $\mathbb{C}^n$, consisting of eigenvectors of A. This means that A is diagonalizable and its rank now equals the multiplicity of the nonzero eigenvalue $\frac{\operatorname{Tr} A}{k}$. By repeating the final part of the previous proof, the conclusion follows.

Solution proposed by Andrei Jelea, contestant. We will use that the rank of a square matrix is at least the number of its non-zero eigenvalues.

Let $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$ and $\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n \neq 0$ be the eigenvalues of A (with r possibly 0). Then $\operatorname{rank} A \geq n - r$. Also, the eigenvalues of $(\operatorname{Tr} A) \cdot I_n - kA$ are $\{\mu_i = \operatorname{Tr} A - k\lambda_i \mid i = 1, 2, \ldots, n\}$, with $\mu_1 = \mu_2 = \ldots = \mu_r = \operatorname{Tr} A \neq 0$. This leads to $\operatorname{rank}((\operatorname{Tr} A) \cdot I_n - kA) \geq r$, hence

$$n = (n-r) + r \leq \operatorname{rank} A + \operatorname{rank}((\operatorname{Tr} A) \cdot I_n - kA) \stackrel{(4)}{=} n.$$

This means that $\operatorname{rank} A = n - r$ and $\operatorname{rank}((\operatorname{Tr} A) \cdot I_n - kA) = r$. Also, $\mu_i = 0$ for all $i \in \{r+1, \ldots, n\}$, otherwise $(\operatorname{Tr} A) \cdot I_n - kA$ has more than r non-zero eigenvalues, which contradicts $\operatorname{rank}((\operatorname{Tr} A) \cdot I_n - kA) = r$. Since $\operatorname{Tr} A = \sum_{i=1}^{n} \lambda_i = \sum_{i=r+1}^{n} \lambda_i$, it follows that

$$0 = \sum_{i=r+1}^{n} \mu_i = (n-r) \operatorname{Tr} A - k \sum_{i=r+1}^{n} \lambda_i = (n-r-k) \operatorname{Tr} A,$$

hence $n - r - k = 0$, which leads to $\operatorname{rank} A = n - r = k$.

Solution proposed by Ştefan Cristian Popa, contestant. Recall that if J is the Jordan canonical form of some matrix $A \in \mathcal{M}_n(\mathbb{C})$, then $\operatorname{rank} A = \operatorname{rank} J = n - p$, where p is the number of Jordan blocks in J corresponding to the eigenvalue 0.

We see, as in the previous proofs, that $k \neq 0$. Write A as $A = PJP^{-1}$, where J is its Jordan canonical form and P is a nonsingular matrix. Since

$(\operatorname{Tr} A) \cdot I_n - kA = (\operatorname{Tr} A) \cdot P \cdot P^{-1} - P \cdot kJ \cdot P^{-1} = P \cdot ((\operatorname{Tr} A) \cdot I_n - kJ) \cdot P^{-1}$

and $\operatorname{Tr} A = \operatorname{Tr} J$, we can write (4) as

$$\operatorname{rank} J + \operatorname{rank}\left((\operatorname{Tr} J)\cdot I_n - kJ\right) = \operatorname{rank} J + \operatorname{rank}\left(J - \frac{\operatorname{Tr} J}{k}\cdot I_n\right) = n. \quad (5)$$

Since $\operatorname{Tr} J = \operatorname{Tr} A \neq 0$, it follows that $J \neq O_n$, hence $\operatorname{rank} J \geq 1$ and, by (5), $\operatorname{rank}\left(J - \frac{\operatorname{Tr} J}{k}\cdot I_n\right) \leq n-1$. Hence, we conclude that $\det\left(J - \frac{\operatorname{Tr} J}{k}\cdot I_n\right) = 0$, which means that 0 is an eigenvalue of the Jordan form $J - \frac{\operatorname{Tr} J}{k}\cdot I_n$, while all the 0 eigenvalues in $J - \frac{\operatorname{Tr} J}{k}\cdot I_n$ correspond to the eigenvalues $\lambda = \frac{\operatorname{Tr} J}{k}$ in J.

Let p and q be the number of Jordan blocks associated to the eigenvalue 0 in J and $J - \frac{\operatorname{Tr} J}{k}\cdot I_n$, respectively ($p$ may be 0); equivalently, q is the number of Jordan blocks associated to the eigenvalue λ in J. Then

$$\operatorname{rank} J = n - p, \qquad \operatorname{rank}\left(J - \frac{\operatorname{Tr} J}{k}\cdot I_n\right) = n - q,$$

hence $n - p + n - q = n$ by (5), which leads to $p + q = n$. This means that the only eigenvalues of J are 0 and λ, each of them belonging to a Jordan block of size 1. In conclusion,

$$\operatorname{Tr} J = p\cdot 0 + q\cdot\lambda = q\cdot\frac{\operatorname{Tr} J}{k} \implies q = k$$

and

$$\operatorname{rank} A = \operatorname{rank} J = n - p = q = k.$$

Solution proposed by Mădălin Mitrofan, contestant. As seen in the previous solutions, $k \neq 0$. Also, $\lambda = \frac{\operatorname{Tr} A}{k} \neq 0$ is an eigenvalue of A, by an argument similar to one presented in the previous proof. Since

$$\operatorname{rank} A = n - \operatorname{rank}\left((\operatorname{Tr} A)\cdot I_n - kA\right) = n - \operatorname{rank}\left(\lambda\cdot I_n - A\right),$$

it follows that $\operatorname{rank} A = q(\lambda)$, the geometric multiplicity of λ. On the other hand, $q(\lambda)$ is the number of blocks in J_A, the Jordan canonical form of A, hence

$$\operatorname{rank} J_A = \operatorname{rank} A = q(\lambda).$$

This happens only when all the blocks in J_A corresponding to λ are of size 1, while the other blocks are all 0, i.e.,

$$J_A = \lambda\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \qquad r = \operatorname{rank} A.$$

Finally,

$$\operatorname{Tr} A = \operatorname{Tr} J_A = r\cdot\lambda = \operatorname{rank} A\cdot\frac{\operatorname{Tr} A}{k},$$

that leads to $\operatorname{rank} A = k$.

Remark. From the seven solutions to this problem, we can summarize several additional conclusions:

(1) $A \cdot ((\operatorname{Tr} A) \cdot I_n - kA) = O_n$; equivalently, $B = \dfrac{k}{\operatorname{Tr} A} \cdot A$ is idempotent ($B^2 = B$), hence $B^m = B$, which translates as $k^{m-1}A^m = (\operatorname{Tr} A)^{m-1} A$ for all $m \geq 1$.

(2) $\operatorname{Ker} A = \operatorname{Im}((\operatorname{Tr} A) \cdot I_n - kA)$, $\operatorname{Im} A = \operatorname{Ker}((\operatorname{Tr} A) \cdot I_n - kA)$ and $\mathbb{C}^n = \operatorname{Ker} A \oplus \operatorname{Ker}((\operatorname{Tr} A) \cdot I_n - kA)$.

(3) A is diagonalizable, with the eigenvalues $\dfrac{\operatorname{Tr} A}{k}$ and, possibly, 0.

Problem 4. Consider $0 < a < T$, $D = \mathbb{R} \setminus \{kT + a \mid k \in \mathbb{Z}\}$, and let $f : D \to \mathbb{R}$ be a $T-$periodic and differentiable function such that $f' > 1$ on $(0, a)$ and which satisfies

$$f(0) = 0, \ \lim_{\substack{x\to a\\ x<a}} f(x) = +\infty \text{ and } \lim_{\substack{x\to a\\ x<a}} \frac{f'(x)}{f^2(x)} = 1.$$

(a) Prove that for every $n \in \mathbb{N}^*$, the equation $f(x) = x$ has an unique solution x_n in the interval $(nT, nT + a)$.

(b) Let $y_n = nT + a - x_n$ and $z_n = \displaystyle\int_0^{y_n} f(x)\,\mathrm{d}x$. Prove that $\lim\limits_{n\to\infty} y_n = 0$ and study the convergence of the series $\displaystyle\sum_{n=1}^{\infty} y_n$ and $\displaystyle\sum_{n=1}^{\infty} z_n$.

Radu Strugariu, Gheorghe Asachi Technical University of Iaşi, Romania

The jury considered this problem to be of medium to high difficulty. Since only two maximum scores were obtained, and another five contestants got close to the complete solution, we consider that the assessment of the jury was right.

Author's solution. (a) By the periodicity of f, it follows that $f(nT) = 0 < nT$ and $\lim\limits_{x \nearrow nT+a} f(x) = +\infty > nT + a$ for all $n \in \mathbb{N}^*$, hence the equation $f(x) = x$ has at least one solution in the interval $(nT, nT + a)$.

Next, consider the function $g(x) = f(x) - x$ on $(nT, nT + a)$ and observe that if there are two solutions of the equation $f(x) = x$ in $(nT, nT + a)$, say $x_n^1 < x_n^2$, then, by Rolle's Theorem, there exists $r_n \in (x_n^1, x_n^2) \subset (nT, nT + a)$ such that $g'(r_n) = f'(r_n) - 1 = 0$, which is in contradiction with $g' = f' - 1 > 0$ on $(nT, nT + a)$ (also, by periodicity).

(b) Observe that f is strictly increasing on $(nT, nT + a)$ for all n. We prove that (y_n) is decreasing. By contradiction, suppose that $y_n < y_{n+1}$ for some n. Then $T + x_n > x_{n+1}$, and by the monotonicity of f, it follows that

$$x_n = f(x_n) = f(x_n + T) > f(x_{n+1}) = x_{n+1},$$

which is an obvious contradiction.

Since $y_n \in (0, a)$ for every n, it follows that (y_n) is convergent, so there exists $\overline{y} = \lim\limits_{n\to\infty} y_n \in [0, a)$. Suppose, by contradiction, that $\overline{y} > 0$. Since $x_n - nT \to a - \overline{y}$ for $n \to \infty$, it follows by the continuity of f on $(-T, a)$ that $f(x_n - nT) \to f(a - \overline{y})$ for $n \to \infty$. At the same time, $f(x_n - nT) = f(x_n) = x_n \to \infty$, hence a contradiction. Therefore, $\lim\limits_{n\to\infty} y_n = 0$.

Next, we prove that

$$\lim_{n\to\infty} ny_n = \frac{1}{T}, \tag{6}$$

hence $\sum\limits_{n=1}^{\infty} y_n$ diverges by a comparison test. For that, observe that

$$\lim_{n\to\infty} ny_n = \frac{1}{T} \lim_{n\to\infty} \frac{nT}{x_n} \cdot x_n y_n = \frac{1}{T} \lim_{n\to\infty} x_n y_n.$$

Moreover,

$$\begin{aligned}\lim_{n\to\infty} x_n y_n &= \lim_{n\to\infty} f(x_n) y_n = \lim_{n\to\infty} f(nT + a - y_n) y_n = \lim_{n\to\infty} f(a - y_n) y_n \\ &= \lim_{n\to\infty} \frac{y_n}{\frac{1}{f(a-y_n)}} = -\lim_{n\to\infty} \frac{(a - y_n) - a}{\frac{1}{f(a-y_n)}}.\end{aligned}$$

But $a - y_n \nearrow a$, so the previous limit is $-\lim\limits_{x\nearrow a} \dfrac{x-a}{\frac{1}{f(x)}} = -\lim\limits_{x\nearrow a} \dfrac{1}{-\frac{f'(x)}{f^2(x)}} = 1$, which concludes (6).

For the second series, observe that for every n, there exists $c_n \in (0, y_n)$ such that $z_n = y_n \cdot f(c_n)$. Since f is increasing on $(0, a)$, it follows that

$$z_n \le y_n f(y_n) = y_n^2 \cdot \frac{f(y_n)}{y_n}. \tag{7}$$

Since f is differentiable at 0 and $\dfrac{f(y_n)}{y_n} \to f'(0) \ge 0$ for $n \to \infty$, it follows that the sequence $\left(\dfrac{f(y_n)}{y_n}\right)_n$ is bounded. By (6) and (7), we conclude that there exist $n_0 \in \mathbb{N}$ and $K > 0$ such that

$$0 \le z_n \le \frac{K}{n^2}, \quad \text{for all } n \ge n_0.$$

By a comparison test, $\sum\limits_{n=1}^{\infty} z_n$ converges.

Solution proposed by Tiberiu Trif. (a) Fix $n \in \mathbb{N}^*$. Then the function $g : [nT, nT + a) \to \mathbb{R}$, $g(x) = f(x) - x$, is continuous and, by the periodicity of f, satisfies $g(nT) = -nT < 0$, $\lim\limits_{x \nearrow nT+a} g(x) = +\infty$. Also, $g'(x) = f'(x) - 1 > 0$, so g is increasing. It follows that the equation $g(x) = 0$ has an unique solution $x_n \in (nT, nT + a)$.

(b) Observe that $f : [0, a) \to [0, +\infty)$ is an increasing bijection, hence $f^{-1} : [0, +\infty) \to [0, a)$ is also an increasing bijection.
Moreover,

$$a - y_n = x_n - nT \Longrightarrow f(a - y_n) = f(x_n - nT) = f(x_n) = x_n,$$

hence $a - y_n = f^{-1}(x_n)$. Since $\lim\limits_{n\to\infty} x_n = +\infty$, it follows that

$$\lim_{n\to\infty} (a - y_n) = \lim_{n\to\infty} f^{-1}(x_n) = \lim_{x\to\infty} f^{-1}(x) = a,$$

so $\lim\limits_{n\to\infty} y_n = 0$.
Since

$$x_n \in (nT, nT + a) \Longrightarrow \lim_{n\to\infty} \frac{x_n}{n} = T,$$

by writing $y_n = a - f^{-1}(x_n)$, it follows that

$$\begin{aligned}
\lim_{n\to\infty} ny_n &= \lim_{n\to\infty} \frac{n}{x_n} \cdot \lim_{n\to\infty} \frac{a - f^{-1}(x_n)}{\frac{1}{x_n}} = \frac{1}{T} \lim_{n\to\infty} \frac{a - f^{-1}(x_n)}{\frac{1}{x_n}} \\
&= \frac{1}{T} \lim_{y\to\infty} \frac{a - f^{-1}(y)}{\frac{1}{y}} \overset{f^{-1}(y)=x}{=\!=\!=\!=} \frac{1}{T} \lim_{x\nearrow a} \frac{a - x}{\frac{1}{f(x)}} = \frac{1}{T} \lim_{x\nearrow a} \frac{-1}{-\frac{f'(x)}{f^2(x)}} \\
&= \frac{1}{T}.
\end{aligned}$$

It follows that $\sum\limits_{n=1}^{\infty} y_n \sim \sum\limits_{n=1}^{\infty} \frac{1}{n}$, i.e., it diverges.

For the second series, observe by the monotonicity of f that

$$z_n \le y_n f(y_n) = \frac{f(y_n)}{y_n} \cdot y_n^2 =: u_n.$$

Then

$$\lim_{n\to\infty} \frac{u_n}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{f(y_n)}{y_n} (ny_n)^2 = f'(0) \cdot \frac{1}{T^2},$$

so $\sum\limits_{n=1}^{\infty} u_n \sim \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, hence $\sum\limits_{n=1}^{\infty} z_n$ is convergent.

Remark. One example of such function f which satisfies the conditions of the problem is $f(x) = \operatorname{tg} x$, for $T = \pi$ and $a = \frac{\pi}{2}$. However, since $y_n =$

$n\pi + \frac{\pi}{2} - x_n \to 0$, the key relation

$$\begin{aligned}\lim_{n\to\infty} ny_n &= \lim_{n\to\infty} \frac{y_n}{\operatorname{tg} y_n} \cdot \lim_{n\to\infty} n \operatorname{tg}\left(n\pi + \frac{\pi}{2} - x_n\right) = \lim_{y\to 0} \frac{y}{\operatorname{tg} y} \cdot \lim_{n\to\infty} \frac{n}{\operatorname{tg} x_n} \\ &= \lim_{n\to\infty} \frac{n}{x_n} = \frac{1}{\pi}\end{aligned}$$

can be much easier obtained in this case.

References

[1] F. Zhang, *Matrix Theory*, 2nd edition, Springer, 2011.

MATHEMATICAL NOTES

An Ermakov type test

Gabriel T. Prăjitură[1)]

Abstract. V. P. Ermakov gave in 1872 a criterion for the convergence of a series with positive terms by employing the exponential function and the integral test. In this note we extended this criterion to more general functions subject to a certain regularity growth condition.

Keywords: positive series, convergence test.

MSC: Primary 37A45, 40A30; Secondary 40E05

Ermakov test [1] is a test for convergence of positive series based on the exponential function and the integral test. A proof can be found in [2] on pages 48–51. We extended this test to more general functions subject to a certain regularity growth condition.

Theorem 1. *Let f be a continuous, positive, decreasing function defined for $x \geq 1$ and g a differentiable, strictly increasing function defined on the same domain such that there is a sequence $(x_n) \subset [1, \infty)$ increasing to ∞ for which $g(x_n) > x_n$ for every $n \geq 1$.*

(i) *If there is a number $q \in (0, 1)$ such that*

$$\frac{f(g(x))g'(x)}{f(x)} \leq q$$

for all sufficiently large x then $\sum_n f(n)$ is convergent.

(ii) *If, for all sufficiently large x*

$$\frac{f(g(x))g'(x)}{f(x)} \geq 1$$

then the series is divergent.

Proof. In both cases we will derive the conclusion from the integral test.

(i) Without loss of generality we can assume that

$$\frac{f(g(x))g'(x)}{f(x)} \leq q$$

is true for every $x \geq x_1$.

Then

$$\int_{g(x_1)}^{g(x)} f(t)\,\mathrm{d}t = \int_{x_1}^{x} f(g(s))g'(s)\,\mathrm{d}s \leq q \int_{x_1}^{x} f(s)\,\mathrm{d}s = q \int_{x_1}^{x} f(t)\,\mathrm{d}t$$

1) Department of Mathematics, The College at Brockport, State University of New York, 350 New Campus Drive *Brockport, New York 14420-2931, USA, gprajitu@brockport.edu

and, for $x_n \geq g(x_1)$,

$$\begin{aligned}(1-q)\int_{g(x_1)}^{g(x_n)} f(t)\,\mathrm{d}t &= \int_{g(x_1)}^{g(x_n)} f(t)\,\mathrm{d}t - q\int_{g(x_1)}^{g(x_n)} f(t)\,\mathrm{d}t \\ &\leq q\int_{x_1}^{x_n} f(t)\,\mathrm{d}t - q\int_{g(x_1)}^{g(x_n)} f(t)\,\mathrm{d}t \\ &= q\left(\int_{x_1}^{x_n} f(t)\,\mathrm{d}t - \int_{g(x_1)}^{g(x_n)} f(t)\,\mathrm{d}t\right) \\ &= q\left(\int_{x_1}^{g(x_1)} f(t)\,\mathrm{d}t - \int_{x_n}^{g(x_n)} f(t)\,\mathrm{d}t\right) \\ &\leq q\int_{x_1}^{g(x_1)} f(t)\,\mathrm{d}t.\end{aligned}$$

Therefore

$$\int_{g(x_1)}^{g(x_n)} f(t)\,\mathrm{d}t \leq \frac{q}{1-q}\int_{x_1}^{g(x_1)} f(t)\,\mathrm{d}t$$

and thus

$$\begin{aligned}\int_{x_1}^{g(x_n)} f(t)\,\mathrm{d}t &= \int_{x_1}^{g(x_1)} f(t)\,\mathrm{d}t + \int_{g(x_1)}^{g(x_n)} f(t)\,\mathrm{d}t \\ &\leq \int_{x_1}^{g(x_1)} f(t)\,\mathrm{d}t + \frac{q}{1-q}\int_{x_1}^{g(x_1)} f(t)\,\mathrm{d}t \\ &= \frac{1}{1-q}\int_{x_1}^{g(x_1)} f(t)\,\mathrm{d}t.\end{aligned}$$

Let $x \geq x_1$ and let $x_n \geq x$. Then

$$\int_{x_1}^{x} f(t)\,\mathrm{d}t \leq \int_{x_1}^{x_n} f(t)\,\mathrm{d}t \leq \frac{1}{1-q}\int_{x_1}^{g(x_1)} f(t)\,\mathrm{d}t.$$

Hence

$$\int_{x_1}^{\infty} f(t)\,\mathrm{d}t$$

exists and thus the series is convergent.

(ii) By considering, if needed, a subsequence, we can assume, without loss of generality, that $x_{n+1} \geq g(x_n)$.

We have

$$\int_{g(x_1)}^{g(x_n)} f(t)\,\mathrm{d}t = \int_{x_1}^{x_n} f(g(s))g'(s)\,\mathrm{d}s \geq \int_{x_1}^{x_n} f(s)\,\mathrm{d}s = \int_{x_1}^{x_n} f(t)\,\mathrm{d}t.$$

Therefore

$$\int_{x_n}^{g(x_n)} f(t)\,\mathrm{d}t \geq \int_{x_1}^{g(x_1)} f(t)\,\mathrm{d}t$$

and thus

$$\int_{x_1}^{g(x_n)} f(t)\,\mathrm{d}t \geq \sum_{k=1}^{n-1} \int_{x_k}^{g(x_k)} f(t)\,\mathrm{d}t \geq n \int_{x_1}^{g(x_1)} f(t)\,\mathrm{d}t,$$

which implies that

$$\int_{x_1}^{\infty} f(t)\,\mathrm{d}t = \infty$$

and so the series is divergent. □

Ermakov test corresponds to $g(x) = e^x$.

The condition about the existence of the sequence (x_n) is necessary in part (i). For instance, if we take $f(x) = 1/x$ and $g(x) = \ln x$ then

$$f(g(x))g'(x) \leq \frac{1}{2} f(x) \iff x \leq \frac{1}{2} x \ln x \iff 2 \leq \ln x,$$

which is true for $x \geq e^2$ but the series is divergent.

The same hypothesis is also necessary in part (ii): if we take $f(x) = 1/x^2$ and $g(x) = \ln x$ then

$$f(g(x))g'(x) \geq f(x) \iff \frac{1}{\ln^2 x}\frac{1}{x} \geq \frac{1}{x^2} \iff x \geq \ln^2 x,$$

which is true for x sufficiently large but the series is convergent.

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