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Recurrent sequences and the asymptotic expansion of a function

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Abstract. Let $a \in \mathbb{R}$ and $f : (a, \infty) \to \mathbb{R}$ be a continuous function such that $f(x) > x$, $\forall x > a$ and $(x_n)_{n \geq 1}$ the sequence defined by the initial condition $x_1 > a$ and the recurrence relation $x_{n+1} = f(x_n)$ for every $n \geq 1$. We prove that if there exist $b_0, b_1, b_2 \in \mathbb{R}$, $b_0 \neq 0$, such that $\lim\limits_{x \to \infty} x^2 \left(f(x) - x - b_0 - \frac{b_1}{x} \right) = b_2$ then

$$C := \lim_{n \to \infty} \left(x_n - b_0 n - \frac{b_1}{b_0} \cdot \ln n \right) \in \mathbb{R}.$$

Moreover, $x_n = b_0 n + \frac{b_1}{b_0} \cdot \ln n + C + \frac{b_1^2}{b_0^3} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right)$. Many and various concrete examples are given.

Keywords: Recurrent sequences, asymptotic expansion of a function, asymptotic expansion of a sequence, Stolz-Cesàro lemma.

MSC: 35C20, 11B37, 40A05, 40A25.

1. Introduction

The concept of asymptotic expansion of a function was introduced by Stieltjes and Poincaré in the year 1886, see [7, 14]. We recall it now. Let $b \in \mathbb{R} \cup \{-\infty\}$, $h : (b, \infty) \to \mathbb{R}$ a function and $(a_n)_{n \geq 0}$ be a sequence of real numbers. A series of the form $a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots$ is called an *asymptotic expansion of the function* h if $\lim\limits_{x \to \infty} \left[h(x) - \left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} \right) \right] x^n = 0$ for every $n \geq 0$, equivalently

$$\lim_{x \to \infty} h(x) = a_0, \ \lim_{x \to \infty} x (h(x) - a_0) = a_1, \ \lim_{x \to \infty} x^2 \left(h(x) - a_0 - \frac{a_1}{x} \right) = a_2,$$

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and so on. Let us note that, for example, if $\lim\limits_{x\to\infty} x^2\left(h(x) - a_0 - \frac{a_1}{x}\right) = a_2$, then $\lim\limits_{x\to\infty} x(h(x) - a_0) = a_1$ and $\lim\limits_{x\to\infty} h(x) = a_0$. For more interesting historical details see [6, pages 536–537].

All our notation and notions are standard. We recall just that, if $(b_n)_{n\in\mathbb{N}}$ is a sequence of real numbers such that there exists $n_0 \in \mathbb{N}$ with $b_n \neq 0$, $\forall n \geq n_0$, and $(a_n)_{n\in\mathbb{N}}$ is another sequence of real numbers, the notation $a_n = o(b_n)$ means $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$. The notation $x_n = a_n + o(b_n)$ means $x_n - a_n = o(b_n)$. If $m \in \mathbb{N}$, $b \in \mathbb{R} \cup \{-\infty\}$ and $f : (b,\infty) \to \mathbb{R}$ is a function, the notation $f(x) = o(x^m)$ as $x \to \infty$ means $\lim\limits_{x\to\infty} \frac{f(x)}{x^m} = 0$, while if $f : (-\delta,\delta) \to \mathbb{R}$ $(\delta > 0)$ the notation $f(x) = o(x^n)$ as $x \to 0$ means $\lim\limits_{x\to 0} \frac{f(x)}{x^m} = 0$; for more details, see [2, 6].

In the paper [12] we have proved the following result.

Theorem 1. *Let $a \in \mathbb{R}$ and $f : (a,\infty) \to \mathbb{R}$ be a continuous function such that $f(x) > x$, $\forall x > a$. Define the sequence $(x_n)_{n\geq 1}$ by the initial condition $x_1 > a$ and the recurrence relation $x_{n+1} = f(x_n)$ for every $n \geq 1$. Then:*
(i) $\lim\limits_{x\to\infty} x_n = \infty$.
(ii) *If there exists $b_0 \in \mathbb{R}$ such that $y = x + b_0$ is an oblique asymptote at the graph of f, then $\lim\limits_{n\to\infty} \frac{x_n}{n} = b_0$.*
(iii) *If there exist $b_0, b_1 \in \mathbb{R}$, $b_0 \neq 0$, such that $\lim\limits_{x\to\infty} x(f(x) - x - b_0) = b_1$, then $\lim\limits_{n\to\infty} \frac{n}{\ln n}\left(\frac{x_n}{n} - b_0\right) = \frac{b_1}{b_0}$, that is $x_n = b_0 n + \frac{b_1}{b_0} \cdot \ln n + o(\ln n)$.*

A simple look of the statement of Theorem 1 show that there is a connection between the asymptotic expansion of the function $h(x) = f(x) - x$ and the asymptotic expansion of the sequence $(x_n)_{n\geq 1}$. Without any claim of completeness let us mention that the asymptotic behavior of some recurrent sequences defined as in Theorem 1 appears in [8, Exercises 173, 174, page 38], [1, Example 11, page 300], [3, pages 154–159], [10, Chapter 2], [13, Theorems 1 and 2], [5, Theorem 3], [11, Proposition 9].

2. The main results

The main purpose of this paper is to complete Theorem 1, thus extending the connection between recurrent sequences and the asymptotic expansion of a function. This is the content of the following theorem.

Theorem 2. *Let $a \in \mathbb{R}$ and $f : (a,\infty) \to \mathbb{R}$ be a continuous function such that $f(x) > x$, $\forall x > a$. Define the sequence $(x_n)_{n\geq 1}$ by the initial condition $x_1 > a$ and the recurrence relation $x_{n+1} = f(x_n)$ for every $n \geq 1$. If there exist $b_0, b_1, b_2 \in \mathbb{R}$, $b_0 \neq 0$, such that $\lim\limits_{x\to\infty} x^2\left(f(x) - x - b_0 - \frac{b_1}{x}\right) = b_2$, then* *there exists*

$$C := \lim_{n\to\infty}\left(x_n - b_0 n - \frac{b_1}{b_0}\cdot \ln n\right) \in \mathbb{R}.$$

Moreover,

$$x_n = b_0 n + \frac{b_1}{b_0}\cdot \ln n + C + \frac{b_1^2}{b_0^3}\cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right)$$

Proof. For every $n \geq 1$ let us define $y_n = x_n - b_0 n - \frac{b_1}{b_0}\cdot \ln n$ and note that

$$\begin{aligned} y_{n+1} - y_n &= x_{n+1} - x_n - b_0 - \frac{b_1}{b_0}\ln\left(1+\frac{1}{n}\right) = \left(f\left(x_n\right) - x_n - b_0 - \frac{b_1}{x_n}\right) \\ &+ b_1\left(\frac{1}{x_n} - \frac{1}{b_0 n}\right) - \frac{b_1}{b_0}\left[\ln\left(1+\frac{1}{n}\right) - \frac{1}{n}\right]. \end{aligned}$$

Therefore we get

$$\begin{aligned} \frac{n^2}{\ln n}\left(y_{n+1} - y_n\right) &= \frac{n^2}{\ln n}\left(f\left(x_n\right) - x_n - b_0 - \frac{b_1}{x_n}\right) + b_1 \cdot \frac{n^2}{\ln n}\left(\frac{1}{x_n} - \frac{1}{b_0 n}\right) \\ &- \frac{b_1}{b_0}\cdot \frac{n^2}{\ln n}\left[\ln\left(1+\frac{1}{n}\right) - \frac{1}{n}\right]. \end{aligned} \tag{1}$$

Since $f(x) > x$, $\forall x > a$, from Theorem 1(i), $\lim_{n\to\infty} x_n = \infty$ and from $\lim_{x\to\infty} x^2\left(f(x) - x - b_0 - \frac{b_1}{x}\right) = b_2$, it follows that

$$\lim_{n\to\infty} x_n^2\left(f\left(x_n\right) - x_n - b_0 - \frac{b_1}{x_n}\right) = b_2.$$

From the hypothesis $\lim_{x\to\infty} x^2\left(f(x) - x - b_0 - \frac{b_1}{x}\right) = b_2 \in \mathbb{R}$ and the equality $f(x) - x - b_0 = x^2\left(f(x) - x - b_0 - \frac{b_1}{x}\right)\cdot\frac{1}{x^2} + \frac{b_1}{x}$ we get $\lim_{x\to\infty}\left(f(x) - x - b_0\right) = 0$. Hence by Theorem 1(ii)

$$\begin{aligned} \frac{n^2}{\ln n}\left(f\left(x_n\right) - x_n - b_0 - \frac{b_1}{x_n}\right) &= x_n^2\left(f\left(x_n\right) - x_n - b_0 - \frac{b_1}{x_n}\right)\cdot\frac{n^2}{x_n^2}\cdot\frac{1}{\ln n} \\ &\longrightarrow b_2 \cdot \frac{1}{b_0^2}\cdot 0 = 0 \text{ as } n \to \infty. \end{aligned} \tag{2}$$

Similarly, from hypothesis and the equality

$$x\left(f(x) - x - b_0\right) - b_1 = x^2\left(f(x) - x - b_0 - \frac{b_1}{x}\right)\cdot\frac{1}{x}$$

we deduce that $\lim_{x\longrightarrow\infty} x\left(f(x) - x - b_0\right) = b_1$, and from Theorem 1(iii) it follows that

$$\frac{n^2}{\ln n}\left(\frac{1}{x_n} - \frac{1}{b_0 n}\right) = -\frac{1}{b_0}\cdot\frac{x_n - b_0 n}{\ln n}\cdot\frac{n}{x_n} \longrightarrow -\frac{b_1}{b_0^3} \text{ as } n \to \infty. \tag{3}$$

We have also

$$\frac{n^2}{\ln n}\left[\ln\left(1+\frac{1}{n}\right)-\frac{1}{n}\right]=\frac{\ln\left(1+\frac{1}{n}\right)-\frac{1}{n}}{\frac{1}{n^2}}\cdot\frac{1}{\ln n}\to-\frac{1}{2}\cdot 0=0 \text{ as } n\to\infty. \quad (4)$$

From the relations (1)–(4) we deduce that

$$\lim_{n\to\infty}\frac{y_{n+1}-y_n}{\frac{\ln n}{n^2}}=-\frac{b_1^2}{b_0^3}. \quad (5)$$

From (5) it follows, in particular, that the sequence $\left(\frac{y_{n+1}-y_n}{\frac{\ln n}{n^2}}\right)_{n\geq 2}$ is bounded, thus there exists $M>0$ such that $|y_{n+1}-y_n|\leq\frac{M\ln n}{n^2}$, $\forall n\geq 2$. Since, by the Cauchy condensation test, the series $\sum\limits_{n=1}^{\infty}\frac{\ln n}{n^2}$ is convergent, from the comparison criterion for series it follows that the series $y_1+\sum\limits_{n=1}^{\infty}(y_{n+1}-y_n)$ is absolutely convergent, hence convergent. Let $C\in\mathbb{R}$ be its sum, that is, $C:=\lim\limits_{n\to\infty}\left(y_1+\sum\limits_{k=1}^{n-1}(y_{k+1}-y_k)\right)$, or equivalently $C=\lim\limits_{n\to\infty}y_n$. By the Stolz-Cesàro lemma, the case $\left[\frac{0}{0}\right]$, see [4], from (5) it follows that

$$\begin{aligned}\lim_{n\to\infty}\frac{\sum\limits_{k=n}^{\infty}(y_{k+1}-y_k)}{\sum\limits_{k=n}^{\infty}\frac{\ln k}{k^2}} &= \lim_{n\to\infty}\frac{\sum\limits_{k=n+1}^{\infty}(y_{k+1}-y_k)-\sum\limits_{k=n}^{\infty}(y_{k+1}-y_k)}{\sum\limits_{k=n+1}^{\infty}\frac{\ln k}{k^2}-\sum\limits_{k=n}^{\infty}\frac{\ln k}{k^2}}\\ &= \lim_{n\to\infty}\frac{y_{n+1}-y_n}{\frac{\ln n}{n^2}}=-\frac{b_1^2}{b_0^3}. \quad (6)\end{aligned}$$

Since for every $n\geq 1$ one has

$$\sum_{k=n}^{\infty}(y_{k+1}-y_k)=\lim_{p\to\infty}\sum_{k=n}^{p}(y_{k+1}-y_k)=\lim_{p\to\infty}(y_{p+1}-y_n)=C-y_n,$$

from (6) it follows that

$$\lim_{n\to\infty}\frac{y_n-C}{\sum\limits_{k=n}^{\infty}\frac{\ln k}{k^2}}=\frac{b_1^2}{b_0^3}. \quad (7)$$

Again an application of the Stolz-Cesàro lemma, the case $\left[\frac{0}{0}\right]$, or [2, Proposition 1, Chapitre V], [9, Proposition 1], [10, chapter V, exercise 5.1], yields that $\lim\limits_{n\to\infty}\frac{\frac{\ln n}{n}}{\sum\limits_{k=n}^{\infty}\frac{\ln k}{k^2}}=1$, so (7) becomes $\lim\limits_{n\to\infty}\frac{y_n-C}{\frac{\ln n}{n}}=\frac{b_1^2}{b_0^3}$, $y_n=C+\frac{b_1^2}{b_0^3}\cdot\frac{\ln n}{n}+o\left(\frac{\ln n}{n}\right)$, i.e., $x_n=b_0n+\frac{b_1}{b_0}\cdot\ln n+C+\frac{b_1^2}{b_0^3}\cdot\frac{\ln n}{n}+o\left(\frac{\ln n}{n}\right)$. □

The following result completes Theorem 2 from [12].

Theorem 3. *Let $\varphi : [0, \infty) \to \mathbb{R}$ be a twice differentiable function such that $\varphi(x) > 0$, $\forall x \geq 0$. Define the sequence $(x_n)_{n\geq 1}$ by the initial condition $x_1 > 0$ and the recurrence relation $x_{n+1} = x_n + \varphi\left(\frac{1}{x_n}\right)$ for every $n \geq 1$. Then there exists $C \in \mathbb{R}$ such that*

$$x_n = \varphi(0)\, n + \frac{\varphi'(0)}{\varphi(0)} \cdot \ln n + C + \frac{[\varphi'(0)]^2}{[\varphi(0)]^3} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right).$$

Proof. Let $f : (0, \infty) \to \mathbb{R}$, $f(x) = x + \varphi\left(\frac{1}{x}\right)$. Obviously, f is continuous and since $\varphi(x) > 0$, $\forall x > 0$, it follows that $f(x) > x$, $\forall x > 0$. From the continuity of φ we have $\lim\limits_{x\to\infty}(f(x) - x) = \lim\limits_{x\to\infty}\varphi\left(\frac{1}{x}\right) = \lim\limits_{t\to 0, t>0}\varphi(t) = \varphi(0) = b_0 > 0$. Since φ is twice differentiable we have

$$\begin{aligned}\lim_{x\to\infty} x\,(f(x) - x - \varphi(0)) &= \lim_{x\to\infty} x\left(\varphi\left(\frac{1}{x}\right) - \varphi(0)\right)\\ &= \lim_{t\to 0, t>0}\frac{\varphi(t) - \varphi(0)}{t} = \varphi'(0) = b_1\end{aligned}$$

and

$$\lim_{x\to\infty} x^2\left(f(x) - x - \varphi(0) - \frac{\varphi'(0)}{x}\right) = \lim_{x\to\infty} x^2\left(\varphi\left(\frac{1}{x}\right) - \varphi(0) - \frac{\varphi'(0)}{x}\right)$$

$$= \lim_{t\to 0, t>0}\frac{\varphi(t) - \varphi(0) - \varphi'(0)\, t}{t^2} = \frac{\varphi''(0)}{2} = b_2.$$

We apply now Theorem 2. □

The following results complete Corollaries 9(i) and 10(i) from [12].

Corollary 4. *Let $\alpha > 1$, $\beta > 0$. Define the sequence $(x_n)_{n\geq 1}$ by the initial condition $x_1 > 0$ and the recurrence relation*

$$x_{n+1} = x_n + \ln\left(\alpha + \frac{\beta}{x_n}\right) \text{ for every } n \geq 1.$$

Then there exists $C \in \mathbb{R}$ such that

$$x_n = (\ln\alpha)\, n + \frac{\beta}{\alpha\ln\alpha} \cdot \ln n + C + \frac{\beta^2}{\alpha^2\ln^3\alpha} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right).$$

Proof. Let $\varphi : [0, \infty) \to (0, \infty)$, $\varphi(x) = \ln(\alpha + \beta x)$. Let us observe that $\varphi'(x) = \frac{\beta}{\alpha + \beta x}$. We apply Theorem 3 for the function φ. □

Corollary 5. *Let $\alpha > 0$, $\beta > 0$. Define the sequence $(x_n)_{n\geq 1}$ by the initial condition $x_1 > 0$ and the recurrence relation*

$$x_{n+1} = x_n + \sqrt{\alpha + \frac{\beta}{x_n}} \text{ for every } n \geq 1.$$

Then there exists $C \in \mathbb{R}$ such that

$$x_n = \sqrt{\alpha} n + \frac{\beta}{2\alpha} \cdot \ln n + C + \frac{\beta^2}{4\alpha^2\sqrt{\alpha}} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right).$$

Proof. Let $\varphi : [0, \infty) \to (0, \infty)$, $\varphi(x) = \sqrt{\alpha + \beta x}$. Let us observe that $\varphi'(x) = \frac{\beta}{2}(\alpha + \beta x)^{-\frac{1}{2}}$. We apply Theorem 3 for the function φ. $\square$

The following result completes [12, Theorem 4].

Theorem 6. *Let $\varphi : [0, \infty) \to \mathbb{R}$ be a three times differentiable function such that $\varphi(x) > 1$, $\forall x > 0$, $\varphi(0) = 1$, $\varphi'(0) \neq 0$. Define the sequence $(x_n)_{n \geq 1}$ by the initial condition $x_1 > 0$ and the recurrence relation*

$$x_{n+1} = x_n \varphi\left(\frac{1}{x_n}\right) \text{ for every } n \geq 1.$$

Then there exists $C \in \mathbb{R}$ such that

$$x_n = \varphi'(0) n + \frac{\varphi''(0)}{2\varphi'(0)} \cdot \ln n + C + \frac{[\varphi''(0)]^2}{4[\varphi'(0)]^3} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right).$$

Proof. Let $f : (0, \infty) \to \mathbb{R}$, $f(x) = x\varphi\left(\frac{1}{x}\right)$. Obviously, f is continuous and since $\varphi(x) > 1$, $\forall x > 0$, it follows that $f(x) > x$, $\forall x > 0$. The continuity of φ implies that $\lim\limits_{x\to\infty} \frac{f(x)}{x} = \lim\limits_{x\to\infty} \varphi\left(\frac{1}{x}\right) = \lim\limits_{t\to 0, t>0} \varphi(t) = \varphi(0) = 1$. Since φ is three times differentiable we have

$$\lim_{x\to\infty} (f(x) - x) = \lim_{x\to\infty} x\left(\varphi\left(\frac{1}{x}\right) - 1\right) = \lim_{t\to 0, t>0} \frac{\varphi(t) - \varphi(0)}{t} = \varphi'(0) = b_0,$$

$$\begin{aligned}\lim_{x\to\infty} x\left(f(x) - x - \varphi'(0)\right) &= \lim_{x\to\infty} x\left(x\varphi\left(\frac{1}{x}\right) - x - \varphi'(0)\right) \\ &= \lim_{t\to 0, t>0} \frac{\varphi(t) - 1 - \varphi'(0)t}{t^2} = \frac{\varphi''(0)}{2} = b_1,\end{aligned}$$

$$\lim_{x\to\infty} x^2\left(f(x) - x - \varphi'(0) - \frac{\varphi''(0)}{2x}\right) = \lim_{x\to\infty} x^2\left(x\varphi\left(\frac{1}{x}\right) - x - \varphi'(0) - \frac{\varphi''(0)}{2x}\right)$$

$$= \lim_{t\to 0, t>0} \frac{\varphi(t) - 1 - \varphi'(0)t - \frac{\varphi''(0)t^2}{2}}{t^3} = \frac{\varphi'''(0)}{3!} = b_2.$$

From Theorem 2 we deduce the evaluation from the statement. $\square$

In the sequel as application of Theorem 6 we will give three concrete examples.

Corollary 7. *Let $\alpha > 0$. Define the sequence $(x_n)_{n\geq 1}$ by the initial condition $x_1 > 0$ and the recurrence relation*

$$x_{n+1} = x_n e^{\sqrt{\frac{1}{x_n} + \alpha^2} - \alpha} \text{ for every } n \geq 1.$$

Then there exists $C \in \mathbb{R}$ *such that*

$$x_n = \frac{n}{2\alpha} + \frac{\alpha - 1}{4\alpha^2} \cdot \ln n + C + \frac{(\alpha - 1)^2}{8\alpha^3} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right).$$

Proof. Let $\varphi : [0, \infty) \to (0, \infty)$, $\varphi(x) = e^{\sqrt{x+\alpha^2}-\alpha}$. Let us observe that $\varphi'(x) = \frac{e^{\sqrt{x+\alpha^2}-\alpha}}{2\sqrt{x+\alpha^2}}$, $\varphi''(x) = \frac{e^{\sqrt{x+\alpha^2}-\alpha}\left(\sqrt{x+\alpha^2}-1\right)}{4(x+\alpha^2)\sqrt{x+\alpha^2}}$, $\varphi'(0) = \frac{1}{2\alpha}$, $\varphi''(0) = \frac{\alpha-1}{4\alpha^3}$. We apply Theorem 6 for the function φ. □

Corollary 8. *Let* $\alpha > 0$, $\beta \geq 0$. *Define the sequence* $(x_n)_{n \geq 1}$ *by the initial condition* $x_1 > 0$ *and the recurrence relation*

$$x_{n+1} = x_n \ln\left(e + \frac{\alpha}{x_n} + \frac{\beta}{x_n^2}\right) \text{ for every } n \geq 1.$$

Then there exists $C \in \mathbb{R}$ *such that*

$$x_n = \frac{\alpha n}{e} + \frac{2\beta e - \alpha^2}{2\alpha e} \cdot \ln n + C + \frac{\left(2\beta e - \alpha^2\right)^2}{4\alpha^3 e} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right).$$

Proof. Let $\varphi : [0, \infty) \to (0, \infty)$, $\varphi(x) = \ln\left(e + \alpha x + \beta x^2\right)$ and note that $\varphi'(x) = \frac{\alpha+2\beta x}{e+\alpha x+\beta x^2}$, $\varphi''(x) = \frac{2\beta\left(e+\alpha x+\beta x^2\right)-(\alpha+2\beta x)^2}{(e+\alpha x+\beta x^2)^2}$, $\varphi'(0) = \frac{\alpha}{e}$, $\varphi''(0) = \frac{2\beta e-\alpha^2}{e^2}$. We apply Theorem 6 for the function φ. □

Corollary 9. *Let* $\alpha > 0$, $\beta \geq 0$. *Define the sequence* $(x_n)_{n \geq 1}$ *by the initial condition* $x_1 > 0$ *and the recurrence relation*

$$x_{n+1} = \sqrt{x_n^2 + \alpha x_n + \beta} \text{ for every } n \geq 1.$$

Then there exists $C \in \mathbb{R}$ *such that*

$$x_n = \frac{\alpha n}{2} - \frac{\alpha^2 - 4\beta}{4\alpha} \cdot \ln n + C + \frac{\left(\alpha^2 - 4\beta\right)^2}{8\alpha^3} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right).$$

Proof. Let us observe that the recurrence relation can be written under the form

$$x_{n+1} = x_n \sqrt{1 + \frac{\alpha}{x_n} + \frac{\beta}{x_n^2}} \text{ for every } n \geq 1.$$

Let $\varphi : [0, \infty) \to (0, \infty)$, $\varphi(x) = \sqrt{1 + \alpha x + \beta x^2}$. Then $\varphi'(x) = \frac{\alpha+2\beta x}{2\sqrt{1+\alpha x+\beta x^2}}$, $\varphi''(x) = \frac{4\beta\left(1+\alpha x+\beta x^2\right)-(\alpha+2\beta x)^2}{4(1+\alpha x+\beta x^2)\sqrt{1+\alpha x+\beta x^2}}$, $\varphi'(0) = \frac{\alpha}{2}$, $\varphi''(0) = \frac{4\beta-\alpha^2}{4}$. We apply Theorem 6 for the function φ. □

The following result completes Corollary 6 from [12].

Proposition 10. (i) *Let $p \geq 1$ be a natural number and $0 < a_1 < a_2 < \cdots < a_p$ real numbers. Define the sequence $(x_n)_{n \geq 1}$ by the initial condition $x_1 > a_p$ and the recurrence relation*

$$x_{n+1} = \frac{x_n^{p+1}}{(x_n - a_1)(x_n - a_2)\cdots(x_n - a_p)} \text{ for every } n \geq 1.$$

Then there exists $C \in \mathbb{R}$ such that

$$x_n = \sigma_1 n + \frac{\sigma_1^2 + \sigma_2}{2\sigma_1} \cdot \ln n + C + \frac{\left(\sigma_1^2 + \sigma_2\right)^2}{4\sigma_1^3} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right)$$

where $\sigma_1 = a_1 + a_2 + \cdots + a_p$, $\sigma_2 = a_1^2 + a_2^2 + \cdots + a_p^2$.
(ii) *Let $p \geq 1$ be a natural number. Define the sequence $(x_n)_{n \geq 1}$ by the initial condition $x_1 > p$ and the recurrence relation*

$$x_{n+1} = \frac{x_n^{p+1}}{(x_n - 1)(x_n - 2)\cdots(x_n - p)} \text{ for every } n \geq 1.$$

Then there exists $C \in \mathbb{R}$ such that

$$x_n = \frac{p(p+1)}{2} \cdot n + \frac{(3p+1)(p+2)}{12} \cdot \ln n + C + \frac{(3p+1)^2 (p+2)^2}{72p(p+1)} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right)$$

Proof. (i) Let $f : (a_p, \infty) \to \mathbb{R}$, $f(x) = \frac{x^{p+1}}{(x-a_1)(x-a_2)\cdots(x-a_p)}$, and note that f is continuous and $f(x) - x > 0$, $\forall x > a_p$. For every $x \in \mathbb{R}$ we have

$$(x - a_1)\cdots(x - a_p) = x^p - s_1 x^{p-1} + s_2 x^{p-2} - s_3 x^{p-3} + \cdots + (-1)^p s_p \quad (8)$$

where $s_1 = a_1 + \cdots + a_p$, $s_2 = \sum_{i<j} a_i a_j, \ldots, s_p = a_1 a_2 \cdots a_p$. It is easy to prove that $\lim_{x \to \infty} (f(x) - x) = s_1 = b_0 > 0$. For every $x > a_p$ we have $f(x) - x - s_1 =$

$$\frac{\left(s_1^2 - s_2\right) x^{p-1} - (s_1 s_2 - s_3) x^{p-2} + \cdots - (-1)^{p-1} (s_1 s_{p-1} - s_p) x - (-1)^p s_1 s_p}{(x - a_1)\cdots(x - a_p)},$$

which gives us that $\lim_{x \to \infty} x(f(x) - x - s_1) = s_1^2 - s_2 = b_1$ and also

$$f(x) - x - s_1 - \frac{s_1^2 - s_2}{x} = \frac{h(x)}{x(x - a_1)\cdots(x - a_p)},$$

where

$$\begin{aligned} h(x) &= \left(s_1^2 - s_2\right) x^p - (s_1 s_2 - s_3) x^{p-1} + \cdots - (-1)^{p-1} (s_1 s_{p-1} - s_p) x \\ &- (-1)^p s_1 s_p - \left(s_1^2 - s_2\right)(x - a_1)\cdots(x - a_p). \end{aligned}$$

From the relation (8) we deduce that

$$\begin{aligned} h(x) &= \left(s_1^2 - s_2\right) x^p - (s_1 s_2 - s_3) x^{p-1} + \cdots - (-1)^{p-1} (s_1 s_{p-1} - s_p) x \\ &- (-1)^p s_1 s_p - \left(s_1^2 - s_2\right)\left(x^p - s_1 x^{p-1} + \cdots + (-1)^p s_p\right) \\ &= \left(s_1^3 - 2 s_1 s_2 + s_3\right) x^{p-1} + \cdots. \end{aligned}$$

It follows then that

$$\begin{aligned}\lim_{x\to\infty} x^2\left(f(x)-x-s_1-\frac{s_1^2-s_2}{x}\right) &= \lim_{x\to\infty}\frac{\left(s_1^3-2s_1s_2+s_3\right)x^{p+1}+\cdots}{x(x-a_1)\cdots(x-a_p)}\\ &= s_1^3-2s_1s_2+s_3=b_2.\end{aligned}$$

From Theorem 2 it follows that

$$\begin{aligned}x_n &= s_1n+\frac{s_1^2-s_2}{s_1}\cdot\ln n+C+\frac{\left(s_1^2-s_2\right)^2}{s_1^3}\cdot\frac{\ln n}{n}+o\left(\frac{\ln n}{n}\right)\\ &= \sigma_1n+\frac{\sigma_1^2+\sigma_2}{2\sigma_1}\cdot\ln n+C+\frac{\left(\sigma_1^2+\sigma_2\right)^2}{4\sigma_1^3}\cdot\frac{\ln n}{n}+o\left(\frac{\ln n}{n}\right).\end{aligned}$$

We have used that $s_1=\sum\limits_{i=1}^{n}a_i=\sigma_1$, $s_2=\frac{\left(\sum\limits_{i=1}^{n}a_i\right)^2-\sum\limits_{i=1}^{n}a_i^2}{2}=\frac{\sigma_1^2-\sigma_2}{2}$, $s_1^2-s_2=\frac{\sigma_1^2+\sigma_2}{2}$.

(ii) Take in (i) $a_i=i$, $i=1,\ldots,p$, so that $\sigma_1=\sum\limits_{i=1}^{p}i=\frac{p(p+1)}{2}$, $\sigma_2=\sum\limits_{i=1}^{p}i^2=\frac{p(p+1)(2p+1)}{6}$, $\sigma_1^2+\sigma_2=\frac{p(p+1)(3p+1)(p+2)}{12}$. □

We need in the sequel the following two results.

Proposition 11. *Let a, b, $c\in\mathbb{R}$. Then for every $r\in\mathbb{R}\setminus\{0\}$*

$$\begin{aligned}&\left(1+\frac{a\ln n}{n}+\frac{b}{n}+\frac{c\ln n}{n^2}+o\left(\frac{\ln n}{n^2}\right)\right)^r=1+\frac{ra\ln n}{n}+\frac{rb}{n}\\ &\quad+\frac{r(r-1)}{2}\cdot\frac{a^2\ln^2 n}{n^2}+r\left((r-1)ab+c\right)\cdot\frac{\ln n}{n^2}+o\left(\frac{\ln n}{n^2}\right).\end{aligned}$$

In particular,

$$\begin{aligned}\frac{1}{1+\frac{a\ln n}{n}+\frac{b}{n}+\frac{c\ln n}{n^2}+o\left(\frac{\ln n}{n^2}\right)} &= 1-\frac{a\ln n}{n}-\frac{b}{n}\\ &\quad+\frac{a^2\ln^2 n}{n^2}+\frac{(2ab-c)\ln n}{n^2}+o\left(\frac{\ln n}{n^2}\right).\end{aligned}$$

Proof. We will use the well-known evaluation $(1+x)^r=1+rx+\frac{r(r-1)}{2}x^2+\frac{r(r-1)(r-2)}{2}x^3+o\left(x^3\right)$ as $x\to 0$. For $x_n=\frac{a\ln n}{n}+\frac{b}{n}+\frac{c\ln n}{n^2}+o\left(\frac{\ln n}{n^2}\right)$ we have $x_n^3=\frac{a^3\ln^3 n}{n^3}+o\left(\frac{\ln^3 n}{n^3}\right)=o\left(\frac{\ln n}{n^2}\right)$ and thus

$$\begin{aligned}&\left(1+\frac{a\ln n}{n}+\frac{b}{n}+\frac{c\ln n}{n^2}+o\left(\frac{\ln n}{n^2}\right)\right)^r=1+r\left(\frac{a\ln n}{n}+\frac{b}{n}+\frac{c\ln n}{n^2}\right)\\ &\qquad+\frac{r(r-1)}{2}\left(\frac{a\ln n}{n}+\frac{b}{n}+\frac{c\ln n}{n^2}\right)^2+o\left(\frac{\ln n}{n^2}\right). \qquad (9)\end{aligned}$$

If we use that

$$\left(\frac{a\ln n}{n}+\frac{b}{n}+\frac{c\ln n}{n^2}\right)^2=\frac{a^2\ln^2 n}{n^2}+\frac{2ab\ln n}{n^2}+o\left(\frac{\ln n}{n^2}\right),$$

from the relation (9), after some simple calculations, we get the evaluation from the statement. $\square$

Proposition 12. *Let α, β, γ, $C\in\mathbb{R}$, $\alpha\neq 0$, and $r\in\mathbb{R}\setminus\{0\}$. If*

$$x_n=\alpha n+\beta\ln n+C+\gamma\cdot\frac{\ln n}{n}+o\left(\frac{\ln n}{n}\right)$$

then

$$\begin{aligned}x_n^r=&\ \alpha^r n^r+\frac{r\beta}{\alpha^{1-r}}\cdot\frac{\ln n}{n^{1-r}}+\frac{rC}{\alpha^{1-r}n^{1-r}}\\&+\frac{r(r-1)\beta^2}{2\alpha^{2-r}}\cdot\frac{\ln^2 n}{n^{2-r}}+\frac{r\left[(r-1)\beta C+\alpha\gamma\right]}{\alpha^{2-r}}\cdot\frac{\ln n}{n^{2-r}}+o\left(\frac{\ln n}{n^{2-r}}\right).\end{aligned}$$

In particular,

$$\frac{1}{x_n}=\frac{1}{\alpha n}-\frac{\beta\ln n}{\alpha^2 n^2}-\frac{C}{\alpha^2n^2}+\frac{\beta^2\ln^2 n}{\alpha^3n^3}+\frac{(2\beta C-\alpha\gamma)\ln n}{\alpha^3n^3}+o\left(\frac{\ln n}{n^3}\right).$$

Proof. We have

$$x_n^r=\alpha^r n^r\left(1+\frac{\beta}{\alpha}\cdot\frac{\ln n}{n}+\frac{C}{\alpha n}+\frac{\gamma}{\alpha}\cdot\frac{\ln n}{n^2}+o\left(\frac{\ln n}{n^2}\right)\right)^r.$$

Since by Proposition 11

$$\begin{aligned}\left(1+\frac{\beta}{\alpha}\cdot\frac{\ln n}{n}+\frac{C}{\alpha n}+\frac{\gamma}{\alpha}\cdot\frac{\ln n}{n^2}+o\left(\frac{\ln n}{n^2}\right)\right)^r&=1+\frac{r\beta}{\alpha}\cdot\frac{\ln n}{n}+\frac{rC}{\alpha n}\\&+\frac{r(r-1)\beta^2}{2\alpha^2}\cdot\frac{\ln^2 n}{n^2}+\frac{r\left[(r-1)\beta C+\gamma\alpha\right]}{\alpha^2}\cdot\frac{\ln n}{n^2}+o\left(\frac{\ln n}{n^2}\right).\end{aligned}$$

after some obvious calculations we obtain the formula from the statement. $\square$

The following application was suggested to us by Corollary 9.

Proposition 13. *Let $\alpha>0$, $\beta\geq 0$. Define the sequence $(y_n)_{n\geq 1}$ by the initial condition $y_1>0$ and the recurrence relation*

$$y_{n+1}=\frac{y_n}{\sqrt{1+\alpha y_n+\beta y_n^2}}\ \text{for every } n\geq 1.$$

Then there exists $C\in\mathbb{R}$ such that

$$\begin{aligned}y_n\ =&\ \frac{2}{\alpha n}+\frac{\alpha^2-4\beta}{\alpha^3}\cdot\frac{\ln n}{n^2}-\frac{4C}{\alpha^2n^2}+\frac{\left(\alpha^2-4\beta\right)^2}{2\alpha^5}\cdot\frac{\ln^2 n}{n^3}\\&-\frac{\left(\alpha^2-4\beta\right)\left(8\alpha C+\alpha^2-4\beta\right)}{2\alpha^5}\cdot\frac{\ln n}{n^3}+o\left(\frac{\ln n}{n^3}\right).\end{aligned}$$

Proof. For every $n \geq 1$ we denote $x_n = \frac{1}{y_n}$. Then $x_1 > 0$ and the recurrence relation becomes

$$x_{n+1} = \sqrt{x_n^2 + \alpha x_n + \beta} \text{ for every } n \geq 1.$$

From Corollary 9 there exists $C \in \mathbb{R}$ such that

$$x_n = \frac{\alpha n}{2} - \frac{\alpha^2 - 4\beta}{4\alpha} \cdot \ln n + C + \frac{\left(\alpha^2 - 4\beta\right)^2}{8\alpha^3} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right)$$

and by Proposition 12

$$\begin{aligned} y_n &= \frac{1}{x_n} = \frac{2}{\alpha n} + \frac{\alpha^2 - 4\beta}{\alpha^3} \cdot \frac{\ln n}{n^2} - \frac{4C}{\alpha^2 n^2} + \frac{\left(\alpha^2 - 4\beta\right)^2}{2\alpha^5} \cdot \frac{\ln^2 n}{n^3} \\ &\quad - \frac{\left(\alpha^2 - 4\beta\right)\left(8\alpha C + \alpha^2 - 4\beta\right)}{2\alpha^5} \cdot \frac{\ln n}{n^3} + o\left(\frac{\ln n}{n^3}\right). \end{aligned}$$

We made the changes $\alpha \to \frac{\alpha}{2}$, $\beta \to -\frac{\alpha^2-4\beta}{4\alpha}$, $\gamma \to \frac{\left(\alpha^2-4\beta\right)^2}{8\alpha^3}$ and use that $\frac{\alpha^2-4\beta}{4\alpha} \cdot \frac{4}{\alpha^2} = \frac{\alpha^2-4\beta}{\alpha^3}$, $\frac{\left(\alpha^2-4\beta\right)^2}{16\alpha^2} \cdot \frac{8}{\alpha^3} = \frac{\left(\alpha^2-4\beta\right)^2}{2\alpha^5}$, $-\frac{\left(\alpha^2-4\beta\right)C}{2\alpha} - \frac{\left(\alpha^2-4\beta\right)^2}{16\alpha^2} = -\frac{\left(\alpha^2-4\beta\right)\left(8\alpha C+\alpha^2-4\beta\right)}{16\alpha^2}$, $-\frac{\left(\alpha^2-4\beta\right)\left(8\alpha C+\alpha^2-4\beta\right)}{16\alpha^2} \cdot \frac{8}{\alpha^3} = -\frac{\left(\alpha^2-4\beta\right)\left(8\alpha C+\alpha^2-4\beta\right)}{2\alpha^5}$. □

The following application was suggested to us by the context considered in Proposition 10. It completes Corollary 7 from [12].

Proposition 14. (i) *Let $p \geq 1$ be a natural number and $0 < a_1 < a_2 < \cdots < a_p$ real numbers. Define the sequence $(y_n)_{n\geq 1}$ by the initial condition $0 < y_1 < \frac{1}{a_p}$ and the recurrence relation*

$$y_{n+1} = y_n\left(1 - a_1 y_n\right)\left(1 - a_2 y_n\right)\cdots\left(1 - a_p y_n\right) \text{ for every } n \geq 1.$$

Then there exists $C \in \mathbb{R}$ such that

$$\begin{aligned} y_n &= \frac{1}{x_n} = \frac{1}{\sigma_1 n} - \frac{\sigma_1^2 + \sigma_2}{2\sigma_1^3} \cdot \frac{\ln n}{n^2} - \frac{C}{\sigma_1^2 n^2} + \frac{\left(\sigma_1^2 + \sigma_2\right)^2}{4\sigma_1^5} \cdot \frac{\ln^2 n}{n^3} \\ &\quad + \frac{\left(\sigma_1^2 + \sigma_2\right)\left(4\sigma_1 C - \sigma_1^2 - \sigma_2\right)}{4\sigma_1^5} \cdot \frac{\ln n}{n^3} + o\left(\frac{\ln n}{n^3}\right) \end{aligned}$$

where $\sigma_1 = a_1 + a_2 + \cdots + a_p$, $\sigma_2 = a_1^2 + a_2^2 + \cdots + a_p^2$.
(ii) *Let $p \geq 1$ be a natural number. Define the sequence $(y_n)_{n\geq 1}$ by the initial condition $0 < y_1 < \frac{1}{p}$ and the recurrence relation*

$$y_{n+1} = y_n\left(1 - y_n\right)\left(1 - 2y_n\right)\cdots\left(1 - p y_n\right) \text{ for every } n \geq 1.$$

Then there exists $C \in \mathbb{R}$ such that

$$y_n = \frac{2}{p\,(p+1)\,n} - \frac{(p+2)\,(3p+1)}{3p^2\,(p+1)^2} \cdot \frac{\ln n}{n^2} - \frac{4C}{p^2\,(p+1)^2\,n^2} + \frac{(p+2)^2\,(3p+1)^2}{18p^3\,(p+1)^3} \cdot \frac{\ln^2 n}{n^3}$$

$$+\frac{2\left(p+2\right)\left(3p+1\right)\left(2C-\frac{(p+2)(3p+1)}{12}\right)}{3p^3\left(p+1\right)^3}\cdot\frac{\ln n}{n^3}+o\left(\frac{\ln n}{n^3}\right).$$

Proof. (i) For every $n\geq 1$ we denote $x_n=\frac{1}{y_n}$. From $0<y_1<\frac{1}{a_p}$ it follows that $x_1>a_p$ and the recurrence relation becomes

$$x_{n+1}=\frac{x_n^{p+1}}{\left(x_n-a_1\right)\left(x_n-a_2\right)\cdots\left(x_n-a_p\right)}\text{ for every }n\geq 1.$$

From Corollary 10 (i) there exists $C\in\mathbb{R}$ such that

$$x_n=\sigma_1\cdot n+\frac{\sigma_1^2+\sigma_2}{2\sigma_1}\cdot\ln n+C+\frac{\left(\sigma_1^2+\sigma_2\right)^2}{4\sigma_1^3}\cdot\frac{\ln n}{n}+o\left(\frac{\ln n}{n}\right)$$

and from Proposition 12

$$\begin{aligned} y_n &= \frac{1}{x_n}=\frac{1}{\sigma_1 n}-\frac{\sigma_1^2+\sigma_2}{2\sigma_1^3}\cdot\frac{\ln n}{n^2}-\frac{C}{\sigma_1^2 n^2}+\frac{\left(\sigma_1^2+\sigma_2\right)^2}{4\sigma_1^5}\cdot\frac{\ln^2 n}{n^3} \\ &\quad+\frac{\left(\sigma_1^2+\sigma_2\right)\left(4\sigma_1 C-\sigma_1^2-\sigma_2\right)}{4\sigma_1^5}\cdot\frac{\ln n}{n^3}+o\left(\frac{\ln n}{n^3}\right). \end{aligned}$$

We have $\alpha=\sigma_1$, $\beta=\frac{\sigma_1^2+\sigma_2}{2\sigma_1}$, $\gamma=\frac{\left(\sigma_1^2+\sigma_2\right)^2}{4\sigma_1^3}$ and thus

$$\frac{2\beta C-\alpha\gamma}{\alpha^3}=\frac{\frac{\left(\sigma_1^2+\sigma_2\right)C}{\sigma_1}-\frac{\left(\sigma_1^2+\sigma_2\right)^2}{4\sigma_1^2}}{\sigma_1^3}=\frac{\left(\sigma_1^2+\sigma_2\right)\left(4\sigma_1 C-\sigma_1^2-\sigma_2\right)}{4\sigma_1^5}.$$

(ii) Take in (i) $a_i=i$, $1\leq i\leq p$. Then $\sigma_1=\frac{p(p+1)}{2}$, $\sigma_2=\frac{p(p+1)(2p+1)}{6}$, so $\sigma_1^2+\sigma_2=\frac{p(p+1)(p+2)(3p+1)}{12}$, $\frac{\sigma_1^2+\sigma_2}{2\sigma_1^3}=\frac{(p+2)(3p+1)}{3p^2(p+1)^2}$, $\frac{\left(\sigma_1^2+\sigma_2\right)^2}{4\sigma_1^5}=\frac{(p+2)^2(3p+1)^2}{18p^3(p+1)^3}$,

$$\begin{aligned} \frac{\left(\sigma_1^2+\sigma_2\right)\left(4\sigma_1 C-\sigma_1^2-\sigma_2\right)}{4\sigma_1^5} &= \frac{(p+2)(3p+1)}{12}\cdot\frac{8}{p^4(p+1)^4} \\ &\quad\cdot\left(2p(p+1)C-\frac{p(p+1)(p+2)(3p+1)}{12}\right) \\ &= \frac{2\left(p+2\right)\left(3p+1\right)\left(2C-\frac{(p+2)(3p+1)}{12}\right)}{3p^3\left(p+1\right)^3}. \end{aligned}$$

□

The case $p=1$ in Proposition 14 was previously studied in [13], [5], and [11].

In the next result we complete the evaluation from [10, exercise 2.8].

Corollary 15. *Define the sequence $\left(x_n\right)_{n\geq 1}$ by the initial condition $x_1>0$ and the recurrence relation $x_{n+1}=x_n+\frac{1}{x_n}$ for every $n\geq 1$. Then there* *exists* $C \in \mathbb{R}$ *such that*

$$x_n = \sqrt{2} \cdot \sqrt{n} + \frac{1}{4\sqrt{2}} \cdot \frac{\ln n}{\sqrt{n}} + \frac{C}{2\sqrt{2} \cdot \sqrt{n}} - \frac{1}{64\sqrt{2}} \cdot \frac{\ln^2 n}{n\sqrt{n}} - \frac{C-1}{8\sqrt{2}} \cdot \frac{\ln n}{n\sqrt{n}} + o\left(\frac{\ln n}{n\sqrt{n}}\right).$$

Proof. Squaring the recurrence relation, we obtain $x_{n+1}^2 = x_n^2 + 2 + \frac{1}{x_n^2}$, $\forall n \geq 1$. For every $n \geq 1$ we denote $x_n^2 = a_n$ and thus $a_{n+1} = a_n + 2 + \frac{1}{a_n} = a_n + \varphi\left(\frac{1}{a_n}\right)$, where $\varphi(x) = 2 + x$. By Theorem 3, there exists $C \in \mathbb{R}$ such that

$$a_n = \varphi(0)\, n + \frac{\varphi'(0)}{\varphi(0)} \cdot \ln n + C + \frac{[\varphi'(0)]^2}{[\varphi(0)]^3} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right)$$

or

$$x_n^2 = a_n = 2n + \frac{\ln n}{2} + C + \frac{1}{8} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right).$$

From Proposition 12 for $r = \frac{1}{2}$ we deduce that

$$\begin{aligned} x_n =& \sqrt{\alpha} \cdot \sqrt{n} + \frac{1}{2} \cdot \frac{\beta}{\alpha^{\frac{1}{2}}} \cdot \frac{\ln n}{n^{\frac{1}{2}}} + \frac{1}{2} \cdot \frac{C}{\alpha^{\frac{1}{2}} n^{\frac{1}{2}}} \\ &+ \frac{\frac{1}{2} \cdot \left(-\frac{1}{2}\right)\beta^2}{2\alpha^{\frac{3}{2}}} \cdot \frac{\ln^2 n}{n^{\frac{3}{2}}} + \frac{\frac{1}{2}\left(-\frac{\beta C}{2} + \alpha\gamma\right)}{\alpha^{\frac{3}{2}}} \cdot \frac{\ln n}{n^{\frac{3}{2}}} + o\left(\frac{\ln n}{n^{\frac{3}{2}}}\right). \end{aligned}$$

or

$$\begin{aligned} x_n =& \sqrt{\alpha}\sqrt{n} + \frac{1}{2} \cdot \frac{\beta}{\sqrt{\alpha}} \cdot \frac{\ln n}{\sqrt{n}} + \frac{1}{2} \cdot \frac{C}{\sqrt{\alpha}\sqrt{n}} \\ &- \frac{\beta^2}{8\alpha\sqrt{\alpha}} \cdot \frac{\ln^2 n}{n\sqrt{n}} - \frac{(\beta C - 2\alpha\gamma)}{4\alpha\sqrt{\alpha}} \cdot \frac{\ln n}{n\sqrt{n}} + o\left(\frac{\ln n}{n\sqrt{n}}\right), \end{aligned}$$

where $\alpha = 2$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{8}$. Then by calculation we get the evaluation from the statement. □

We need latter the following result which establishes a natural connection between two asymptotic evaluations of some functions.

Proposition 16. *Let* $A > 0$ *and* $g : [0, A) \to \mathbb{R}$ *be a function with the property that there exist* a_1, a_2, $a_3 \in \mathbb{R}$ *such that* $g(x) = x + a_1 x^2 + a_2 x^3 + a_3 x^4 + o(x^4)$ *if* $x \to 0$, $x > 0$. *Then*

$$\frac{1}{g(x)} = \frac{1}{x} - a_1 + \left(a_1^2 - a_2\right) x - \left(a_1^3 - 2a_1 a_2 + a_3\right) x^2 + o\left(x^2\right) \ \text{ as } x \to 0.$$

In particular,

$$\frac{1}{g\left(\frac{1}{x}\right)} = x - a_1 + \frac{a_1^2 - a_2}{x} - \frac{a_1^3 - 2a_1 a_2 + a_3}{x^2} + o\left(\frac{1}{x^2}\right) \ \text{ as } x \to \infty.$$

Proof. We have $\frac{1}{g(x)} = \frac{1}{x} \cdot \frac{1}{1 + a_1 x + a_2 x^2 + a_3 x^3 + o(x^3)}$. Since as is well-known

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + o\left(u^3\right) \ \text{ as } u \to 0,$$

we will have

$$\frac{1}{1+a_1x+a_2x^2+a_3x^3+o\left(x^3\right)} = 1-\left(a_1x+a_2x^2+a_3x^3\right)$$
$$+\left(a_1x+a_2x^2+a_3x^3\right)^2 - a_1^3x^3+o\left(x^3\right).$$

Since $\left(a_1x+a_2x^2+a_3x^3+o\left(x^3\right)\right)^2 = a_1^2x^2+2a_1a_2x^3+o\left(x^3\right)$ we deduce that

$$\frac{1}{1+a_1x+a_2x^2+a_3x^3+o\left(x^3\right)} = 1-\left(a_1x+a_2x^2+a_3x^3\right)$$
$$+\left(a_1^2x^2+2a_1a_2x^3\right)-a_1^3x^3+o\left(x^3\right)$$
$$=1-a_1x+\left(a_1^2-a_2\right)x^2-\left(a_1^3-2a_1a_2+a_3\right)x^3+o\left(x^3\right).$$

(Another proof is to verify by calculus that

$$\lim_{x\to 0}\frac{\frac{1}{1+a_1x+a_2x^2+a_3x^3+o(x^3)}-1+a_1x-\left(a_1^2-a_2\right)x}{x^3} = -a_1^3+2a_1a_2-a_3.)$$

Then we get

$$\frac{1}{g\left(x\right)} = \frac{1}{x}-a_1+\left(a_1^2-a_2\right)x-\left(a_1^3-2a_1a_2+a_3\right)x^2+o\left(x^2\right) \text{ as } x\to 0.$$

□

The following theorem is a natural companion of Theorem 2.

Theorem 17. *Let $A>0$, $g:[0,A)\to[0,\infty)$ be a continuous function such that $0<g(x)<x$, $\forall\ 0<x<A$. Define the sequence $(x_n)_{n\geq 1}$ by the initial condition $x_1\in(0,A)$ and the recurrence relation*

$$x_{n+1}=g(x_n) \textit{ for every } n\geq 1.$$

If there exist real numbers a_1, a_2, a_3, $a_1\neq 0$, such that

$$g\left(x\right)=x+a_1x^2+a_2x^3+a_3x^4+o\left(x^4\right) \textit{ as } x\to 0, x>0$$

then there exists $C\in\mathbb{R}$ such that

$$x_n = -\frac{1}{a_1}\cdot\frac{1}{n}+\frac{a_1^2-a_2}{a_1^3}\cdot\frac{\ln n}{n^2}-\frac{C}{a_1^2n^2}-\frac{\left(a_1^2-a_2\right)^2}{a_1^5}\cdot\frac{\ln^2 n}{n^3}$$
$$+\frac{\left(a_1^2-a_2\right)\left(2a_1C+a_1^2-a_2\right)}{a_1^5}\cdot\frac{\ln n}{n^3}+o\left(\frac{\ln n}{n^3}\right).$$

Proof. Let $f:\left(\frac{1}{A},\infty\right)\to\mathbb{R}$, $f\left(x\right)=\frac{1}{g\left(\frac{1}{x}\right)}$. Then f is continuous and from $0<g(x)<x$, $\forall 0<x<A$, it follows that $f\left(x\right)>x$, $\forall x>\frac{1}{A}$. For every $n\geq 1$ we define $v_n=\frac{1}{x_n}$. Then $v_1>\frac{1}{A}$ and the recurrence relation becomes $v_{n+1}=f\left(v_n\right)$ for every $n\geq 1$. Since $g\left(x\right)=x+a_1x^2+a_2x^3+a_3x^4+o\left(x^4\right)$ as $x\to 0$,

$x>0$, from Proposition 16 it follows that $f(x)=x+b_0+\frac{b_1}{x}+\frac{b_2}{x^2}+o\left(\frac{1}{x^2}\right)$ as $x\to\infty$, where

$$b_0=-a_1,\quad b_1=a_1^2-a_2,\quad b_2=-a_1^3+2a_1a_2-a_3.$$

From Theorem 2 we have

$$v_n=b_0n+\frac{b_1}{b_0}\cdot\ln n+C+\frac{b_1^2}{b_0^3}\cdot\frac{\ln n}{n}+o\left(\frac{\ln n}{n}\right)$$

and by Proposition 12

$$\frac{1}{v_n}=\frac{1}{b_0n}-\frac{b_1\ln n}{b_0^3n^2}-\frac{C}{b_0^2n^2}+\frac{b_1^2\ln^2 n}{b_0^5n^3}+\frac{b_1\left(2b_0C-b_1\right)\ln n}{b_0^5n^3}+o\left(\frac{\ln n}{n^3}\right).$$

Since $x_n=\frac{1}{v_n}$ we deduce that

$$\begin{aligned} x_n \;=\;& -\frac{1}{a_1}\cdot\frac{1}{n}+\frac{a_1^2-a_2}{a_1^3}\cdot\frac{\ln n}{n^2}-\frac{C}{a_1^2n^2}-\frac{\left(a_1^2-a_2\right)^2}{a_1^5}\cdot\frac{\ln^2 n}{n^3}\\ &+\frac{\left(a_1^2-a_2\right)\left(2a_1C+a_1^2-a_2\right)}{a_1^5}\cdot\frac{\ln n}{n^3}+o\left(\frac{\ln n}{n^3}\right).\end{aligned}$$

□

In the next result we complete the evaluation from [10, exercise 2.3].

Corollary 18. *Let $(x_n)_{n\geq 1}$ be the sequence defined by the initial condition $x_1>0$ and the recurrence relation $x_{n+1}=\ln(1+x_n)$ for every $n\geq 1$. Then there exists $C\in\mathbb{R}$ such that*

$$x_n=\frac{2}{n}+\frac{2}{3}\cdot\frac{\ln n}{n^2}-\frac{4C}{n^2}+\frac{2}{9}\cdot\frac{\ln^2 n}{n^3}-\frac{2\left(12C+1\right)}{9}\cdot\frac{\ln n}{n^3}+o\left(\frac{\ln n}{n^3}\right).$$

Proof. As is well known $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+o\left(x^4\right)$ as $x\to 0$, i.e., $a_1=-\frac{1}{2}$, $a_2=\frac{1}{3}$, $a_3=-\frac{1}{4}$. Then $\frac{a_1^2-a_2}{a_1^3}=\frac{8}{12}=\frac{2}{3}$, $\frac{\left(a_1^2-a_2\right)^2}{a_1^5}=-\frac{2}{9}$, $\frac{\left(a_1^2-a_2\right)\left(2a_1C+a_1^2-a_2\right)}{a_1^5}=\frac{32}{12}\left(-C-\frac{1}{12}\right)=-\frac{2(12C+1)}{9}$. We apply Theorem 17. □

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On the number of dividers of square-free rational integers and quadratic fields of class-number 1

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Abstract. Throughout the paper d denotes a square-free rational integer, $K = \mathbb{Q}(\sqrt{d})$ a quadratic field and $\mathbb{A}_{\sqrt{D}}$ its ring of integers. Our goal is to prove that if $\mathbb{A}_{\sqrt{D}}$ is a Unique Factorization Domain (UFD), then d is a prime rational integer or d has only two prime factors which are not congruent to 1 modulo 4.

Keywords: quadratic field; unique factorization domain
MSC: 11A51

1. Introduction

This article is the result of several years of collaboration between the two authors. Now, following the passing of Valentin Tănase, I feel it is my duty to share our results regarding a classical problem of Number Theory.

Throughout the paper d denotes a square-free rational integer, $K = \mathbb{Q}(\sqrt{d})$ a quadratic field and $\mathbb{A}_{\sqrt{D}}$ its ring of integers.

Definition 1. *For any rational integer n, we say that n* is represented by the norm *(in $\mathbb{A}_{\sqrt{D}}$) if there is $\alpha \in \mathbb{A}_{\sqrt{D}}$ with $N(\alpha) = \pm n$.*

We also assume that the next statement is well known.

Proposition 2. *Let p be an odd rational prime which does not divide d. Then:*

(1) p remains prime in $\mathbb{A}_{\sqrt{D}}$ iff d is not a quadratic residue modulo p.

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(2) p is irreducible in $\mathbb{A}_{\sqrt{D}}$ iff p is not represented by the norm (in $\mathbb{A}_{\sqrt{D}}$).

We also assume that the properties of the Legendre symbol and the law of quadratic reciprocity are very well known to our readers.

2. A Quick Example

For a quick illustration of our proof, let's consider $d = 15$ and let's take a quick look at the next diophantine equation:

$$x^2 - 15y^2 = \pm 7. \tag{1}$$

Assuming that the equation has rational solutions, then obviously ± 7 would be quadratic residue modulo 3 and 5. However we can immediately notice that neither 7 nor -7 is a quadratic residue modulo 5 and therefore the above equation has no solutions. We've just proved that 7 is not represented by the norm in $\mathbb{A}_{\sqrt{15}}$ and therefore 7 *is irreducible in* $\mathbb{A}_{\sqrt{15}}$.

On the other hand 15 is a quadratic residue modulo 7 and therefore 7 is not prime in $\mathbb{A}_{\sqrt{15}}$. By all means $\mathbb{A}_{\sqrt{15}}$ is not a UFD.

3. Main Result

This very basic criterion for identifying irreducible elements that are not primes in $\mathbb{A}_{\sqrt{D}}$ will be successfully exploited in order to prove our main theorem. The next lemma is a generalization of the above criterion.

Lemma 3. *Let p be a rational prime which does not divide d and is represented by the norm, and let q be an odd rational integer that divides d. Then either p or $-p$ is a quadratic residue modulo q.*

Proof. Notice that p is represented by the norm if and only if one of the next two diophantine equations has rational solutions:

$$x^2 - dy^2 = \pm p \quad \text{if } d \equiv 2, 3 \bmod 4, \text{or} \tag{2}$$

$$x^2 - dy^2 = \pm 4p \quad \text{if } d \equiv 1 \bmod 4. \tag{3}$$

For the former we have that $x^2 \equiv \pm p \bmod q$, $(\forall) q \mid d$, which will lead us to the desired conclusion.

For the latter we have that $x^2 \equiv \pm 4p \bmod q$, $(\forall) q \mid d$, and because q is odd we have $(2^{-1}x)^2 \equiv \pm p \bmod q$, which again proves our conclusion. $\square$

Remark 4. The same criterion could be restricted only to the rational primes q that divide d using the Legendre symbol.

If p is represented by the norm then

$$\left(\frac{\pm p}{q}\right) = 1, (\forall) q \textit{ odd prime}, q \mid d. \tag{4}$$

Corollary 5. *Let p be a prime rational integer, which does not divide d and is represented by the norm. Then either p or $-p$ is a quadratic residue modulo all q, where q is any odd rational integer that divides d.*

Theorem 6. *If $\mathbb{A}_{\sqrt{D}}$ is a UFD, then d is a prime rational integer or d has only two prime factors which are not congruent to 1 modulo 4.*

Proof. Let $d = d_1 \cdots d_k$ with $d_i > 0, (\forall) i = \overline{1,k}$, be the factorization of d as a product of rational primes.

(I) Let us assume that d is composed and $d_1 \equiv 1 \bmod 4$.

Let α be a rational integer which is coprime with d_1 and is not a quadratic residue modulo d_1 (i.e., $\left(\frac{\alpha}{d_1}\right) = -1$).

If d is even we can assume that $d_2 = 2$ and consider the following system of simultaneous congruences:

$$\begin{aligned} x \equiv & \ 5 \bmod 8, & (5)\\ x \equiv & \ \alpha \bmod d_1, & (6)\\ x \equiv & \ 1 \bmod d_i, i = \overline{3,k}. & (7) \end{aligned}$$

By Chinese Remainder Theorem the above system of congruences has rational solutions and let x_0 be a solution. Since $\mathrm{lcm}(8, d_1, d_2, ..., d_k) = 4d$, all rational integers $n = x_0 + 4dn, (\forall) n \in \mathbb{Z}$, are also valid solutions. On the other hand, since x_0 and $4d$ are coprime, by Dirichlet's Prime Number Theorem there is a rational odd prime p that satisfies the above system of simultaneous congruences. To conclude, we found out a rational prime p with the following properties:

$$\left(\frac{p}{d_i}\right) = \begin{cases} -1 & \text{if } i = 1, \\ 1 & \text{if } i > 2. \end{cases} \tag{8}$$

Using the Legendre symbol properties we have

$$\left(\frac{-p}{d_i}\right) = (-1)^{\frac{d_i-1}{2}} \left(\frac{p}{d_i}\right) = \begin{cases} -1 & \text{if } i = 1, \\ (-1)^{\frac{d_i-1}{2}} & \text{if } i > 2, \end{cases} \tag{9}$$

since $d_1 \equiv 1 \bmod 4$.

We have just proved that $\left(\frac{\pm p}{d_1}\right) = -1$ and from the previous lemma, p is not represented by norm and therefore p is irreducible in $\mathbb{A}_{\sqrt{D}}$.

Our next goal is to prove that p is not a prime element of $\mathbb{A}_{\sqrt{D}}$ (which is true iff $\left(\frac{d}{p}\right) = 1$). Applying the quadratic reciprocity law we get

$$\left(\frac{d_i}{p}\right) = (-1)^{\frac{p-1}{2}\frac{d_i-1}{2}} \left(\frac{p}{d_i}\right) = \begin{cases} \left(\frac{p}{d_i}\right) & \text{if } i \neq 2, \\ -1 & \text{if } i = 2, \end{cases} \tag{10}$$

since $p \equiv 5 \bmod 8$ and obviously $\left(\frac{d_2}{p}\right) = \left(\frac{2}{p}\right) = -1$.

We are now ready to compute:

$$\left(\frac{d}{p}\right) = \prod_{i=1}^{k}\left(\frac{d_i}{p}\right) = \left(\frac{2}{p}\right)\prod_{i\neq 2}^{k}\left(\frac{d_i}{p}\right) = (-1)\prod_{i\neq 2}^{k}\left(\frac{d_i}{p}\right) = (-1)(-1)\prod_{i=3}^{k}\left(\frac{d_i}{p}\right)$$
$$= 1$$

and therefore we were able to find an element $p \in \mathbb{A}_{\sqrt{D}}$ which is irreducible but is not prime. Hence $\mathbb{A}_{\sqrt{D}}$ is not a UFD.

If d **is odd** let β be a rational integer, which is coprime with d_2 and is not a quadratic residue modulo d_2 (i.e., $\left(\frac{\beta}{d_2}\right) = -1$). Let us consider now the following system of simultaneous congruences:

$$x \equiv \ 1 \bmod 4, \tag{11}$$
$$x \equiv \ \alpha \bmod d_1, \tag{12}$$
$$x \equiv \ \beta \bmod d_2, \tag{13}$$
$$x \equiv \ 1 \bmod d_i, i = \overline{3,k}. \tag{14}$$

Similarly we can prove that there is a rational odd prime p which is also solution to the above system of simultaneous congruences. Obviously, it holds

$$\left(\frac{p}{d_i}\right) = \begin{cases} -1 & \text{if } i \leq 2, \\ \ \ 1 & \text{if } i > 2. \end{cases} \tag{15}$$

At the same time one has

$$\left(\frac{-p}{d_i}\right) = (-1)^{\frac{d_i-1}{2}}\left(\frac{p}{d_i}\right). \tag{16}$$

We notice that $\left(\frac{\pm p}{d_1}\right) = -1$ (since $d \equiv 1 \bmod 4$), and by the previous lemma p is not represented by norm and therefore p is irreducible in $\mathbb{A}_{\sqrt{D}}$.

Applying the quadratic reciprocity law we get

$$\left(\frac{d_i}{p}\right) = (-1)^{\frac{p-1}{2}\frac{d_i-1}{2}}\left(\frac{p}{d_i}\right) = \left(\frac{p}{d_i}\right) \tag{17}$$

since $p \equiv 1 \bmod 4$. Therefore, we get

$$\left(\frac{d}{p}\right) = \prod_{i=1}^{k}\left(\frac{d_i}{p}\right) = \prod_{i=1}^{k}\left(\frac{p}{d_i}\right) = \left(\frac{p}{d_1}\right)\left(\frac{p}{d_2}\right)\prod_{i=3}^{k}\left(\frac{p}{d_i}\right) = (-1)(-1)\cdot 1\cdots 1$$
$$= 1.$$

Again we were able to find an element $p \in \mathbb{A}_{\sqrt{D}}$ which is irreducible but is not prime, and therefore $\mathbb{A}_{\sqrt{D}}$ is not a UFD.

(II) Let us suppose now that d has at least three rational prime factors (i.e., $k \geq 3$) which are not congruent to 1 modulo 4.

Let α be a rational integer which is coprime with d_1 and is not a quadratic residue modulo d_1 (i.e., $\left(\frac{\alpha}{d_1}\right) = -1$).

If d is even we may assume that $d_2 = 2$ and let us consider the following system of simultaneous congruences:

$$x \equiv \quad 5 \bmod 8, \tag{18}$$

$$x \equiv \quad \alpha \bmod d_1, \tag{19}$$

$$x \equiv \quad 1 \bmod d_i, i = \overline{3, k}. \tag{20}$$

We know by now that there is an odd rational prime p which is a solution of the above system of simultaneous congruences. Hence,

$$\left(\frac{p}{d_i}\right) = \begin{cases} -1 & \text{if } i = 1, \\ 1 & \text{if } i > 2. \end{cases} \tag{21}$$

Using the Legendre symbol properties we have

$$\left(\frac{-p}{d_i}\right) = (-1)^{\frac{d_i-1}{2}} \left(\frac{p}{d_i}\right) = \begin{cases} 1 & \text{if } i = 1, \\ -1 & \text{if } i > 2, \end{cases} \tag{22}$$

since $d_i \equiv 3 \bmod 4$, $(\forall) i = \overline{1, k}, i \neq 2$.

Let us notice now that p is not represented by the norm. If that would be the case, then $\pm p$ would be quadratic residue modulo q, $(\forall) q \mid d$ (see Corollary 5). The identities above however show that neither $\left(\frac{p}{d_i}\right) = 1$ nor $\left(\frac{-p}{d_i}\right) = 1 (\forall) i = \overline{1, k}, i \neq 2$ (here we used the assumption that $k > 2$ and therefore $\left(\frac{-p}{d_3}\right) = -1$). Hence p is irreducible in $\mathbb{A}_{\sqrt{D}}$.

Applying the quadratic reciprocity law we get

$$\left(\frac{d_i}{p}\right) = (-1)^{\frac{p-1}{2}\frac{d_i-1}{2}} \left(\frac{p}{d_i}\right) = \begin{cases} -1 & \text{if } i = 2, \\ \left(\frac{p}{d_i}\right) & \text{if } i \neq 2, \end{cases} \tag{23}$$

since $p \equiv 5 \bmod 8$ and $d_2 = 2$.

Finally, we get

$$\left(\frac{d}{p}\right) = \prod_{i=1}^{k} \left(\frac{d_i}{p}\right) = \left(\frac{2}{p}\right) \prod_{i \neq 2}^{k} \left(\frac{d_i}{p}\right) = (-1) \prod_{i \neq 2}^{k} \left(\frac{p}{d_i}\right) = (-1)(-1) \prod_{i=3}^{k} \left(\frac{p}{d_i}\right)$$
$$= 1.$$

Again we were able to find an element $p \in \mathbb{A}_{\sqrt{D}}$ which is irreducible but is not prime and therefore $\mathbb{A}_{\sqrt{D}}$ is not a UFD.

If d is odd let β be a rational integer which is coprime with d_2 and is not a quadratic residue modulo d_2 (i.e., $\left(\frac{\beta}{d_2}\right) = -1$). Let us consider now the following system of simultaneous congruences:

$$x \equiv 1 \bmod 4, \tag{24}$$
$$x \equiv \alpha \bmod d_1, \tag{25}$$
$$x \equiv \beta \bmod d_2, \tag{26}$$
$$x \equiv 1 \bmod d_i, i = \overline{3,k}. \tag{27}$$

Similarly we can prove that there is a rational odd prime p which is also solution to the above system of simultaneous congruences. Obviously,

$$\left(\frac{p}{d_i}\right) = \begin{cases} -1 & \text{if } i \leq 2, \\ 1 & \text{if } i > 2. \end{cases} \tag{28}$$

At the same time one has

$$\left(\frac{-p}{d_i}\right) = (-1)^{\frac{d_i-1}{2}}\left(\frac{p}{d_i}\right) = \begin{cases} 1 & \text{if } i \leq 2, \\ -1 & \text{if } i > 2. \end{cases} \tag{29}$$

Because neither p nor $-p$ is a quadratic residue modulo all d_i with $i = \overline{1,k}$, we have that p is irreducible in $\mathbb{A}_{\sqrt{D}}$.

Applying the quadratic reciprocity law we get

$$\left(\frac{d_i}{p}\right) = (-1)^{\frac{p-1}{2}\frac{d_i-1}{2}}\left(\frac{p}{d_i}\right) = \left(\frac{p}{d_i}\right) \tag{30}$$

since $p \equiv 1 \bmod 4$.

Finally,

$$\left(\frac{d}{p}\right) = \prod_{i=1}^{k}\left(\frac{d_i}{p}\right) = \prod_{i=1}^{k}\left(\frac{p}{d_i}\right) = \left(\frac{p}{d_1}\right)\left(\frac{p}{d_2}\right)\prod_{i=3}^{k}\left(\frac{p}{d_i}\right) = (-1)(-1)\cdot 1 \cdots 1$$
$$= 1,$$

and again we were able to find an element $p \in \mathbb{A}_{\sqrt{D}}$ which is irreducible but is not prime, and therefore $\mathbb{A}_{\sqrt{D}}$ is not a UFD. □

Remark 7. There are more powerful results regarding the structure of the ideal classes group, in the case of quadratic fields. See, for example, Theorem 105 and Theorem 106 from [2], where it is proven that the number of elements of order less than or equal to 2 from group $C_{Q(\sqrt{d})}$ may be 2^{t-1} or 2^{t-2}, where t is the number of prime factors of the discriminant D of $\sqrt{d}$.

Additionally, in Theorem 2.18 from [1] it is proved that in the case of a quadratic imaginary field with $d = -n \equiv 2, 3 \bmod 4$, we have $h(d) = h(-4n) = 1 \iff n = 1, 2, 3, 7$.

However, our results are obtained following a different approach than the aforementioned textbooks, an elementary one.

Acknowledgments. Many thanks to the referee for indicating the textbooks of D. Hilbert and D. Cox, where these results are presented.

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Traian Lalescu national mathematics contest for university students, 2019 edition

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Abstract. This note deals with the problems proposed at the 2019 edition of the Traian Lalescu mathematics contest for university students, hosted by the Technical University of Cluj-Napoca between May 9th and May 11th, 2019.

Keywords: Rank, eigenvalue, equivalent matrices, connectedness, change of variable, integrals, series

MSC: Primary 15A03; Secondary 15A21, 26D15.

17 students, representing five universities from Braşov, Bucharest, Cluj-Napoca, Craiova and Timişoara, participated in Section A. Problem 1 in this section was solved by more than half of the participants, whilst few contestants managed to tackle the other problems, and even fewer gave complete solutions for these.

A number of 27 students participated in Section B, first and second year, technical section-electric profile. These students were selected from the following six universities: Alexandru I. Cuza University of Iaşi, Gheorghe Asachi Technical University of Iaşi, Ferdinand I Military Technical Academy of Bucharest, Politehnica University of Timişoara, Politehnica University of Bucharest, and Technical University of Cluj-Napoca. Analyzing the scores which were obtained by the contestants, the problems were correctly ordered, respecting the difficulty. Problems 1 and 2, one of algebra and the other one of analysis, were accessible. Half of the students obtained high results for the first two problems. Problem 3 was a difficult problem of mathematical analysis (integral calculus). Problem 4 was the most difficult problem in competition and was not solved completely by any contestant.

Many of the students are former participants and medalists of the national high school mathematics olympiad.

We present in the sequel the problems proposed in Sections A and B of the contest and their solutions.

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Section A

Problem 1. One considers the points $A(0,0,1)$, $B(6,0,0)$ in $\mathbb{R}^3$, and the straight line

$$d : \begin{cases} x - y = 0, \\ z = 0. \end{cases}$$

Find the coordinates of the point $M \in d$ for which the sum of the distances to A and B is minimal.

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Solution 1. Since $OA \perp d$, the distance from any point $P \in d$ to A equals the distance from P to any point of the circle that has centre O, radius OA, and is perpendicular to d.

We consider the point $C\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$. Then $\vec{i}+\vec{j} \perp \overrightarrow{OA}$ and $\vec{i}+\vec{j} \perp \overrightarrow{OC}$, so $d \perp (AOC)$. Consequently, C belongs to the aforementioned circle, so for any point $P \in d$ we have $PA = PC$.

Since C lies in the plane spanned by d and B and d separates B and C, we derive, using the triangle inequality, that the point M we are looking for is at the intersection of the lines d and BC.

Thus, there is a real constant λ such that $\overrightarrow{BM} = \lambda\overrightarrow{BC}$, i.e.,

$$(x_M - 6)\vec{i} + y_M\vec{j} = -\lambda\left(\frac{1}{\sqrt{2}} + 6\right)\vec{i} + \frac{\lambda}{\sqrt{2}}\vec{j},$$

whence $x_M = 6 - \lambda\left(\frac{1}{\sqrt{2}} + 6\right)$ and $y_M = \frac{\lambda}{\sqrt{2}}$.

Consequently, $M \in d$ if and only if $6 - \lambda\left(\dfrac{1}{\sqrt{2}} + 6\right) = \dfrac{\lambda}{\sqrt{2}}$, which yields $\lambda = \dfrac{3\sqrt{2}}{1+3\sqrt{2}}$.

Therefore, the coordinates of the point M are $\left(\dfrac{3}{1+3\sqrt{2}}, \dfrac{3}{1+3\sqrt{2}}, 0\right)$.

Solution 2. Consider an arbitrary point $Q_t(t,t,0) \in d$ and define a function $D : \mathbb{R} \to \mathbb{R}$ by $D(t) = Q_tA + Q_tB = \sqrt{2t^2+1} + \sqrt{(t-6)^2+t^2}$. Then D is everywhere differentiable and

$$D'(t) = \frac{2t}{\sqrt{2t^2+1}} + \frac{2t-6}{\sqrt{(t-6)^2+t^2}}.$$

The equation $D'(t) = 0$ does obviously not admit the solution $t = 0$, so it is equivalent to

$$\sqrt{\frac{(t-6)^2+t^2}{2t^2+1}} = \frac{3-t}{t},$$

and thus its solutions will lie in the interval $(0,3]$. On this interval, the equation $D'(t)=0$ is equivalent to $t^2(2t^2-12t+36)=(2t^2+1)(3-t)^2$ and further to $17t^2+6t-9=0$. The only solution this last equation has in the interval $(0,3]$ is $\tau=\frac{3}{1+3\sqrt{2}}$. Consequently, D has τ as its only stationary point. Since $\lim\limits_{t\to\pm\infty}=\infty$, τ is the point of global minimum of D.

Thus, the point we are looking for is $M\left(\dfrac{3}{1+3\sqrt{2}}, \dfrac{3}{1+3\sqrt{2}}, 0\right)$.

Problem 2. Let R be a commutative unitary ring. Two matrices $A,B\in\mathcal{M}_2(R)$ are called *equivalent* if there exist invertible matrices U, V in $\mathcal{M}_2(R)$ such that $B=UAV$.

Show that the matrices $\begin{pmatrix} X & Y \\ 0 & Z\end{pmatrix}$ and $\begin{pmatrix} X & 0 \\ Y & Z\end{pmatrix}$ from $\mathcal{M}_2(\mathbb{C}[X,Y,Z])$ are not equivalent.

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Solution. Assume that the given matrices are equivalent. Then there exist two invertible matrices $U=\begin{pmatrix} f & g \\ h & j\end{pmatrix}$, $V=\begin{pmatrix} F & G \\ H & J\end{pmatrix}$ in $\mathcal{M}_2(\mathbb{C}[X,Y,Z])$ such that

$$U\begin{pmatrix} X & Y \\ 0 & Z\end{pmatrix}=\begin{pmatrix} X & 0 \\ Y & Z\end{pmatrix}V.$$

Then

$$\begin{aligned} Xf &= XF, \\ Yf+Zg &= XG, \\ Xh &= YF+ZH, \\ Y(h-G) &= Z(J-j), \end{aligned} \tag{1}$$

and from the last equation we conclude that there is $\Phi\in\mathbb{C}[X,Y,Z]$ such that $h=G+Z\Phi$ and $J=j+Y\Phi$.

We have $\det U=fj-g(G+Z\Phi)$.

From the second equation in (1) we get $f\in(X,Z)$ (here (X,Z) denotes the ideal generated by X and Z in $\mathbb{C}[X,Y,Z]$) and $g\in(X,Y)$, so $\det U\in(X,Y,Z)$, and therefore $\det U$ is not invertible in $\mathbb{C}[X,Y,Z]$, a contradiction.

Remark. It is well known that two matrices (of the same size) over a principal ideal domain are equivalent if and only if the ideals generated by their $r\times r$ minors are equal, for all $r\geq 1$.

In our problem the matrices have the same ideals generated by minors, namely (X,Y,Z) and (XZ), but are not equivalent. The reason is that the ring $\mathbb{C}[X,Y,Z]$ is not principal.

Problem 3. Let $n \geq 2$ be a positive integer and let $f : \mathbb{R}^n \to \mathbb{R}$,

$$f(x_1, \ldots, x_n) = \begin{cases} \dfrac{\sum\limits_{1 \leq i < j \leq n} x_i x_j}{\sum\limits_{i=1}^{n} x_i^2} & \text{for } (x_1, \ldots, x_n) \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\} \\ a & \text{for } (x_1, \ldots, x_n) = (0, \ldots, 0). \end{cases}$$

Find $a \in \mathbb{R}$ such that the graph of the function f is connected.

Gabriel Mincu, University of Bucharest

Solution. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\}$. Then

$$\sum_{i=1}^{n} x_i^2 \geq -2 \sum_{1 \leq i < j \leq n} x_i x_j,$$

so

$$-\frac{1}{2} \leq \frac{\sum\limits_{1 \leq i < j \leq n} x_i x_j}{\sum\limits_{i=1}^{n} x_i^2}.$$

The inequality

$$\sum_{1 \leq i < j \leq n} (x_i - x_j)^2 \geq 0$$

yields

$$(n-1) \sum_{i=1}^{n} x_i^2 \geq 2 \sum_{1 \leq i < j \leq n} x_i x_j,$$

whence

$$\frac{\sum\limits_{1 \leq i < j \leq n} x_i x_j}{\sum\limits_{i=1}^{n} x_i^2} \leq \frac{n-1}{2}.$$

Consequently,

$$f(\mathbb{R}^n \setminus \{(0, \ldots, 0)\}) \subset \left[-\frac{1}{2}, \frac{n-1}{2}\right]. \tag{2}$$

The value $-\frac{1}{2}$ is taken at $(1, 1, \ldots, 1, 1-n)$, whilst the value $\frac{n-1}{2}$ is taken at $(1, 1, \ldots, 1)$.

The function f is continuous on $\mathbb{R}^n \setminus \{(0, \ldots, 0)\}$, so $f(\mathbb{R}^n \setminus \{(0, \ldots, 0)\})$ is connected. Taking into account relation (2), we get

$$f\left(\mathbb{R}^n \setminus \{(0, \ldots, 0)\}\right) = \left[-\frac{1}{2}, \frac{n-1}{2}\right].$$

If $a \in \mathbb{R} \setminus \left[-\frac{1}{2}, \frac{n-1}{2}\right]$ and d is the distance from a to the interval $\left[-\frac{1}{2}, \frac{n-1}{2}\right]$, we have

$$\begin{aligned} G_f = &\left(G_f \cap \left(\mathbb{R}^n \times \left(a - \frac{d}{2}, a + \frac{d}{2}\right)\right)\right) \\ &\cup \left(G_f \cap \left(\mathbb{R}^n \times \left(-\frac{1}{2} - \frac{d}{2}, \frac{n-1}{2} + \frac{d}{2}\right)\right)\right), \end{aligned}$$

so G_f is not connected.

If $a \in \left[-\frac{1}{2}, \frac{n-1}{2}\right]$, we first notice that, since the restriction of f to $\mathbb{R}^n \setminus \{(0, \ldots, 0)\}$ is continuous, its graph is path-connected.
Since $f(\mathbb{R}^n \setminus \{(0, \ldots, 0)\}) = \left[-\frac{1}{2}, \frac{n-1}{2}\right]$, there exists $u \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\}$ such that $f(u) = a$. Thus,

$$\lim_{x \to 0} \left(tu_1, \ldots, tu_n, f\left(tu_1, \ldots, tu_n\right)\right) = (0, \ldots, 0, f(u)) = (0, \ldots, 0, a),$$

so $\gamma : [0,1] \to \mathbb{R}^{n+1}$, $\gamma(t) = \left(tu_1, \ldots, tu_n, f\left(tu_1, \ldots, tu_n\right)\right)$ is continuous. Moreover, $\mathrm{Im}\gamma \subset G_f$ and $\gamma(0) = (0, \ldots, 0, a)$. Therefore G_f is also path-connected, and thus it is connected.

We conclude that G_f is connected if and only if $a \in \left[-\frac{1}{2}, \frac{n-1}{2}\right]$.

Problem 4. (a) Let $n \in \mathbb{N}^*$. Compute

$$I_n = \int_0^1 \frac{\ln(1-x) + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}}{x^{n+1}} \, \mathrm{d}x.$$

(b) Let $a \in \mathbb{R}$ and $b > 0$. Prove that the integral

$$J(a,b) = \int_0^\infty \left[2 + (x+a) \ln \frac{x}{x+b}\right] \mathrm{d}x$$

is convergent if and only if $a = 1$ and $b = 2$, and in this case compute $J(1,2)$.

Ovidiu Furdui, Technical University of Cluj-Napoca

Solution. (a) We have

$$
\begin{aligned}
I_n &= -\int_0^1 \frac{\sum\limits_{i=n+1}^{\infty} \frac{x^i}{i}}{x^{n+1}} \mathrm{d}x = -\int_0^1 \sum_{i=n+1}^{\infty} \frac{x^{i-n-1}}{i} \mathrm{d}x \\
&\overset{(*)}{=} -\sum_{i=n+1}^{\infty} \frac{1}{i(i-n)} = -\sum_{j=1}^{\infty} \frac{1}{j(j+n)} \\
&= -\frac{1}{n} \sum_{j=1}^{\infty} \left(\frac{1}{j} - \frac{1}{j+n}\right) = -\frac{1}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).
\end{aligned}
$$

(b) We change the variable $\frac{x}{x+b} = y$ and get

$$
\begin{aligned}
J(a,b) &= b \int_0^1 \left[2 + \frac{by + a(1-y)}{1-y} \ln y\right] \frac{1}{(1-y)^2} \mathrm{d}y \\
&\overset{1-y=t}{=} b \int_0^1 \left[2 + \frac{b-(b-a)t}{t} \ln(1-t)\right] \frac{1}{t^2} \mathrm{d}t.
\end{aligned}
$$

An easy computation shows that

$$
\begin{aligned}
&2 + \frac{b-(b-a)t}{t} \ln(1-t) = \\
&= 2 - b + \frac{b-2a}{2} t + b \frac{\ln(1-t) + t + \frac{t^2}{2}}{t} - (b-a)\left(\ln(1-t) + t\right).
\end{aligned}
$$

Then we have

$$
J(a,b) = b \int_0^1 \left[\frac{2 - b + \frac{b-2a}{2} t}{t^2} + b \frac{\ln(1-t) + t + \frac{t^2}{2}}{t^3} - (b-a) \frac{\ln(1-t) + t}{t^2}\right] \mathrm{d}t \tag{3}
$$

Since the integrals

$$
I_1 = \int_0^1 \frac{\ln(1-t) + t}{t^2} \mathrm{d}t \quad \text{and} \quad I_2 = \int_0^1 \frac{\ln(1-t) + t + \frac{t^2}{2}}{t^3} \mathrm{d}t
$$

are convergent, their values being calculated in (a), we have that $J(a,b)$ is convergent if and only if the integral

$$
\int_0^1 \frac{2 - b + \frac{b-2a}{2} t}{t^2} \mathrm{d}t
$$

is convergent.

We have

$$\int_0^1 \frac{2-b+\frac{b-2a}{2}t}{t^2}\mathrm{d}t = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{2-b+\frac{b-2a}{2}t}{t^2}\mathrm{d}t$$
$$= \lim_{\varepsilon\to 0^+}\left(-\frac{2-b}{t}+\frac{b-2a}{2}\ln t\right)\Big|_\varepsilon^1$$
$$= b-2-\lim_{\varepsilon\to 0^+}\frac{(b-2a)\varepsilon\ln\varepsilon-2(2-b)}{2\varepsilon}.$$

The foregoing integral is convergent if and only if the limit

$$\lim_{\varepsilon\to 0^+}\frac{(b-2a)\varepsilon\ln\varepsilon-2(2-b)}{2\varepsilon}$$

exists and is finite. But this limit is finite if and only if $2-b=0$ and $b-2a=0$, that is, $b=2$ and $a=1$.

In this case we have, from (3), that the value of integral $J(1,2)$ equals

$$\begin{aligned} J(1,2) &= 4\int_0^1 \frac{\ln(1-t)+t+\frac{t^2}{2}}{t^3}\mathrm{d}t - 2\int_0^1 \frac{\ln(1-t)+t}{t^2}\mathrm{d}t \\ &= 4I_2-2I_1 \\ &= 4\left(-\frac{3}{4}\right)-2(-1) \\ &= -1. \end{aligned}$$

SECTION B

Problem 1. For any $a\in\mathbb{C}, a\neq 0$, and any matrix $A\in\mathcal{M}_n(\mathbb{C}), n\geq 2$, show that

$$\operatorname{rank}(aA-A^2)=\operatorname{rank}A+\operatorname{rank}(aI_n-A)-n.$$

Is the statement true for $a=0$? Justify!

Bogdan Sebacher, Ferdinand I Military Technical Academy Bucharest

For this problem the students gave five full solutions and five partial solutions.

Solution 1. [Author's solution] For $a=0$ and $A=O_n$ the relation is not true. We prove the case $a\neq 0$. With the substitution $A=aB$ the relation becomes

$$\operatorname{rank}(B-B^2)=\operatorname{rank}B+\operatorname{rank}(I_n-B)-n. \tag{4}$$

Consider the linear map $T:\mathbb{C}^n\to\mathbb{C}^n$, $Tx=Bx$. In the canonical basis, this map has the attached matrix $M_T=B$ and then $\operatorname{rank}B=\operatorname{rank}T$, $\operatorname{rank}(I_n-B)=\operatorname{rank}(I-T)$, $\operatorname{rank}(B-B^2)=\operatorname{rank}(T-T^2)$, where I is the identity operator.

Using dimension theorem for T we obtain

$$n = \dim(\operatorname{Im} T) + \dim(\operatorname{Ker} T) = \operatorname{rank} T + \operatorname{def} T.$$

Therefore, relation (4) becomes

$$\operatorname{def}(T - T^2) = \operatorname{def} T + \operatorname{def}(I - T). \tag{5}$$

In order to prove (5), it is enough to show that

$$\operatorname{Ker}(T - T^2) = \operatorname{Ker} T \oplus \operatorname{Ker}(I - T),$$

that is, for any $x \in \operatorname{Ker}(T-T^2)$ there exist a unique $x_1 \in \operatorname{Ker} T$ and a unique $x_2 \in \operatorname{Ker}(I-T)$ such that $x = x_1 + x_2$. If there are such x_1, x_2 we will obtain the relations

$$(T - T^2)(x) = 0 \Leftrightarrow T(x) = T^2(x),$$
$$T(x_1) = 0 \text{ and } (I - T)(x_2) = 0 \Leftrightarrow T(x_2) = x_2.$$

From the relations $x = x_1 + x_2$ and $T(x) = T(x_1) + T(x_2) = x_2$ it follows

$$x_2 = T(x), x_1 = x - T(x),$$

which are uniquely determined. Next we show that $x_1 \in \operatorname{Ker} T$ and $x_2 \in \operatorname{Ker}(I - T)$. We have

$$T(x_1) = T(x) - T^2(x) = 0,\ T(x_2) = T^2(x) = T(x) = x_2,$$

which conclude the proof.

Solution 2. [Vasile Pop] We prove relation (4) by using elementary transformations of block matrices. We start with the matrix

$$M = \left(\begin{array}{c|c} B & O \\ \hline O & I_n - B \end{array}\right)$$

and make the following successive elementary transformations (we denote by C_i and L_i the transformation using column and line $i \in \{1, 2\}$, respectively):

$$N = L_2 L_1 M C_1 C_2 = \left(\begin{array}{c|c} B - B^2 & O \\ \hline O & I_n \end{array}\right),$$

where $C_1 = \left(\begin{array}{c|c} I_n & I_n \\ \hline O & I_n \end{array}\right)$, $L_1 = \left(\begin{array}{c|c} I_n & O \\ \hline I_n & I_n \end{array}\right)$, $L_2 = \left(\begin{array}{c|c} I_n & -B \\ \hline O & I_n \end{array}\right)$, $C_2 = \left(\begin{array}{c|c} I_n & O \\ \hline -I_n & I_n \end{array}\right)$.

The elementary matrices C_1, C_2, L_1, L_2 have determinant 1, therefore $\operatorname{rank} M = \operatorname{rank} N$. It follows

$$\operatorname{rank} B + \operatorname{rank}(I_n - B) = \operatorname{rank}(B - B^2) + \operatorname{rank} I_n = \operatorname{rank}(B - B^2) + n,$$

hence relation (4) is proved.

Solution 3. [Vasile Pop] If J_B is the Jordan canonical form of matrix B and P is the similarity transformation matrix, then $B = P J_B P^{-1}$. Since $\operatorname{rank} J_B = \operatorname{rank} B$, relation (4) becomes

$$\operatorname{rank}(J_B - J_B^2) = \operatorname{rank} J_B + \operatorname{rank}(I_n - J_B) - n. \tag{6}$$

Let $J_B = \left(\begin{array}{c|c|c|c} J_{\lambda_1} & O & \dots & O \\ \hline O & J_{\lambda_2} & \dots & O \\ \hline O & O & \ddots & O \\ \hline O & O & \dots & J_{\lambda_p} \end{array}\right)$ be the canonical Jordan form of the matrix B. Since

$$\operatorname{rank}\left(\begin{array}{c|c|c|c} B_1 & O & \dots & O \\ \hline O & B_2 & \dots & O \\ \hline O & O & \ddots & O \\ \hline O & O & \dots & B_p \end{array}\right) = \operatorname{rank} B_1 + \dots + \operatorname{rank} B_p,$$

the relation (6) reduces to the same relation for Jordan cells. If

$$J_\lambda = \begin{pmatrix} \lambda & 1 & \dots & 0 \\ 0 & \lambda & \ddots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & \dots & \lambda \end{pmatrix}$$

is a Jordan cell of dimension k, we will prove that

$$\operatorname{rank}(J_\lambda - J_\lambda^2) = \operatorname{rank} J_\lambda + \operatorname{rank}(I_k - J_\lambda) - k. \tag{7}$$

We have

$$I_k - J_\lambda = \begin{pmatrix} 1-\lambda & 1 & \dots & 0 \\ 0 & 1-\lambda & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1-\lambda \end{pmatrix},$$

$$J_\lambda - J_{\lambda^2} = \begin{pmatrix} \lambda-\lambda^2 & 1-2\lambda & 1 & \dots & 0 & 0 \\ 0 & \lambda-\lambda^2 & 1-2\lambda & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \lambda-\lambda^2 & \dots & 1-2\lambda & 1 \\ 0 & 0 & 0 & \dots & \lambda-\lambda^2 & 1-2\lambda \\ 0 & 0 & 0 & \dots & 0 & \lambda-\lambda^2 \end{pmatrix},$$

so that

- for $\lambda = 1$ relation (7) is $k-1 = k + (k-1) - k$.
- for $\lambda = 0$ relation (7) becomes $k-1 = (k-1) + k - k$.
- for $\lambda \neq 1, \lambda \neq 0$ relation (7) is $k = k + k - k$.

All the three cases are true.

Solution 4. [Cornel Băeţica] This problem is a particular case of equality in the Sylvester's rank inequality. An immediate consequence of Theorem 2.6 from F. Zhang, *Matrix Theory. Basic Results and Techniques*, Springer, 2011, is the following:

Let X and Y be complex matrices of sizes $m\times n$ and $n\times p$, respectively. Then $\operatorname{rank}(XY)=\operatorname{rank}(X)+\operatorname{rank}(Y)-n$ *if and only if* $\operatorname{Ker}X\subseteq\operatorname{Im}Y$.

If we set $X=aI_n-A$ (with $a\neq 0$) and $Y=A$, then we get the desired result.

Problem 2. Let $(a_n)_{n\geq 3}$ be a decreasing sequence of positive numbers. Show that the series

$$\sum_{n=3}^{\infty}\frac{a_n}{n\,(\ln n)^2}\quad\text{and}\quad\sum_{n=3}^{\infty}\left(\frac{a_n}{\ln(n-1)}-\frac{a_{n+1}}{\ln(n+1)}\right)$$

are convergent.

Mircea Rus, Technical University of Cluj-Napoca

The contestants have given five complete solutions and one partial solution for this problem.

Solution 1. [Author's solution] Using comparison test or Abel's test, we can notice that the series with positive terms $(A):\displaystyle\sum_{n=3}^{\infty}\frac{a_n}{n\,(\ln n)^2}$ is convergent since the sequence (a_n) is decreasing (so, bounded), and $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n\,(\ln n)^2}$ is convergent (the convergence of the last series can be obtained, by example, using condensation test).

For the second series, we write

$$\begin{aligned}\frac{a_n}{\ln(n-1)}-\frac{a_{n+1}}{\ln(n+1)}&=\left(\underbrace{\frac{a_n}{\ln(n-1)\ln n}}_{b_n}-\underbrace{\frac{a_{n+1}}{\ln n\ln(n+1)}}_{b_{n+1}}\right)\ln n\\&=(b_n-b_{n+1})\ln n,\end{aligned}$$

where, obviously, $b_n=\dfrac{a_n}{\ln(n-1)\ln n}$ $(n\geq 3)$ decreases to 0.

Also,

$$\frac{a_n}{n\,(\ln n)^2}=\frac{b_n}{n}\cdot\frac{\ln(n-1)}{\ln n},$$

such that series $(A):\displaystyle\sum_{n=3}^{\infty}\frac{a_n}{n\,(\ln n)^2}$ and $(B):\displaystyle\sum_{n=3}^{\infty}\frac{b_n}{n}$ have the same nature, hence the series B is convergent.

We use a similar approach to the summation rule of Abel and we show that the series $(C):\displaystyle\sum_{n=3}^{\infty}(b_n-b_{n+1})\ln n$ has the same nature as B (hence, is convergent). To this end, consider the sequence of positive numbers

$$c_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \quad (n \geq 1),$$

convergent to Euler's constant γ. Then $\dfrac{1}{k} = c_k - c_{k-1} + (\ln k - \ln(k-1))$ and

$$\begin{aligned} B_n &= \sum_{k=3}^{n} \frac{b_k}{k} = \sum_{k=3}^{n} b_k c_k - \sum_{k=3}^{n} b_k c_{k-1} + \sum_{k=3}^{n} b_k \ln k - \sum_{k=3}^{n} b_k \ln(k-1) \\ &= \sum_{k=3}^{n} b_k c_k - \sum_{k=2}^{n-1} b_{k+1} c_k + \sum_{k=3}^{n} b_k \ln k - \sum_{k=2}^{n-1} b_{k+1} \ln k \\ &= b_n c_n - b_3 c_2 + \sum_{k=3}^{n-1} (b_k - b_{k+1}) c_k + b_n \ln n - b_3 \ln 2 + \sum_{k=3}^{n-1} (b_k - b_{k+1}) \ln k \\ &= C_{n-1} + \underbrace{b_n c_n}_{\downarrow \atop 0} + \underbrace{\frac{a_n}{\ln(n-1)}}_{\downarrow \atop 0} - b_3 (c_2 + \ln 2) + \sum_{k=3}^{n-1} (b_k - b_{k+1}) c_k. \end{aligned}$$

Since (b_n) is decreasing and (c_n) is convergent (hence bounded), it follows that the series (with positive terms) $\sum_{k=3}^{\infty} (b_k - b_{k+1}) c_k$ is convergent. This means that the series B and C have the same nature, hence C is also convergent.

Solution 2. [Alternative statement and solution by Mircea Ivan, Technical University of Cluj-Napoca]

Let $(a_n)_{n \geq 3}$ be a bounded sequence. Show that the series

$$\sum_{n=3}^{\infty} \frac{a_n}{n (\ln n)^2} \quad \text{and} \quad \sum_{n=3}^{\infty} \left(\frac{a_n}{\ln(n-1)} - \frac{a_{n+1}}{\ln (n+1)} \right)$$

are convergent.

Assume $|a_n| \leq M$ for all $n \geq 3$. By comparison test we have

$$\sum_{n=3}^{\infty} \left| \frac{a_n}{n (\ln n)^2} \right| \leq \sum_{n=3}^{\infty} \frac{M}{n (\ln n)^2},$$

hence the first series is convergent.

We have

$$\begin{aligned} \sum_{n=3}^{\infty} \left(\frac{a_n}{\ln(n-1)} - \frac{a_{n+1}}{\ln (n+1)} \right) &= \sum_{n=3}^{\infty} a_n \left(\frac{1}{\ln(n-1)} - \frac{1}{\ln (n+1)} \right) \\ &\quad + \sum_{n=3}^{\infty} \left(\frac{a_n}{\ln n} - \frac{a_{n+1}}{\ln (n+1)} \right), \end{aligned}$$

where
$$\sum_{n=3}^{\infty} a_n \left| \frac{1}{\ln(n-1)} - \frac{1}{\ln(n+1)} \right| \leq \frac{M}{\ln 2}$$
and
$$\sum_{n=3}^{\infty} \left(\frac{a_n}{\ln n} - \frac{a_{n+1}}{\ln(n+1)} \right) = \frac{a_3}{\ln 3}.$$

Problem 3. Let $f : [a, \infty) \to (0, \infty)$ be a continuous and monotone function such that $\int_a^\infty f(x)\,\mathrm{d}x$ is convergent. The graph of f, together with the lines $y = 0$ and $x = a$, delimit a domain of area $A > 0$.

a) Show that, for any $n \in \mathbb{N}^*$, there are points $x_1, \ldots, x_n \in [a, \infty)$ such that the lines of equations $x = x_k$, $k = \overline{1, n}$, parallel to Oy, split the above domain in parts of equal areas.

b) Let $(a_n)_{n\geq 1}$ be a sequence given by
$$a_n = \frac{1}{n} \sum_{k=1}^{n} f(x_k), \quad \forall n \geq 1.$$
Show that V, the volume of the body obtained by rotating the graph of the function f around Ox, is finite and
$$\lim_{n\to\infty} a_n = \frac{V}{\pi A}.$$

Radu Strugariu, Gheorghe Asachi Technical University of Iaşi

For this difficult problem just one contestant gave a full solution. Three partial solutions were also given.

Solution. [Author's solution]

a) We consider the function $F : [a, \infty) \to [0, A)$ given by $F(x) = \int_a^x f(t)\,\mathrm{d}t$. This is continuous, strictly increasing on $[a, \infty)$ and satisfies $\lim_{x\to\infty} F(x) = A$. We take $x_1 = a$. Since F is strictly increasing and has the intermediate value property, it will achieve, successively, the values
$$\frac{A}{n}, \frac{2A}{n}, \ldots, \frac{(n-1)A}{n}$$
in the points $x_2 < x_3 < \cdots < x_n$. It follows $\int_{x_n}^\infty f(t)\,\mathrm{d}t = \dfrac{A}{n}$.

b) Since f is monotone, there exists $\lim_{x\to\infty} f(x)$. Since $\int_a^\infty f(x)\,\mathrm{d}x$ is convergent, it follows that
$$\lim_{x\to\infty} f(x) = 0.$$
So, there exists $\delta > a$ such that, for any $x > \delta$, we have
$$|f(x)| \leq 1.$$
But f is continuous on $[a, \delta]$, hence bounded. It follows that f is bounded on $[a, \infty)$.

Since it is bounded and has positive values, we obtain

$$V = \pi \int_a^\infty f^2(x)\ \mathrm{d}x \le M\pi \int_a^\infty f(x)\ \mathrm{d}x < \infty,$$

where by $M \in (0, \infty)$ we denote the supremum of the function f on $[a, \infty)$.

Since F is continuous and strictly increasing on $[a, \infty)$, it is bijective, and its inverse F^{-1} has the same properties on the interval $[0, A)$.

Moreover

$$F(x_1) = 0,\ F(x_2) = \frac{A}{n},\ F(x_3) = \frac{2A}{n}, \ldots, F(x_n) = \frac{(n-1)A}{n}.$$

Then

$$x_k = F^{-1}\left(\frac{(k-1)A}{n}\right),\ \forall k = \overline{1, n},$$

hence, we can write

$$a_n = \frac{1}{n}\sum_{k=1}^n f(x_k) = \frac{1}{A}\cdot\frac{A}{n}\sum_{k=1}^n f\left(F^{-1}\left(\frac{(k-1)A}{n}\right)\right).$$

Notice the fact that the function $f \circ F^{-1} : [0, A) \to (0, \infty)$ is continuous and bounded, hence we can compute its Riemann integral on $[0, A]$ (we can add an arbitrary value in A). Ignoring the factor $\frac{1}{A}$ in the above formula, we have a Riemann sum of the function $f \circ F^{-1}$. It follows that

$$\lim_{n\to\infty} a_n = \frac{1}{A}\int_0^A f\left(F^{-1}(x)\right)\ \mathrm{d}x.$$

By changing the variable $F^{-1}(x) = y$, it follows that $x = F(y)$, hence $\mathrm{d}x = f(y)\ \mathrm{d}y$, and then

$$\lim_{n\to\infty} a_n = \frac{1}{A}\int_a^\infty f^2(y)\ \mathrm{d}y = \frac{V}{\pi A}.$$

Problem 4. Let $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $A^2 + B^2 = 2AB$.

a) Prove that $(A - B)^n = O_n$.

b) Prove that A and B have the same eigenvalues.

Vasile Pop, Technical University of Cluj-Napoca

Mihai Opincariu, Avram Iancu National College, Brad

This problem was the most difficult one, only four partial solutions were given by contestants.

Solution 1. [Authors' solution]

a) We have the sequence of equivalences

$$A^2+B^2 = 2AB \Leftrightarrow A^2+B^2-AB-BA = AB-BA \Leftrightarrow (A-B)^2 = AB-BA.$$

We denote $C = A - B$ (equivalently, $A = B + C$) and from the above relation we obtain

$$C^2 = AB - BA = (B + C)B - B(B + C) = CB - BC = [C, B] \overset{\text{not}}{=} D.$$

We get $D^{k+1} = (CB - BC)C^{2k} = CBC^{2k} - BC^{2k+1} = [C, BC^{2k}]$ hence $\mathrm{Tr}(D^k) = 0$ for any $k \in \mathbb{N}^*$. We obtain that all eigenvalues of D are 0.

Finally we get $D^n = O_n \Leftrightarrow C^{2n} = O_n \Leftrightarrow C^n = O_n \Leftrightarrow (A - B)^n = O_n$.

b) From the given relation we get

$$(A - \lambda I_n)^2 + (B - \lambda I_n)^2 = 2(A - \lambda I_n)(B - \lambda I_n),$$

for any $\lambda \in \mathbb{C}$. This implies

$$(A - \lambda I_n)(2B - A - \lambda I_n) = (B - \lambda I_n)^2$$

and

$$(B - \lambda I_n)(2A - B - \lambda I_n) = (A - \lambda I_n)^2.$$

Now, by taking determinants we obtain $f_A(\lambda) = 0 \Leftrightarrow f_B(\lambda) = 0$, where f_A and f_B is the characteristic polynomial of A and B, respectively. Thus, A and B have the same eigenvalues.

Solution 2. [Cornel Băeţica]

a) If X is a square matrix over a field of characteristic zero, and X commutes with one of its commutators, say $[X, Y]$, then $[X, Y]$ is nilpotent. (This is a well known result of Jacobson from 1935.)

Now set $X = A - B$ and $Y = B$. We have $X^2 = [X, Y]$, and it is obvious that X commutes with $[X, Y]$, so $[X, Y] = (A - B)^2$ is nilpotent. Thus $A - B$ is nilpotent, and therefore $(A - B)^n = O_n$.

b) If λ is an eigenvalue of B, then there exists $x \in \mathbb{C}^n$, $x \neq 0$, such that $Bx = \lambda x$. Since $A^2 + B^2 = 2AB$, we get $A^2x + B^2x = 2ABx$, which implies $(A - \lambda I_n)^2 x = 0$, and thus λ is an eigenvalue of A, too.

For the converse take the transpose in $A^2 + B^2 = 2AB$ and recall that the eigenvalues of a matrix are equal to the eigenvalues of its transpose.

Remark 1. Although A and B have the same eigenvalues, it does not follow that A is equivalent to B and neither A is similar to B, as we can see from the following example:

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^2 + B^2 = 2AB = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

but

$$\operatorname{rank} A = 1 \neq 2 = \operatorname{rank} B.$$

Remark 2. One can prove even more: If $A, B \in \mathcal{M}_n(\mathbb{C})$ have the property $A^2 + B^2 = 2AB$, then A, B are simultaneously triangularizable. In particular, it follows that A and B have the same characteristic polynomials.

MATHEMATICAL NOTES

A remarkable series on fixed points of $\tan z$

Robert Bosch[1)]

Abstract. In this note we consider the function $\tan z$ over complex numbers, showing their fixed points are real numbers, and also, there are infinitely many. Later, we show that it holds

$$\sum_{k=1}^{\infty} \frac{1}{x_k^2} = \frac{1}{10},$$

where x_k are the positive fixed points of $\tan x$. The source of this problem is not well determined, the author proposed it to the journal *Mathematical Reflections* as problem U223. Our solution is very similar to the one found by *G.R.A.20 Problem Solving Group, Roma, Italy.*

Keywords: $\tan z$, complex numbers, fixed points.
MSC: 33B10.

1. The fixed points of $\tan z$ are real numbers

In this section we shall prove that if $\tan z = z$, with $z = a - bi$ a complex number, then $b = 0$. For sake of contradiction, assume $b \neq 0$. We know that

$$\tan z = \frac{e^{2iz} - 1}{i(e^{2iz} + 1)}.$$

So, $\tan z = z$ is equivalent to

$$\begin{aligned} a \sin 2a + (1-b)\cos 2a &= \frac{b+1}{e^{2b}}, \\ (1-b)\sin 2a - a\cos 2a &= \frac{a}{e^{2b}}. \end{aligned}$$

From these equations we obtain

$$\begin{aligned} \sin 2a &= \frac{2a}{(a^2 + (b-1)^2)e^{2b}}, \\ \cos 2a &= \frac{(1 - a^2 - b^2)}{(a^2 + (b-1)^2)e^{2b}}. \end{aligned}$$

From $\sin^2 2a + \cos^2 2a = 1$, we deduce

$$e^{4b} = \frac{a^2 + (b+1)^2}{a^2 + (b-1)^2}.$$

So, we can assume without loss of generality $a \geq 0$ and $b > 0$. We claim that it holds

$$1 \geq \frac{\sin 2a}{2a} = \frac{e^{4b} - 1}{4b \cdot e^{2b}} > 1,$$

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a contradiction. Thus, $b = 0$. Note that for $b > 0$ one has

$$\frac{e^{4b}-1}{4b\cdot e^{2b}} > 1 \Leftrightarrow e^{2b} > 2b + \sqrt{4b^2+1}.$$

Setting $x = 2b$, all that we need to prove is that for $x > 0$ the following inequality holds

$$e^x > x + \sqrt{x^2+1}.$$

Note that this is stronger than the classical $e^x > x + 1$. For the proof, let $f(x) = e^x - x - \sqrt{x^2+1}$. We want to show that $f(x) > f(0)$ for $x > 0$. So, it is enough to prove the function $f(x)$ is strictly increasing on $(0, +\infty)$. This is clear because its first derivative satisfies

$$f'(x) = e^x - 1 - \frac{x}{\sqrt{x^2+1}} > x - \frac{x}{\sqrt{x^2+1}} > 0.$$

2. There are infinitely many fixed points for $\tan x$

Clearly $x = 0$ is a solution to the equation $\tan x = x$. Since $\tan(-x) = -\tan x = -x$, we may search for positive solutions only. Consider the following intervals

$$I_k = \left((2k-3)\frac{\pi}{2}, (2k-1)\frac{\pi}{2}\right) \quad \text{for} \quad k \geq 2.$$

We shall prove there is precisely one solution in each I_k. Denote $f(x) = \tan x - x$, and $g(x) = \sin x - x\cos x$. Clearly, the function $g(x)$ is continuous on I_k and satisfies

$$\begin{aligned} g\left((2k-3)\frac{\pi}{2}\right)\cdot g\left((2k-1)\frac{\pi}{2}\right) &= \sin\left((2k-3)\frac{\pi}{2}\right)\cdot \sin\left((2k-1)\frac{\pi}{2}\right) \\ &= (-1)^k \cdot (-1)^{k+1} \\ &= (-1)^{2k+1} \\ &= -1 < 0. \end{aligned}$$

So, by Bolzano's theorem, there is $x_k \in I_k$ such that $g(x_k) = 0$, and hence $f(x_k) = 0$. Now, let us prove there is only one fixed point x_k in each I_k. Suppose, by contradiction, $f(x_k) = f(x'_k) = 0$, for x_k and x'_k in I_k. The function $f(x)$ is derivable on I_k, its first derivative is $f'(x) = \frac{1}{\cos^2 x} - 1$, clearly non-negative, thus $f(x)$ is increasing. By Rolle's theorem, there is ξ_k with $x_k < \xi_k < x'_k$ and $f'(\xi_k) = 0$. Observe that the first derivative is zero only for integer multiples of π, therefore $\xi_k = m\pi$, for m a positive integer. But, $f(m\pi) = -m\pi < 0$, which is a contradiction because the function $f(x)$ is increasing.

3. Problem U223

Problem U223. *(Mathematical Reflections)* Let $(x_k)_{k\geq 1}$ be the positive solutions of the equation $\tan x = x$. Show that

$$\sum_{k=1}^{\infty} \frac{1}{x_k^2} = \frac{1}{10}.$$

Solution. Let us consider the entire complex function

$$f(z) = \sin z - z \cos z.$$

The zeroes of $f(z)$ are x_k, $-x_k$, and 0. These are all simple with the exception of 0 whose order is 3. By the Weierstrass factorization theorem we have

$$f(z) = \frac{z^3}{3} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{x_k^2}\right).$$

On the other hand, by expanding $f(z)$ at 0 we have that

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} - z \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \frac{z^3}{3} \sum_{n=0}^{\infty} \frac{6(n+1)}{(2n+3)!} (-1)^n z^{2n}.$$

By comparing these two expressions we find that

$$S_n := \sum_{k_1 < k_2 < \cdots < k_n} \frac{1}{x_{k_1}^2 x_{k_2}^2 \cdots x_{k_n}^2} = \frac{6(n+1)}{(2n+3)!}.$$

In particular, for $n = 1$ we obtain

$$S_1 = \sum_{k=1}^{\infty} \frac{1}{x_k^2} = \frac{6 \cdot 2}{5!} = \frac{1}{10}.$$

PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of November 2020**.

PROPOSED PROBLEMS

496. Calculate the integral:

$$\int_0^\infty \frac{\arctan x}{\sqrt{x^4+x^2+1}}\mathrm{d}x.$$

Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.

497. Let $n \geq 4$ and let $a_1, \ldots, a_n$ be nonzero real numbers such that $\frac{1}{a_1} + \cdots + \frac{1}{a_n} = 0$. Prove that

$$\left(\frac{1}{a_1^2} + \cdots + \frac{1}{a_n^2}\right) \sum_{1\leq i<j\leq n} (a_i - a_j)^2 \geq n^3.$$

When do we have equality?

Proposed by Leonard Giugiuc, Traian National College, Drobeta Turnu Severin, Romania.

498. Let $A, B \in M_n(\mathbb{C})$ be two matrices such that

$$A^2 - B^2 - I_n = \frac{1}{3}(AB - BA).$$

Prove that:

(i) $\det(A^2 - B^2) = \det(A - B)\det(A + B) = 1$.

(ii) $(AB - BA)^n = 0$.

Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

499. Let $a, b \geq 0$. Calculate

$$\lim_{n\to\infty} \sqrt{n} \int_0^{\frac{\pi}{2}} \sqrt{a \sin^{2n} x + b \cos^{2n} x}\,\mathrm{d}x.$$

Proposed by Ovidiu Furdui, Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

500. Let C be a simplex in $\mathbb{R}^n$ with the vertices $A_1, \dots, A_{n+1}$ and let M be a point in the interior of C. For every $1 \le i < j \le n+1$ we denote by $A_{i,j}$ the point where hyperplane generated by M and $A_1, \dots, \hat{A}_i, \dots, \hat{A}_j, \dots, A_n$ intersects the edge A_iA_j of C. We denote by D the convex hull of $\{A_{i,j} : 1 \le i < j \le n+1\}$.

Prove that $\operatorname{vol} D \le (1 - \frac{n+1}{2^n}) \operatorname{vol} C$, and the equality is reached if and only if M is the centroid of C.

Proposed by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Costel Bălcău, University of Piteşti, and Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.

501. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then $f(x+y) - f(x) \ge yf'(x)\ \forall x, y \in \mathbb{R}$ if and only if $n\,(f(x+1/n) - f(x)) \ge f'(x)\ \forall x \in \mathbb{R}$ and for every positive integer n.

Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

502. Let $m \ge 0$ be an integer. Evaluate the series

$$\sum_{k=1}^{\infty} \frac{(x^m \log x)^{(k+m)}}{k!}, \quad x > 1,$$

where $f^{(i)}$ is the derivative of order i of f.

Proposed by Mircea Ivan, Technical University of Cluj-Napoca, Romania.

503. The Poincaré half-space model of the non-Euclidean n-dimensional space is the upper half-space $\mathbb{H}_n = \{(x,y) \mid x \in \mathbb{R}^{n-1},\ y > 0\}$. We regard an element $x \in \mathbb{R}^{n-1}$ as a column vector, i.e., as an element of $\mathcal{M}_{n-1,1}(\mathbb{R})$.

Then the group of positively oriented isometries of $\mathbb{H}$ is made of the functions $f_{\alpha,A,a} : \mathbb{H} \to \mathbb{H}$, with $\alpha > 0$, $A \in O^+(n-1)$ and $a \in \mathbb{R}^{n-1}$, given by $(x,y) \mapsto \alpha(Ax + a, y)$, and the functions $g_{\alpha,A,r,a} : \mathbb{H} \to \mathbb{H}$, with $\alpha > 0$, $A \in O^-(n-1)$ and $r, a \in \mathbb{R}^{n-1}$, given by $(x,y) \mapsto \alpha\left(\frac{A(x-r)}{|x-r|^2+y^2} + b, \frac{y}{|x-r|^2+y^2}\right)$.

Give a direct proof of the fact that if G is the set of all $f_{\alpha,A,a}$ and all $g_{\alpha,A,r,a}$ then $(G, \circ)$ is a group.

Here $O(n-1)$ is the orthogonal group, $O(n-1) = \{A \in \mathcal{M}_{n-1}(\mathbb{R}) \mid A^TA = I_{n-1}\}$. We have $O(n-1) = O^+(n-1) \cup O^-(n-1)$, where $O^{\pm}(n-1) = \{A \in O(n-1) \mid \det A = \pm 1\}$.

If $x = (x_1, \dots, x_{n-1})^T \in \mathbb{R}^{n-1}$ then $|x|$ denotes its Euclidean length, $|x|^2 = x_1^2 + \cdots + x_{n-1}^2$.

Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.

SOLUTIONS

482. Let $x \in \mathbb{R}$. Prove the series

$$\sum_{n=0}^{\infty} 3^n \left(\sin x - x + \frac{1}{3!}x^3 - \cdots - \frac{\sin\frac{n\pi}{2}}{n!}x^n\right),$$

converges absolutely and calculate its sum.

`Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

Solution by the author. The answer is $\sin x \cos(2x)$. First we prove the series converges absolutely. We apply the Maclaurin formula of order n to the function $f(x) = \sin x$ and we have that

$$\sin x - x + \frac{1}{3!}x^3 - \cdots - \frac{\sin\frac{n\pi}{2}}{n!}x^n = R_n(x),$$

where

$$R_n(x) = \frac{\sin^{(n+1)}(\theta_x x)}{(n+1)!}x^{n+1} = \frac{\sin\left(\theta_x x + \frac{(n+1)\pi}{2}\right)}{(n+1)!}x^{n+1},$$

for some $\theta_x \in (0,1)$. It follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left| 3^n \left(\sin x - x + \frac{1}{3!}x^3 - \cdots - \frac{\sin\frac{n\pi}{2}}{n!}x^n\right) \right| \\
&= \sum_{n=0}^{\infty} \left| 3^n \frac{\sin\left(\theta_x x + \frac{(n+1)\pi}{2}\right)}{(n+1)!}x^{n+1} \right| \\
&\le \frac{1}{3}\sum_{n=0}^{\infty} \frac{(3|x|)^{n+1}}{(n+1)!} \\
&= \frac{1}{3}\left(\mathrm{e}^{3|x|} - 1\right),
\end{aligned}$$

which shows the series converges absolutely.

Now we calculate its sum. We have

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} 3^n \left(\sin x - x + \frac{1}{3!}x^3 - \cdots - \frac{\sin \frac{n\pi}{2}}{n!} x^n \right) \\
&= \sin x + \sum_{n=1}^{\infty} 3^n \left(\sin x - x + \frac{1}{3!}x^3 - \cdots - \frac{\sin \frac{n\pi}{2}}{n!} x^n \right) \\
&\overset{n-1=m}{=\!=\!=} \sin x + \sum_{m=0}^{\infty} 3^{m+1} \left(\sin x - x + \frac{1}{3!}x^3 - \cdots - \frac{\sin \frac{(m+1)\pi}{2}}{(m+1)!} x^{m+1} \right) \\
&= \sin x + 3 \sum_{m=0}^{\infty} 3^m \left(\sin x - x + \frac{1}{3!}x^3 - \cdots - \frac{\sin \frac{m\pi}{2}}{m!} x^m \right) \\
&\quad - \sum_{m=0}^{\infty} 3^{m+1} \frac{\sin \frac{(m+1)\pi}{2}}{(m+1)!} x^{m+1} \\
&= \sin x + 3f(x) - \sum_{m=0}^{\infty} \frac{\sin^{(m+1)}(0)}{(m+1)!} (3x)^{m+1} \\
&= \sin x + 3f(x) - \sin(3x),
\end{aligned}
$$

and it follows that $2f(x) = \sin(3x) - \sin x$. Concluding, $f(x) = \sin x \cos(2x)$ for all $x \in \mathbb{R}$.

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain. Since $\sin x - \sum_{k=0}^{n} \sin \frac{k\pi}{2} \frac{x^k}{k!} = \sum_{k=n+1}^{\infty} \sin \frac{k\pi}{2} \frac{x^k}{k!}$, our series writes as

$$
S = \sum_{n=0}^{\infty} 3^n \sum_{k=n+1}^{\infty} \sin \frac{k\pi}{2} \frac{x^k}{k!}.
$$

To solve the problem, we prove that the double series S is absolutely convergent and we calculate its sum.

Assuming that S converges absolutely, we have

$$\begin{aligned}S &= \sum_{k=1}^{\infty} \sin\frac{k\pi}{2}\frac{x^k}{k!}\sum_{n=0}^{k-1} 3^n \\ &= \sum_{k=1}^{\infty} \sin\frac{k\pi}{2}\frac{x^k}{k!}\left(\frac{3^k-1}{2}\right) \\ &= \frac{1}{2}\left(\sum_{k=1}^{\infty} \sin\frac{k\pi}{2}\frac{(3x)^k}{k!} - \sum_{k=1}^{\infty} \sin\frac{k\pi}{2}\frac{x^k}{k!}\right) \\ &= \frac{\sin(3x)-\sin(x)}{2}.\end{aligned}$$

To prove that S is absolutely convergent we note that $\left|\sin\frac{k\pi}{2}\frac{x^k}{k!}\right| \le \frac{|x|^k}{k!}$ so it is enough to show that $T<\infty$, where $T = \sum_{n=0}^{\infty} 3^n \sum_{k=n+1}^{\infty} \frac{|x|^k}{k!}$. By the same calculations as for S, but with the factor $\sin\frac{k\pi}{2}$ ignored and with x replaced by $|x|$, one gets

$$T = \frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{(3|x|)^k}{k!} - \sum_{k=1}^{\infty}\frac{|x|^k}{k!}\right) = \frac{1}{2}((e^{3|x|}-1)-(e^{|x|}-1)) = \frac{e^{3|x|}-e^{|x|}}{2} < \infty.$$

In conclusion, the proposed series is absolutely convergent and its sum is $\frac{\sin 3x - \sin x}{2}$.

Solution by Nicuşor Minculete, Braşov, Romania, and Daniel Văcaru, Piteşti, Romania. If we consider the Taylor expansion of $f(x) = \sin x$ we have

$$\begin{aligned}\sin x - x + \frac{1}{3!}x^3 - \cdots - \frac{\sin\frac{\pi}{2}}{n!}x^n &= f(x) - \sum_{k=0}^{n} f^{(k)}(0)\frac{x^k}{k!} \\ &= \int_0^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t.\end{aligned}$$

We will determine

$$S := \sum_{n\ge 0} a^n \int_0^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t.$$

For this problem we need the case $a=3$.

Since $|f^{(n+1)}(t)| \leq 1\ \forall t$, we have $\int_0^x |f^{(n+1)}(t)\frac{(x-t)^n}{n!}|\mathrm{d}t \leq \frac{|x|^{n+1}}{(n+1)!}$, so our series S is absolutely convergent. We have

$$S = \int_0^x g(t)\mathrm{d}t, \quad \text{where} \quad g(t) = \sum_{n\geq 0} f^{(n+1)}(t)\frac{a^n(x-t)^n}{n!}.$$

Let $y = ax - (a-1)t$, then $a^n(x-t)^n = (ax-at)^n = (y-t)^n$, so that

$$g(t) = \sum_{n\geq 0}(f')^{(n)}(t)\frac{(y-t)^n}{n!} = f'(y) = f'(ax-(a-1)t).$$

Therefore

$$S = \int_0^x f'(ax-(a-1)t)\mathrm{d}t = \frac{f(ax)-f(x)}{a-1}.$$

In our case, $f(x) = \sin x$ and $a = 3$, so $S = \frac{\sin 3x - \sin x}{2} = \sin x \cos 2x$.

Note of the Editor. Note that in their proof Nicuşor Minculete and Daniel Văcaru prove that the sum of the series writes as $\int_0^x f'(ax-(a-1)t)\mathrm{d}t$. Since $f(x) = \sin x$ and $a = 3$, this means the sum is $\int_0^x \cos(3x-2t)\mathrm{d}t$.

We received a solution from Ulrich Abel, from Technische Hochschule Mittelhessen, Germany, who arrives to the same formula, but in a different way. He too writes the remainder of the Taylor series in the integral form, but then he continues as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} 3^n\left(\sin x - \frac{x}{1!} + \frac{x^3}{3!} - \cdots - f^{(n)}(0)\frac{x^n}{n!}\right)\\
=\ &\sum_{n=0}^{\infty} 3^n \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t\\
=\ &\sum_{n=0}^{\infty} 3^{2n} \int_0^x \frac{(x-t)^{2n}}{(2n)!}(-1)^n\cos(t)\,\mathrm{d}t\\
&\quad + \sum_{n=0}^{\infty} 3^{2n+1}\int_0^x \frac{(x-t)^{2n+1}}{(2n+1)!}(-1)^{n+1}\sin(t)\,\mathrm{d}t\\
=\ &\int_0^x (\cos(3(x-t))\cos(t) - \sin(3(x-t))\sin(t))\,\mathrm{d}t\\
=\ &\int_0^x \cos(3x-2t)\,\mathrm{d}t = \frac{-1}{2}\sin(3x-2t)\Big|_{t=0}^{x} = \frac{\sin(3x)-\sin(x)}{2}.
\end{aligned}$$

483. Suppose that $0 < a_1 \leq a_2 \leq \dots$ and $\sum_{n=1}^{\infty} 1/a_n < \infty$. Let $A := \{n \in \mathbb{N}^* : a_n < n \log n\}$. Prove that A has logarithmic density 0, that is, $\lim_{x\to\infty} \frac{1}{\log x} \sum_{k\leq x, k\in A} \frac{1}{k} = 0$.

`Proposed by George Stoica, New Brunswick, Canada.`

Solution by the author. Let $B := \{m \in \mathbb{N}^* : (2^{m-1}, 2^m] \cap A \neq \emptyset\}$. Suppose $m \in B$. Then there exists an integer $n_m \in A$ with $2^{m-1} < n_m \leq 2^m$. Then for $n_m/2 < k \leq n_m$ we have $a_k \leq a_{n_m} < n_m \log n_m$, so

$$(*) \qquad \sum_{n_m/2<k\leq n_m} \frac{1}{a_k} \geq \frac{n_m - [n_m/2]}{n_m \log n_m} \geq \frac{1}{2\log n_m} \geq \frac{1}{2m\log 2}.$$

Note that if $m, m' \in B$ with $m' - m \geq 2$ then $2n_m \leq 2^{m+1} \leq 2^{m'-1} < n_{m'}$ so $(n_m/2, n_m] \cap (n_{m'}/2, n_{m'}] = \emptyset$. It follows that every $k \geq 1$ belongs to at most two intervals of the type $(n_m/2, n_m]$. Thus, when we sum the inequality (*) over $m \geq 1$, we get

$$2\sum_{k=1}^{\infty} \frac{1}{a_k} \geq \sum_{m\in B} \frac{1}{2m\log 2}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{a_k} < \infty$, this implies that $\sum_{m\in B} \frac{1}{m} < \infty$.

We also have

$$\sum_{2^{m-1}<k\leq 2^m,\, k\in A} \frac{1}{k} \leq \begin{cases} 2^{m-1} \cdot \frac{1}{2^{m-1}} = 1 & \text{if } m \in B, \\ 0 & \text{if } m \notin B. \end{cases}$$

If $y := (\log x)/(\log 2)$ then

$$\sum_{k\leq x, k\in A} \frac{1}{k} = \sum_{k\leq 2^y,\, k\in A} \frac{1}{k} \leq \sum_{m\leq y+1} \left(\sum_{k\in A,\, 2^{m-1}<k\leq 2^m} \frac{1}{k} \right)$$
$$\leq \frac{1}{y\log 2} \sum_{m\in B,\, m\leq y+1} 1 = \frac{1}{y\log 2} f(y+1),$$

where $f(y) := \sum_{m\in B,\, m\leq y} 1$.

We have

$$\frac{f(N) - f(N/2)}{N} \leq \sum_{m\in B,\, n/2<m\leq N} \frac{1}{m} \longrightarrow 0 \text{ as } N \longrightarrow \infty.$$

Thus $f(y)/y \longrightarrow 0$ as $y \longrightarrow \infty$, hence

$$\limsup_{x\to\infty} \frac{1}{\log x} \sum_{k\leq x, k\in A} \frac{1}{k} \leq \limsup_{y\to\infty} \frac{f(y+1)}{y} = 0.$$

As the $\liminf$ of the expression in the conclusion is always ≥ 0, the problem now follows.

484. Prove that for a continuous nonconstant function $f : \mathbb{R} \to \mathbb{R}$, the following conditions are equivalent:

(1) $f(x) - f(y) \in \mathbb{Q}$ for all $x, y \in \mathbb{R}$ such that $x - y \in \mathbb{Q}$;

(2) $f(x) - f(y) \in \mathbb{R} \setminus \mathbb{Q}$ for all $x, y \in \mathbb{R}$ such that $x - y \in \mathbb{R} \setminus \mathbb{Q}$;

(3) there exist $a \in \mathbb{Q}^*$ and $b \in \mathbb{R}$ such that $f(x) = ax + b$ for all $x \in \mathbb{R}$.

`Proposed by Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

Solution by the author. Assuming (3), then $f(x) - f(y) = a(x-y)$ for all $x, y \in \mathbb{R}$, and the implications (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are straightforward.

Conversely, assume either (1) or (2), and let $A = \mathbb{Q}$, or $A = \mathbb{R} \setminus \mathbb{Q}$ respectively.

Fix $y \in A$ and define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x+y) - f(x)$ for all $x \in \mathbb{R}$. Then $g(\mathbb{R}) \subseteq A$, and $g(\mathbb{R})$ is an interval (possibly degenerated) since g is continuous, hence g must be constant. It follows that $g(x) = g(0)$ for all $x \in \mathbb{R}$, hence

$$f(x+y) - f(x) = f(y) - f(0) \quad \text{for all } x \in \mathbb{R},\ y \in A. \tag{4}$$

Next, fix $x \in \mathbb{R}$ and let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(y) = f(x+y) - f(y)$ for all $y \in \mathbb{R}$. By (4), $h(y) = h(0)$ for all $y \in A$ and by using the continuity of h and the density of A in $\mathbb{R}$, it follows that $h(y) = h(0)$ for all $y \in \mathbb{R}$.

Thus one has

$$f(x+y) = f(x) + f(y) - f(0) \quad \text{for all } x, y \in \mathbb{R},$$

or, equivalently,

$$f(x+y) - f(0) = (f(x) - f(0)) + (f(y) - f(0)) \quad \text{for all } x, y \in \mathbb{R}.$$

Using the continuity and additivity of

$$F : \mathbb{R} \to \mathbb{R}, \quad F(x) = f(x) - f(0) \quad \text{for all } x \in \mathbb{R},$$

it follows that $F(x) = ax$ for all $x \in \mathbb{R}$, with $a \in \mathbb{R}$ constant, hence

$$f(x) = ax + b \quad \text{for all } x \in \mathbb{R}$$

where $b = f(0) \in \mathbb{R}$. Since f is nonconstant, it follows that $a \neq 0$.

Finally, under the assumption (1), we have that

$$a = f(1) - f(0) \in \mathbb{Q},$$

while, under the assumption (2), we have that

$$f\left(\frac{1}{a}\right) - f(0) = 1 \in \mathbb{Q},$$

hence $\dfrac{1}{a} \notin \mathbb{R} \setminus \mathbb{Q}$, leading to $a \in \mathbb{Q}$ and concluding the proof.

We received the same solution from Daniel Văcaru, Maria Teiuleanu Economic College, Piteşti, Romania.

485. Assume that ABC is a triangle with $a \geq b \geq c$, where the angle A has a fixed value. We denote by Σ the sum

$$\sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}}.$$

Then the only possible values of A are $\pi/3 \leq A < \pi$ and we have:

(i) The smallest possible value Σ is

$$\frac{4\sin\frac{A}{2}+\sqrt{2\left(1-\sin\frac{A}{2}\right)}}{\sqrt{2\sin\frac{A}{2}}}.$$

(ii) If $\pi/3 \leq A < \pi/2$ then the largest possible value of Σ is

$$\frac{4\cos A+\sqrt{2\left(1-\cos A\right)}}{\sqrt{2\cos A}}.$$

If $\pi/2 \leq A < \pi$ then there is no finite upper bound for Σ.

`Proposed by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania.`

Solution by the author. Since $a \geq b \geq c$ we have $A \geq B \geq C$. Since A is the largest angle of a triangle, we have $\pi/3 \leq A < \pi$.

Let $x = \frac{b+c-a}{a}$, $y = \frac{c+a-b}{b}$, $z = \frac{a+b-c}{c}$, so that $\Sigma = \sqrt{x}+\sqrt{y}+\sqrt{z}$. We have the well known relations:

$$x = \frac{2\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}, \quad y = \frac{2\sin\frac{C}{2}\sin\frac{A}{2}}{\sin\frac{B}{2}}, \quad z = \frac{2\sin\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{C}{2}}.$$

It follows that

$$yz = 4\sin^2\frac{A}{2},\ zx = 4\sin^2\frac{B}{2},\ xy = 4\sin^2\frac{C}{2},\ xyz = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Hence the identity $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1$ writes as $xy+yz+zx+xyz = 4$.

Since $\frac{\pi}{2} > \frac{A}{2} \geq \frac{B}{2} \geq \frac{C}{2} > 0$ we have $yz \geq zx \geq xy$, whence $x \leq y \leq z$.

Let $k = yz = 4\sin^2\frac{A}{2}$. Since A is fixed, so is k. We have $\pi/3 \leq A < \pi$, so that $4\sin^2\frac{\pi}{6} \leq k < 4\sin^2\frac{\pi}{2}$, i.e., $1 \leq k < 4$.

Let $y+z = 2s$. We also have $yz = k$ so, by the AM-GM inequality, $s \geq \sqrt{k}$.

We have $x(y+z+yz) = 4-yz$, i.e., $x(2s+k) = 4-k$, so $x = \frac{4-k}{2s+k}$. We also have $\sqrt{y}+\sqrt{z} = \sqrt{y+z+2\sqrt{yz}} = \sqrt{2s+2\sqrt{k}}$. Hence

$$\Sigma = \sqrt{2s+2\sqrt{k}}+\sqrt{\frac{4-k}{2s+k}} = f_k(s),$$

where $f_k : [\sqrt{k}, \infty) \to \mathbb{R}$, $f_k(t) = \sqrt{2t + 2\sqrt{k}} + \sqrt{\frac{4-k}{2t+k}}$.

We claim that f_k is strictly increasing. We have $f_k'(t) = \frac{1}{\sqrt{2t+2\sqrt{k}}} - \sqrt{\frac{4-k}{(2t+k)^3}}$. We must prove that $f_k'(t) > 0$, i.e., $(2t+k)^3 > 2(t+\sqrt{k})(4-k)$, $\forall t \ge \sqrt{k}$. Since $k \ge 1$ we have $(2t+k)^3 \ge (2t+1)^3$ and $3 \ge 4-k$. Hence it suffices to prove that $(2t+1)^3 > 6(t+\sqrt{k})$, i.e., $8t^3+12t^2+6t+1 > 6t+6t\sqrt{k}$. But this follows from $12t^2 > 12t > 12\sqrt{k} > 6\sqrt{k}$. (We have $t > \sqrt{k} \ge 1$.)

We are now ready to solve (i). Since f_k is increasing, the smallest value of $\Sigma = f_k(s)$ is obtained when we take s minimal, i.e., when $s = \sqrt{k} = 2\sin\frac{A}{2}$. Then $f_k(\sqrt{k}) = \sqrt{2\sqrt{k}+2\sqrt{k}} + \sqrt{\frac{4-k}{2\sqrt{k}+k}}$. But $\sqrt{2\sqrt{k}+2\sqrt{k}} = \sqrt{8\sin\frac{A}{2}}$ and

$$\sqrt{\frac{4-k}{2\sqrt{k}+k}} = \sqrt{\frac{4-k}{\sqrt{k}(2+\sqrt{k})}} = \sqrt{\frac{2-\sqrt{k}}{\sqrt{k}}} = \sqrt{\frac{2-2\sin\frac{A}{2}}{2\sin\frac{A}{2}}}.$$

In conclusion, the smallest possible value of Σ is

$$\sqrt{8\sin\frac{A}{2}} + \sqrt{\frac{2-2\sin\frac{A}{2}}{2\sin\frac{A}{2}}} = \frac{4\sin\frac{A}{2} + \sqrt{2\left(1-\sin\frac{A}{2}\right)}}{\sqrt{2\sin\frac{A}{2}}}.$$

To see when this minimal value of Σ is reached, recall that $k = yz$ and $2s = y+z$, so $s = \sqrt{k}$ happens precisely when $y = z = \sqrt{k}$. This is equivalent to $zx = xy$, i.e., $2\sin^2\frac{B}{2} = 2\sin^2\frac{C}{2}$, i.e., $B = C = \frac{\pi-A}{2}$. Note that $A \ge \pi/3$ implies $A \ge \frac{\pi-A}{2}$, so $A \ge B = C$. Hence the condition $a \ge b \ge c$ is fulfilled.

For (ii), in order to obtain large values of $\Sigma = f_k(s)$ we need large values of s. Therefore we must find the largest eligible value of s. Since $y+z = 2s$, $yz = k$ and $y \le z$, we have $y = s - \sqrt{s^2-k}$, $z = s + \sqrt{s^2-k}$. Since also $x = \frac{4-k}{2s+k}$, the condition that $x \le y$ writes as $\frac{4-k}{2s+k} \le s - \sqrt{s^2-k}$, which is equivalent to $(2s+k)(s-\sqrt{s^2-k}) \ge 4-k$, i.e., $g_k(s) \ge 4-k$, where $g_k : [\sqrt{k}, \infty) \to \mathbb{R}$, $g_k(t) = (2t+k)(t-\sqrt{t^2-k})$. Moreover, we have $x = y$ if and only if $g_k(s) = 4-k$.

We claim that g_k is strictly decreasing. We have $g_k'(t) = 2(t-\sqrt{t^2-k}) + (2t+k)(1-\frac{t}{\sqrt{t^2-k}}) = (t-\sqrt{t^2-k})(2-\frac{2t+k}{\sqrt{t^2-k}})$, which is < 0 for $t > \sqrt{k}$ since the first factor of the product is always positive and the second is always negative. Hence g_k is decreasing on its domain $[\sqrt{k}, \infty)$. Also note that $g_k(\sqrt{k}) = (2\sqrt{k}+k)\sqrt{k} = 2k + k\sqrt{k} \ge 3k \ge 4-k$, as $k \ge 1$. We also have $\lim_{t\to\infty} g_k(t) = \lim_{t\to\infty}(2t+k)\frac{k}{t+\sqrt{t^2-k}} = k$. Hence $\lim_{t\to\infty} g_k(t)$ is $\ge 4-k$ or $< 4-k$ when $k \ge 2$ or $k < 2$, respectively. As $k = 4\sin^2\frac{A}{2}$, the two cases correspond to $A \ge \pi/2$ and $A < \pi/2$. We consider the two cases separately.

a. $\pi/3 \le A < \pi/2$, i.e., $k < 2$. Then $g_k(\sqrt{k}) \ge 4-k > \lim_{t\to\infty} g_k(t)$. Since g_k is decreasing and continuous, there is a unique $t_0 \in [\sqrt{k}, \infty)$ such that $g_k(t_0) = 4 - k$. We have $g_k(t) \geq 4 - k$ if and only if $t \in [\sqrt{k}, t_0]$. Hence the largest value of $s \in [\sqrt{k}, \infty)$ with $g_k(s) \geq 4 - k$ is $s = t_0$, so the largest value of Σ is achieved for $s = t_0$.

If $s = t_0$ then $g_k(s) = 4 - k$, which is equivalent to $x = y$. Since $yz = k$, we have $z = \frac{k}{y} = \frac{k}{x}$. If we replace $y = x$ and $z = \frac{k}{x}$, the identity $xy + yz + zx + xyz = 4$ writes as $x^2 + k + k + kx = 4$, i.e., $x^2 + kx + 2k - 4 = 0$. But the roots of $X^2 + kX + 2k - 4 = 0$ are -2 and $2 - k$. Since $x > 0$, we have $x = 2 - k$. It follows that $y = 2 - k$ and $z = \frac{k}{2-k}$ and so $\Sigma = \sqrt{x} + \sqrt{y} + \sqrt{z} = 2\sqrt{2-k} + \sqrt{\frac{k}{2-k}}$. But $2 - k = 2 - 4\sin^2 \frac{A}{2} = 2\cos A$, so $k = 2(1 - \cos A)$. Thus the largest possible value of Σ is

$$2\sqrt{2-k} + \sqrt{\frac{k}{2-k}} = \frac{4\cos A + \sqrt{2(1-\cos A)}}{\sqrt{2\cos A}}.$$

In order to achieve this maximal value we need that $x = y$, which is equivalent to $yz = zx$, i.e., $4\sin^2 \frac{A}{2} = 4\sin^2 \frac{B}{2}$. So we have $A = B$ and $C = \pi - 2A$. Since $\pi/3 \leq A < \pi/2$ we have $A \geq \pi - 2A > 0$. Hence $A = B \geq C > 0$. Hence the condition $a \geq b \geq c$ is fulfilled.

b. $\pi/2 \leq A < \pi$, i.e., $k \geq 2$. Then $\lim_{t\to\infty} g_k(t) \geq 4 - k$. Since g_k is strictly decreasing, we have $g_k(s) > 4 - k$ $\forall s \in [\sqrt{k}, \infty)$ so there are no restrictions on s. Since obviously $\lim_{s\to\infty} f_k(s) = \infty$, the value of $\Sigma = f_k(s)$ can be arbitrarily large, i.e., there is no finite upper bound.

Alternatively, for every $M \geq 1$ we may consider the triangle ABC, where $c = 1$, $b = M$ and A is our given value, $\pi/2 \leq A < \pi$. Since $M \geq 1$, we have $b \geq c$ and, since $A \geq \pi/2$, we have $a > b = M$. Thus the condition that $a \geq b \geq c$ is fulfilled. Since $a > M$, $b = M$, $c = 1$ we have

$$\Sigma > \sqrt{\frac{a+b-c}{c}} > \sqrt{\frac{M+M-1}{1}} = \sqrt{2M-1}.$$

So if $M \to \infty$ then $\Sigma \to \infty$, so Σ can be arbitrarily large.

Solution by Marian Cucoaneş, Eremia Grigorescu Technical Highschool, Mărăşeşti, Vrancea, Romania. If we put $p = \frac{1}{2}(a + b + c)$ then we have the well known formula $\sin \frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}$ and similarly for $\sin \frac{B}{2}$ and $\sin \frac{C}{2}$. From these we get

$$\frac{\sin(A/2)\sin(B/2)}{\sin(C/2)} = \frac{p-c}{c} = \frac{a+b-c}{2c},$$

so

$$\sqrt{\frac{a+b-c}{c}} = \sqrt{\frac{2\sin(A/2)\sin(B/2)}{\sin(C/2)}}.$$

Together with the similar relations, this implies that

$$\begin{aligned}\frac{\Sigma}{\sqrt{2}} &= \sqrt{\frac{\sin(A/2)\sin(B/2)}{\sin(C/2)}} + \sqrt{\frac{\sin(A/2)\sin(C/2)}{\sin(B/2)}} + \sqrt{\frac{\sin(B/2)\sin(C/2)}{\sin(A/2)}} \\ &= \sqrt{\sin(A/2)}\left(\sqrt{\frac{\sin(B/2)}{\sin(C/2)}} + \sqrt{\frac{\sin(C/2)}{\sin(B/2)}}\right) + \frac{\sqrt{\sin(B/2)\sin(C/2)}}{\sqrt{\sin(A/2)}}.\end{aligned}$$

We denote $k = \sin\frac{A}{2}$ and $x = \left(\sqrt{\frac{\sin(B/2)}{\sin(C/2)}} + \sqrt{\frac{\sin(C/2)}{\sin(B/2)}}\right)^2$. From the AM-GM inequality we get $x \geq 4$, with equality iff $\sin(B/2) = \sin(C/2)$, i.e., iff $B = C$.

We have $a \geq b \geq c$, so $A \geq B \geq C$, which implies that $A \geq \frac{\pi}{3}$. From $\frac{\pi}{3} \leq A < \pi$ we get $\frac{\pi}{6} \leq \frac{A}{2} < \frac{\pi}{2}$, so $\frac{1}{2} \leq \sin\frac{A}{2} < 1$, i.e., $k \in [\frac{1}{2}, 1)$.

We have

$$\begin{aligned}x &= \frac{\sin(B/2)}{\sin(C/2)} + \frac{\sin(B/2)}{\sin(C/2)} + 2 = 2 + \frac{\sin^2(B/2) + \sin^2(C/2)}{\sin(B/2)\sin(C/2)} \\ &= 2 + \frac{2 - \cos B - \cos C}{2\sin(B/2)\sin(C/2)}.\end{aligned}$$

Since $\frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}$, we have $\cos\frac{B+C}{2} = \sin\frac{A}{2}$. We also have

$$\cos\frac{B-C}{2} = \cos\frac{B+C}{2} + 2\sin\frac{B}{2}\sin\frac{C}{2} = \sin\frac{A}{2} + 2\sin\frac{B}{2}\sin\frac{C}{2}.$$

Therefore

$$\cos B + \cos C = 2\cos\frac{B+C}{2}\cos\frac{B-C}{2} = 2\sin\frac{A}{2}\left(\sin\frac{A}{2} + 2\sin\frac{B}{2}\sin\frac{C}{2}\right).$$

Then the formula for x writes as

$$\begin{aligned}x &= 2 + \frac{2 - 2\sin(A/2)(\sin(A/2) + 2\sin(B/2)\sin(C/2)}{2\sin(B/2)\sin(C/2)} \\ &= 2 + \frac{1 - k^2 - 2k\sin(B/2)\sin(C/2)}{\sin(B/2)\sin(C/2)} \\ &= 2 - 2k + \frac{1 - k^2}{\sin(B/2)\sin(C/2)},\end{aligned}$$

so that

$$\sqrt{\sin(B/2)\sin(C/2)} = \sqrt{\frac{1-k^2}{x + 2k - 2}}.$$

It follows that $\frac{\Sigma}{\sqrt{2}} = f(x)$, where $f : [4, \infty)$ is defined by formula

$$f(x) = \sqrt{kx} + \frac{\sqrt{1-k^2}}{\sqrt{k}\sqrt{x + 2k - 2}}.$$

We have

$$f'(x) = \frac{\sqrt{k}}{2\sqrt{x}} - \frac{\sqrt{1-k^2}}{2\sqrt{k}} \cdot \frac{1}{\sqrt{(x+2k-2)^3}} = \frac{k\sqrt{(x+2k-2)^3} - \sqrt{(1-k^2)x}}{2\sqrt{kx}\sqrt{(x+2k-2)^3}}.$$

It follows that for every $x \in [4, \infty)$ $f'(x)$ has the same sign as $g(x)$, where $g : [4, \infty) \to \mathbb{R}$ is given by

$$g(x) = (k\sqrt{(x+2k-2)^3})^2 - (\sqrt{(1-k^2)x})^2 = k^2(x+2k-2)^3 - (1-k^2)x.$$

We have

$$\begin{aligned} g'(x) &= 3k^2(x+2k-2)^2 - (1-k^2) = k^2\left(3(x+2k-2)^2 - \frac{1}{k^2} + 1\right) \\ &\geq k^2\left(3\left(4 + 2\cdot\frac{1}{2} - 2\right)^2 - \frac{1}{(1/2)^2} + 1\right) = 24k^2 > 0. \end{aligned}$$

(because $x \geq 4$ and $k \geq \frac{1}{2}$.)

It follows that for every $x \in [4, \infty)$ it holds

$$g(x) \geq g(4) = k^2(2k+2)^3 - 4(1-k^2) = 4(k+1)(2k^2(k+1)^2 + k - 1) > 0.$$

(We have $k \geq \frac{1}{2}$, so $g(4) \geq 4\left(\frac{1}{2}+1\right)\left(2\cdot\left(\frac{1}{2}\right)^2\left(\frac{1}{2}+1\right)^2 + \frac{1}{2} - 1\right) = \frac{15}{4}$.)

Hence for every $x \in [4, \infty)$ we have $f(x) \geq f(4)$, whence

$$\Sigma \geq \sqrt{2}f(4) = \sqrt{2}\left(2\sqrt{k} + \frac{\sqrt{1-k^2}}{\sqrt{k}\sqrt{2k+2}}\right) = \frac{4k + \sqrt{2(1-k)}}{\sqrt{2k}}.$$

Since $k = \sin\frac{A}{2}$, this minimal value writes as $\Sigma = \frac{4\sin(A/2) + \sqrt{2 - 2\sin(A/2)}}{\sqrt{2\sin(A/2)}}$. The minimum is reached if $x = 4$, i.e., if $B = C = \frac{\pi - A}{2}$. Since $A \geq \frac{\pi}{3}$, we have $A \geq \frac{\pi - A}{2}$, so the condition $A \geq B \geq C$ is satisfied.

Suppose now that $\frac{\pi}{3} \leq A < \frac{\pi}{2}$ and $A \geq B \geq C$. In this case we have $k = \sin\frac{A}{2} \in [\frac{1}{2}, \frac{\sqrt{2}}{2})$.

Moreover, $B - C = B - (\pi - A - B) = A + 2B - \pi \leq 3A - \pi < \frac{\pi}{2}$, with equality iff $A = B$. Then $0 \leq \frac{B-C}{2} \leq \frac{3A-\pi}{2} < \frac{\pi}{4} < \pi$, which implies that $\cos\frac{B-C}{2} \geq \cos\frac{3A-\pi}{2} = \cos\frac{\pi - 3A}{2} = \sin\frac{3A}{2}$, with equality iff $B - C = 3A - \pi$, i.e., iff $A = B$. We also have $\cos\frac{B+C}{2} = \cos\left(\frac{\pi}{2} - \frac{A}{2}\right) = \sin\frac{A}{2}$. It follows that

$$\begin{aligned} \sin\frac{B}{2}\sin\frac{C}{2} &= \frac{1}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right) \geq \frac{1}{2}\left(\sin\frac{3A}{2} - \sin\frac{A}{2}\right) \\ &= \sin\frac{A}{2}\cos A = k(1-2k^2), \end{aligned}$$

with equality iff $A = B$. Then

$$\begin{aligned} x &= 2 - 2k + \frac{1-k^2}{\sin(B/2)\sin(C/2)} \leq 2 - 2k + \frac{1-k^2}{k(1-2k^2)} \\ &= \frac{(1-k)^2(1+2k)^2}{k(1-2k^2)} =: x_0. \end{aligned}$$

Since f is strictly increasing, we have $f(x) \leq f(x_0)$ with equality iff $x = x_0$, which is equivalent to $A = B$.

To compute $f(x_0)$, note that $x_0 + 2k - 2 = \frac{1-k^2}{k(1-2k^2)}$ and therefore

$$\begin{aligned} f(x_0) &= \sqrt{kx_0} + \frac{\sqrt{1-k^2}}{\sqrt{k(x_0+2k-2)}} = \frac{(1-k)(1+2k)}{\sqrt{1-2k^2}} + \frac{\sqrt{1-k^2}}{\sqrt{\frac{k(1-k^2)}{k(1-2k^2)}}} \\ &= \frac{1+k-2k^2}{\sqrt{1-2k^2}} + \sqrt{1-2k^2} = 2\sqrt{1-2k^2} + \frac{k}{\sqrt{1-2k^2}}. \end{aligned}$$

We have $\cos A = 1 - 2\sin^2\frac{A}{2} = 1 - 2k^2$ so $k = \sqrt{\frac{1-\cos A}{2}}$. Hence

$$\begin{aligned} \max\Sigma &= \sqrt{2}f(x_0) = \sqrt{2}\left(2\sqrt{\cos A} + \sqrt{\frac{1-cosA}{2\cos A}}\right) \\ &= \frac{4\cos A + \sqrt{2(1-\cos A)}}{\sqrt{2\cos A}} \end{aligned}$$

and the maximum is reached when $B = A$ and $C = \pi - 2A$. Since $A \geq \frac{\pi}{3}$, we have $\pi - 2A \leq A$, so $A \geq B \geq C$ is satisfied.

If $\frac{\pi}{2} \leq A < \pi$ then for every $0 < C \leq \frac{\pi-A}{2}$ we take $B = \pi - A - C$ and we get a triangle with $A \geq B \geq C$. We have $x = g(c) := \sqrt{\frac{\sin(B/2)}{\sin(C/2)} + \frac{\sin(C/2)}{\sin(B/2)}}$. Since $\lim_{C\to 0} g(c) = \infty$ and $\lim_{x\to\infty} f(x) = \infty$, we have that $\Sigma = \sqrt{2}f(x)$ has no finite upper bound.

Note from the Editor. We also received a solution for the part (i) of the problem from Yury Yucra Limachi, from Puno, Peru. He obtained the same formula for Σ, in terms of $\sin\frac{A}{2}$, $\sin\frac{B}{2}$ and $\sin\frac{C}{2}$, as in Marian Cucoaneş's proof, but then he considers its square, $F(A,B,C) = \Sigma^2$, which no longer has square roots. Then, after some lengthy calculations, he proves that $F(A,B,C) \geq F(A, \frac{B}{2}, \frac{C}{2})$, therefore proving that the minimum for $F(A,B,C)$, and so for Σ, is reached when $B = C = \frac{\pi-A}{2}$.

486. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$f(-x) = x + \int_0^x \sin t\, f(x-t)\,\mathrm{d}t, \quad \forall x \in \mathbb{R}.$$

`Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

Solution by the authors. We prove that $f(x) = -\frac{\sin(\sqrt{2}x)}{2\sqrt{2}} - \frac{x}{2}$.
Using the substitution $x - t = u$ the equation becomes

$$f(-x) = x + \int_0^x \sin(x-u)\, f(u)\, \mathrm{d}u. \tag{1}$$

Since the right hand side of the preceding equation is a differentiable function, f being continuous, we get that the left hand side of the equation is a differentiable function and hence f is differentiable.

First we observe that equation (1) can be written as

$$f(-x) = x + \sin x \int_0^x \cos u\, f(u)\, \mathrm{d}u - \cos x \int_0^x \sin u\, f(u)\, \mathrm{d}u. \tag{2}$$

Taking derivatives of both sides of equation (2) we get that

$$-f'(-x) = 1 + \cos x \int_0^x \cos u\, f(u)\, \mathrm{d}u + \sin x \int_0^x \sin u\, f(u)\, \mathrm{d}u, \quad \forall x \in \mathbb{R}. \tag{3}$$

Since the right hand side of equation (3) is a differentiable function we get that f' is also differentiable and it follows by differentiation both sides of the preceding equation that

$$f''(-x) = -\sin x \int_0^x \cos u\, f(u)\, \mathrm{d}u + \cos x \int_0^x \sin u\, f(u)\, \mathrm{d}u + f(x), \quad \forall x \in \mathbb{R}. \tag{4}$$

Adding equations (2) and (4) we get that

$$f''(-x) + f(-x) = x + f(x), \quad \forall x \in \mathbb{R}. \tag{5}$$

Replacing x by $-x$ in (5) we get that

$$f''(x) + f(x) = -x + f(-x), \quad \forall x \in \mathbb{R}. \tag{6}$$

It follows, adding (5) and (6), that $f''(x) + f''(-x) = 0$, $\forall x \in \mathbb{R}$. This implies that $f'(x) - f'(-x) = \mathcal{C}$, for some $\mathcal{C} \in \mathbb{R}$. Taking $x = 0$ one has that $\mathcal{C} = 0$ and it follows that $f'(x) - f'(-x) = 0$, $\forall x \in \mathbb{R}$. This equation implies that $f(x) + f(-x) = \mathcal{C}_1$, for some $\mathcal{C}_1 \in \mathbb{R}$. Since $f(0) = 0$, we get that $\mathcal{C}_1 = 0$ and we have that $f(x) + f(-x) = 0$, $\forall x \in \mathbb{R}$. Thus, $f(-x) = -f(x)$ and we obtain, based on equality (6), that $f''(x) + 2f(x) = -x$. The solution of this non homogeneous second order differential equation with constant coefficients is given by $f(x) = a\cos(\sqrt{2}x) + b\sin(\sqrt{2}x) - \frac{x}{2}$, $a, b \in \mathbb{R}$. Since $f(0) = 0$ and $f'(0) = -1$, we get that $a = 0$ and $b = -\frac{1}{2\sqrt{2}}$, and it follows that $f(x) = -\frac{\sin(\sqrt{2}x)}{2\sqrt{2}} - \frac{x}{2}$.

One can check that the function $f(x) = -\frac{\sin(\sqrt{2}x)}{2\sqrt{2}} - \frac{x}{2}$ verifies the integral equation in the statement of the problem. The problem is solved.

487. We consider the complex matrices A and B of dimensions $m \times n$ and $n \times m$, where $m > n \geq 2$. Let $C = AB$. If 0 is an eigenvalue of C of order $m - n$ and $C^{k+1} = \lambda C^k$ for some $k \geq 1$ and $\lambda \in \mathbb{C} \setminus \{0\}$, then determine BA.

Proposed by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania, and Costel Bălcău, University of Piteşti, Piteşti, Romania.

Solution by the authors. Let f and g be the characteristic polynomials of AB and BA, respectively. It is well known that $f(X) = X^{m-n}g(X)$. Since, by hypothesis, 0 is a root of order $m - n$ of f, 0 is not a root of g. Thus BA is invertible.

We have:

$$(BA)^{k+1} = B(AB)^k A = BC^k A \text{ and } (BA)^{k+2} = B(AB)^{k+1} A = BC^{k+1} A.$$

But $C^{k+1} = \lambda C^k$, so $(BA)^{k+2} = \lambda (BA)^{k+1}$. Since BA is invertible, this implies $BA = \lambda I_n$.

488. Let $0 \leq a < b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) = f(b)$. Then there exist $\alpha, \beta \in [a, b]$ such that $f(\alpha) = f(\beta)$ and $\alpha/\beta \notin \mathbb{Q}$.

Proposed by George Stoica, New Brunswick, Canada.

Solution by the author. We may assume that f is not constant on $[a, b]$. Thus there is some $c \in (a, b)$ such that $f(c) \neq f(a)$. We assume that $f(c) > f(a)$ (a similar argument applies if $f(c) < f(a)$), and define a new function $g : [f(a), f(c)] \to \mathbb{R}$ by

$$g(y) = \frac{\inf\{[a, c] \cap f^{-1}(y)\}}{\sup\{[c, b] \cap f^{-1}(y)\}}.$$

As y increases, the Intermediate Value Theorem implies that the numerator of the above function strictly increases, while the denominator strictly decreases. Therefore g is injective. We deduce that $g(y)$ is irrational for some $y \in [f(a), f(c)]$. Set $\alpha = \inf\{[a, c] \cap f^{-1}(y)\}$ and $\beta = \sup\{[c, b] \cap f^{-1}(y)\}$. Since $[a, c] \cap f^{-1}(y)$ is closed, we see that $\alpha \in [a, c] \cap f^{-1}(y)$, and thus $f(\alpha) = y$. Similarly, $f(\beta) = y$. As $\alpha/\beta \notin \mathbb{Q}$, the proof is complete.

Note from the Editor. We received essentially the same proof from Leonard Giugiuc, Traian National College, Drobeta Turnu Severin, Romania.

Solution by Filip Munteanu, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Cluj-Napoca, Romania. First we prove that there are $s, t \in (a, b)$, $s < t$, such that $f(s) = f(t)$. If f is a constant then we have nothing to prove. If f is not constant then, according to Weierstrass' Theorem, f has a minimum m and a maximum M and they are distinct. In particular, we have either $m \neq f(a)$ or $M \neq f(a)$. Assume that $m \neq f(a)$, i.e., that $m < f(a) = f(b)$. Let z be such that $f(z) = m$ and let's consider

$w \in (m, f(a))$, so that $f(z) = m < w < f(a) = f(b)$. We have $a < z < b$, $f(a) > w > f(z)$ and $f(z) < w < f(b)$. Since f is continuous, there exist $s \in (0, z)$ and $t \in (z, b)$ such that $f(s) = f(t) = w$. The case when $M \neq f(a)$ is similar.

If $s/t \notin \mathbb{Q}$, we can choose $\alpha = s$ and $\beta = t$. Otherwise there exists an integer $n > 1$ such that $r = \left(\frac{s}{t}\right)^{1/n} \notin \mathbb{Q}$. (We write $\frac{s}{t} > 1$ as $\frac{s}{t} = p_1^{r_1} \cdots p_k^{r_k}$ for pairwise distinct prime numbers p_i and nonzero integers r_i, and we take n to be any integer not dividing $\gcd\{r_1, \ldots, r_k\}$.) We have $r > 1$ and $t = sr^n$.

Let's define the function $g : \left[s, sr^{n-1}\right] \to \mathbb{R}$, $g(x) = f(xr) - f(x)$ and note that g is continuous. In addition, we have

$$g(s) + g(sr) + g(sr^2) + \cdots + g(sr^{n-1}) = f(sr^n) - f(s) = f(t) - f(s) = 0.$$

From here one may deduce that g has some zeroes (otherwise, since g is continuous, it would follow that either $g > 0$ or $g < 0$, which contradicts the relation from above).

Let c be a zero for g. We can choose $\alpha = c$ and $\beta = cr$.

Note from the Editor. This is a late solution to a problem published in the the 1–2/2018 issue of GMA and solved in the 1–2/2019 issue.

472. Let $a, b, c \in [0, \frac{\pi}{2}]$ such that $a+b+c = \pi$. Prove the following inequality:

$$\sin a + \sin b + \sin c \geq 2 + 4 \left| \sin\left(\frac{a-b}{2}\right) \sin\left(\frac{b-c}{2}\right) \sin\left(\frac{c-a}{2}\right) \right|.$$

Proposed by Leonard Giugiuc, Traian National College, Drobeta Turnu Severin, Romania and Jiahao He, South China University of Technology, People's Republic of China.

Solution by Marian Dincă. Rewrite the inequality to be proved in the equivalent form

$$\sin a + \sin b + \sin c \geq 2 + |\sin(a-b) + \sin(b-c) + \sin(c-a)|.$$

Recall, $a, b, c \in [0, \frac{\pi}{2}]$, with $a + b + c = \pi$.

By symmetry, we assume that $a \geq b \geq c$. Then

$$\begin{aligned} |\sin(a-b) + \sin(b-c) + \sin(c-a)| &= |\sin(a-b) + \sin(b-c) - \sin(a-c)| \\ &= \sin(a-b) + \sin(b-c) - \sin(a-c). \end{aligned}$$

Here we used the fact that the sine function is non-negative and subadditive on $[0, \pi/2]$, i.e., $\sin x + \sin y \geq \sin(x+y)$ if $x, y \geq 0$, with $x + y \leq \pi/2$.

Then our inequality writes as

$$\sin a + \sin b + \sin c \geq 2 + \sin(a-b) + \sin(b-c) - \sin(a-c)$$

or, equivalently,

$$\sin a + \sin b + \sin c + \sin(a-c) \geq \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin(a-b) + \sin(b-c)|.$$

We now use the following result, which is a variant [2, 3] of Karamata's inequality [1].

Let $f : I \to \mathbb{R}$ be a concave and increasing function, where $I \subseteq \mathbb{R}$ is an interval. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be two sequences in I such that $b_1 \leq \cdots \leq b_n$ and $a_1 + \cdots + a_k \leq b_1 + \cdots + b_k$ for $k = 1, \ldots, n$.

Then we have $f(a_1) + \cdots + f(a_n) \leq f(b_1) + \cdots + f(b_n)$.

Apply this result for $f : [0, \pi/2] \to \mathbb{R}$, $f(x) = \sin x$, which is increasing and concave, $n = 4$, $(a_1, a_2, a_3, a_4) = (\min(a-b, b-c), \max(a-b, b-c), \frac{\pi}{2}, \frac{\pi}{2})$, and $(b_1, b_2, b_3, b_4) = (\min(c, a-c), \max(c, a-c), b, a)$.

Since $0 \leq a, b, c \leq \frac{\pi}{2}$ and $a + b + c = \pi$, we have $a \leq \frac{\pi}{2} \leq b + c$. We get similar inequalities if we permute a, b, c.

We now verify that a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 satisfy the conditions of the theorem.

Obviously $\min(c, a-c) \leq \max(c, a-c)$. Since $b \geq c$ and $b + c \geq a$, so $b \geq a - c$, we have $b \geq \max(c, a-c)$. Also $a \geq b$. Hence $b_1 \leq b_2 \leq b_3 \leq b_4$.

We have $b + c \geq a$ and $a \geq b$ so $a - c \leq b$ and $b - c \leq a - c$. It follows that $\min(a-b, b-c) \leq \min(c, a-c)$, i.e., $a_1 \leq b_1$. Also $a_1 + a_2 = (a-b) + (b-c) = a - c$ and $b_1 + b_2 = c + a - c = a$. We have $a - c \leq a$, i.e.. $a_1 + a_2 \leq b_1 + b_2$. Also $a_1 + a_2 + a_3 = a - c + \frac{\pi}{2}$ and $b_1 + b_2 + b_3 = a + b$. Since $b + c \geq a$, we have $a - c + \frac{\pi}{2} \leq a + b$, i.e., $a_1 + a_2 + a_3 \leq b_1 + b_2 + b_3$. Finally $a_1 + a_2 + a_3 + a_4 = a - c + \frac{\pi}{2} + \frac{\pi}{2} = a - c + \pi$ and $b_1 + b_2 + b_3 + b_4 = a + b + a = 2a + b$. Since $a + b + c = \pi$, we have $a - c + \pi = 2a + b$, i.e., $a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$.

So all the conditions of the theorem are fulfilled and we have

$$f(a_1) + f(a_2) + f(a_3) + f(a_4) \leq f(b_1) + f(b_2) + f(b_3) + f(b_4).$$

But

$$f(a_1) + f(a_2) + f(a_3) + f(a_4) = \sin(a-b) + \sin(b-c) + \sin\frac{\pi}{2} + \sin\frac{\pi}{2}$$

and

$$f(b_1) + f(b_2) + f(b_3) + f(b_4) = \sin c + \sin(a-c) + \sin b + \sin a.$$

Thus we have the desired inequality.

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