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A special constant and series with zeta values and harmonic numbers

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Abstract. In this paper we demonstrate the importance of a mathematical constant which is the value of several interesting numerical series involving harmonic numbers, zeta values, and logarithms. We also evaluate in closed form a number of numerical and power series.

Keywords: Harmonic numbers, skew-harmonic numbers, Riemann zeta function, Hurwitz zeta function, digamma function.

MSC: 11B34, 11M06, 33B15, 40C15.

1. Introduction

The purpose of this paper is to demonstrate the importance of the constant

$$M = \int_0^1 \frac{\psi(t+1)+\gamma}{t}\,\mathrm{d}t \quad \approx 1.257746$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function and $\gamma = -\psi(1)$ is Euler's constant. Theorem 2 in Section 2 shows that M is the numerical value of many interesting and important series involving logarithms, harmonic numbers H_n and zeta values $\zeta(n)$. The harmonic numbers are defined by

$$H_n = \psi(n+1) + \gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \;\; n \geq 1, \quad H_0 = 0,$$

with generating function

$$-\frac{\ln(1-t)}{1-t} = \sum_{n=1}^{\infty} H_n\, t^n, \quad |t| \; < 1.$$

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We also use the skew-harmonic numbers given by

$$H_n^- = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{n}, \quad n \geq 1, \quad H_0^- = 0.$$

Riemann's zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \operatorname{Re} s > 1.$$

We consider series with terms $H_n(\zeta(n) - 1)$, $H_n(\zeta(n) - \zeta(n+1))$, $H_n^-(\zeta(n) - \zeta(n+1))$ and similar ones. Related power series whose coefficients are $H_n\,\zeta(n+1, a)$ and $H_n^-\,\zeta(n+1, a)$, where $\zeta(s, a)$ is the Hurwitz zeta function, are also studied. Some series are evaluated in closed form in terms of known constants and special functions. Other interesting series will be presented in integral form. The main results of this paper are Theorem 2 in Section 2, Theorem 9 in Section 3, and Theorem 12 in Section 4.

2. A mosaic of series

A classical theorem rooted in the works of Christian Goldbach (1690–1764) says that

$$\sum_{n=2}^{\infty} (\zeta(n) - 1) = 1$$

(see [1], [9], and [12, p. 142]; a short proof is given in the Appendix). The terms $\zeta(n) - 1$ of the above series decrease like the powers of $\frac{1}{2}$. The following lemma was proved in [7, p. 51]. It will be needed throughout the paper. (For convenience, a proof is given in the Appendix.)

Lemma 1. *For every $n \geq 2$ it holds*

$$\frac{1}{2^n} < \zeta(n) - 1 < \frac{1}{2^n}\frac{n+1}{n-1}.$$

The lemma shows that the power series with general term $(\zeta(n) - 1)x^n$ is convergent in the disk $|x| < 2$. The following representation is true (see, for instance, [12, p. 173, (139)])

$$\sum_{n=2}^{\infty} (\zeta(n) - 1)\, x^{n-1} = 1 - \gamma - \psi(2 - x). \tag{1}$$

Integration of this series yields (cf. [12, p. 173, (135)])

$$\sum_{n=2}^{\infty} \frac{\zeta(n) - 1}{n}\, x^n = (1 - \gamma)x + \ln \Gamma(2 - x) \tag{2}$$

and in particular, when $x = 1$, this turns into a classical result published by Euler in 1776 (see [11])

$$\sum_{n=2}^{\infty} \frac{\zeta(n) - 1}{n} = 1 - \gamma. \tag{3}$$

Putting $x = -1$ in (2), we find

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n) - 1}{n} = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} = \gamma - 1 + \ln \Gamma(3),$$

so that (as $\ln 2 = \ln \Gamma(3)$)

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} = \gamma, \tag{4}$$

which is another representation of γ often used by Euler.

Cases when the series can be evaluated in explicit closed form are intertwined with similar cases when this is not possible (unless we accept integral form). Series of the form

$$\sum_{n=2}^{\infty} \frac{\zeta(n) - 1}{n + p}, \qquad p = 1, 2,$$

have been studied and evaluated in a number of papers — see, for instance, [9], [12, pp. 213–219, (469), (474), (517), (518)] and also [13], [14]. At the same time the series

$$\sum_{n=2}^{\infty} \frac{\zeta(n+1) - 1}{n}, \qquad \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\zeta(n+1) - 1}{n}$$

resist evaluation in closed form in terms of recognized constants. However, they can be evaluated in integral form and we want to point out one simple integral which appears in several interesting cases as evidenced by Theorem 2 below. Namely, consider the constant

$$M = \int_0^1 \frac{\psi(t+1) + \gamma}{t} \, \mathrm{d}t.$$

Theorem 2. *With M as defined above, the following statements hold:*

(a) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(n+1) = M$;

(b) $\sum\limits_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n}\right) = M$;

(c) $\sum\limits_{n=1}^{\infty} \frac{\ln(n+1)}{n(n+1)} = M$;

(d) $\sum\limits_{n=1}^{\infty} H_n \left(\zeta(n+1) - 1\right) = M$;

(*e*) $\sum\limits_{n=2}^{\infty} H_n\,(\zeta(n)-1) = M + 1 - \gamma$;

(*f*) $\sum\limits_{n=1}^{\infty} \frac{1}{n}\,(n - \zeta(2) - \zeta(3) - \cdots - \zeta(n)) = M$
(the first term in this series is just 1*);*

(*g*) $\sum\limits_{n=1}^{\infty} H_n^-\,(\zeta(n+1) - \zeta(n+2)) = M - \ln 2$;

(*h*) $\displaystyle\int_0^{\infty} \frac{\mathrm{Ein}(x)}{e^x - 1}\,\mathrm{d}x = M$, *where* $\mathrm{Ein}(x)$ *is the modified exponential integral, an entire function defined by*

$$\mathrm{Ein}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n!n};$$

(*i*) $\displaystyle\int_0^1 \frac{(1-u)\ln(1-u)}{u\ln u}\,\mathrm{d}u = M$;

(*j*) $\sum\limits_{n=1}^{\infty} \frac{1}{n2^n} \left\{ \sum\limits_{k=1}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{k-1}\zeta(k+1) \right\} = M$.

Before proving the theorem we want to mention that the evaluations (a), (b), (c), and (i) are not new (for instance, they are listed on p. 142 in [3]). The series (c) appears in important applications in number theory — see [5] and [8]. The entire paper [4] is dedicated to that series.

Proof. Starting from the well-known Taylor expansion for $\psi(1+x)+\gamma$

$$\sum_{n=1}^{\infty} (-1)^{n-1}\zeta(n+1)x^n = \psi(1+x) + \gamma, \tag{5}$$

we divide both sides by x and integrate from 0 to 1 to prove (a). Then (a) implies (b):

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\zeta(n+1)}{n} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left\{ \sum_{k=1}^{\infty} \frac{1}{k^{n+1}} \right\} \\
&= \sum_{k=1}^{\infty} \frac{1}{k} \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{1}{k}\right)^n \right\} \\
&= \sum_{k=1}^{\infty} \frac{1}{k} \ln\left(1 + \frac{1}{k}\right).
\end{aligned}$$

Next we prove that (d) follows from (b). Starting with

$$\sum_{n=1}^{\infty} H_n\,(\zeta(n+1)-1) = \sum_{n=1}^{\infty} H_n \left\{ \sum_{k=2}^{\infty} \frac{1}{k^{n+1}} \right\} = \sum_{k=2}^{\infty} \frac{1}{k} \left\{ \sum_{n=1}^{\infty} H_n \left(\frac{1}{k}\right)^n \right\}$$

and using the generating functions for the harmonic numbers, we continue this way:

$$= \sum_{k=2}^{\infty} \frac{1}{k} \left\{ \frac{-1}{1-\frac{1}{k}} \ln\left(1-\frac{1}{k}\right) \right\} = \sum_{k=2}^{\infty} \frac{1}{k-1} \ln\left(\frac{k}{k-1}\right)$$
$$= \sum_{m=1}^{\infty} \frac{1}{m} \ln\left(\frac{m+1}{m}\right) = M.$$

Now starting from the first sum in the last equation we write

$$M = \sum_{k=2}^{\infty} \frac{1}{k} \left\{ \frac{-1}{1-\frac{1}{k}} \ln\left(1-\frac{1}{k}\right) \right\} = \sum_{k=2}^{\infty} \frac{1}{k-1} \left\{ -\ln\left(1-\frac{1}{k}\right) \right\}$$
$$= \sum_{k=2}^{\infty} \frac{1}{k-1} \left\{ \sum_{m=1}^{\infty} \frac{1}{m\,k^m} \right\} = \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \sum_{k=2}^{\infty} \frac{1}{k^n(k-1)} \right\}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n} \left(n - \zeta(2) - \zeta(3) - \cdots - \zeta(n)\right)$$

by using the evaluation (see [7, p. 51])

$$\sum_{k=2}^{\infty} \frac{1}{k^n(k-1)} = n - \zeta(2) - \zeta(3) - \cdots - \zeta(n).$$

Thus (f) is also proved. Equation (e) follows from (d) by writing

$$\sum_{n=2}^{\infty} H_n\left(\zeta(n)-1\right) = \sum_{n=2}^{\infty} \left(H_{n-1} + \frac{1}{n}\right)\left(\zeta(n)-1\right)$$
$$= \sum_{m=1}^{\infty} H_m\left(\zeta(m+1)-1\right) + \sum_{n=2}^{\infty} \frac{1}{n}\left(\zeta(n)-1\right)$$

and then applying (3).

Next we prove (h). From the representation

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\,\mathrm{d}x, \quad \operatorname{Re} s > 1,$$

we find with $s = n+1$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\zeta(n+1)}{n} = \int_0^{\infty} \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n!n} \right\} \frac{\mathrm{d}x}{e^x-1} = \int_0^{\infty} \frac{\operatorname{Ein}(x)}{e^x-1}\,\mathrm{d}x.$$

The exponential integral has the representation

$$\operatorname{Ein}(x) = \int_0^{x} \frac{1-e^{-t}}{t}\,\mathrm{d}t$$

and integrating by parts we find

$$\begin{aligned}\int_0^\infty \frac{\mathrm{Ein}(x)}{e^x-1}\,\mathrm{d}x &= \int_0^\infty \frac{\mathrm{Ein}(x)\,\mathrm{d}(1-e^{-x})}{1-e^{-x}}\\ &= \mathrm{Ein}(x)\ln(1-e^{-x})\big|_0^\infty - \int_0^\infty \mathrm{Ein}'(x)\ln(1-e^{-x})\,\mathrm{d}x\\ &= -\int_0^\infty \frac{1-e^{-x}}{x}\ln(1-e^{-x})\,\mathrm{d}x.\end{aligned}$$

Here $\mathrm{Ein}(x)\ln(1-e^{-x})\big|_0^\infty = 0$ since $\mathrm{Ein}(x)$ grows like $\ln x$ when $x\to\infty$ and also $\lim\limits_{x\to 0} x\ln(1-e^{-x})=0$. In the last integral we make the substitution $u=e^{-x}$ to find

$$-\int_0^\infty \frac{1-e^{-x}}{x}\ln(1-e^{-x})\,\mathrm{d}x = \int_0^1 \frac{(1-u)\ln(1-u)}{u\ln u}\,\mathrm{d}u$$

and (i) is proved.

We shall now prove the implication $(a)\Longrightarrow(g)$ by using Abel's lemma for transformation of series (see, for example, [10, Exercise 10, p. 78]):

Lemma 3. *Let $\{a_n\}$ and $\{b_n\}$, $n\ge p$, be two sequences of complex numbers and let $A_n=a_p+a_{p+1}+\cdots+a_n$. Then for every $n>p$ we have*

$$\sum_{k=p}^{n} a_k b_k = b_n A_n + \sum_{k=p}^{n-1} A_k(b_k-b_{k+1}).$$

We take here $p=1$, $a_k=\frac{(-1)^{k-1}}{k}$, and $b_k=\zeta(k+1)-1$. Then $A_k=H_k^-$ and we find

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\big(\zeta(k+1)-1\big) = H_n^-\big(\zeta(n+1)-1\big)+\sum_{k=1}^{n-1}H_k^-\big(\zeta(k+1)-\zeta(k+2)\big).$$

Setting $n\to\infty$ and using the estimate from Lemma 1 and also the fact that $|H_n^-|\le H_n$ and $H_n\sim\ln n$ at infinity, we come to the equation

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\big(\zeta(k+1)-1\big) = \sum_{k=1}^{\infty}H_k^-\big(\zeta(k+1)-\zeta(k+2)\big).$$

According to (a) we have

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\big(\zeta(k+1)-1\big) = M-\ln 2$$

and (g) follows.

The implication $(b)\Longrightarrow(c)$ also follows from Abel's lemma: we take $p=1$ and

$$a_k=\ln\left(1+\frac{1}{k}\right)=\ln(k+1)-\ln(k),\quad b_k=\frac{1}{k}.$$

Here $A_n = \ln(n+1)$ and Lemma 3 provides the equation

$$\sum_{k=1}^{n} \frac{1}{k} \ln\left(1+\frac{1}{k}\right) = \frac{\ln(n+1)}{n} + \sum_{k=1}^{n-1} \frac{\ln(k+1)}{k(k+1)}$$

which clearly shows the relation between (b) and (c).

Lastly we prove (j). The proof is based on Euler's series transformation [2]. Given a power series $f(x) = a_0 + a_1 x + \cdots$ we have for sufficiently small $|t|$

$$\frac{1}{1-t} f\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} t^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\}.$$

We take $f(x) = \psi(x+1) + \gamma$ with the expansion (5), where $a_0 = f(0) = 0$. Using the substitution $x = \frac{t}{1-t}$ we compute

$$\begin{aligned}
\int_0^1 \frac{\psi(x+1)+\gamma}{x}\,\mathrm{d}x &= \int_0^{1/2} \frac{1}{1-t} f\left(\frac{t}{1-t}\right) \frac{\mathrm{d}t}{t} \\
&= \int_0^{1/2} \sum_{n=1}^{\infty} t^n \left\{ \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \zeta(k+1) \right\} \frac{\mathrm{d}t}{t} \\
&= \sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \zeta(k+1)
\end{aligned}$$

after integrating term by term.

It is easy to see that for $n \geq 1$ it holds

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \zeta(k+1) = \int_0^{\infty} \frac{1 - L_n(x)}{e^x - 1}\,\mathrm{d}x,$$

where $L_n(x)$ are the Laguerre polynomials.

The proof of the theorem is complete. □

Remark 4. Applying Lemma 3 for $p = 1$, $a_k = \frac{1}{k}$, and $b_k = \zeta(k+1) - 1$ we find

$$\sum_{k=1}^{\infty} \frac{\zeta(k+1)-1}{k} = \sum_{k=1}^{\infty} H_n\big(\zeta(k+1) - \zeta(k+2)\big).$$

On the other hand, by (1) we have

$$\sum_{k=1}^{\infty} \frac{\zeta(k+1)-1}{k} = \int_0^1 \frac{1-\gamma-\psi(2-t)}{t}\,\mathrm{d}t.$$

Remark 5. This is a note on the series in (f). Let

$$S_n = \frac{1}{2^n} + \frac{1}{3^n 2} + \frac{1}{4^n 3} + \cdots = \sum_{k=2}^{\infty} \frac{1}{k^n(k-1)} = n - \zeta(2) - \zeta(3) - \cdots - \zeta(n).$$

Obviously,

$$\frac{1}{2^n} < S_n < \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots = \zeta(n) - 1 < \frac{1}{2^n} \cdot \frac{n+1}{n-1}.$$

That is, we have the same estimate as in Lemma 1

$$\frac{1}{2^n} < S_n < \frac{1}{2^n} \cdot \frac{n+1}{n-1}.$$

It is interesting to compare equation (g) from the above theorem with the following result.

Proposition 6. *We have*

$$\sum_{n=2}^{\infty} H_n^- \big(\zeta(n) - \zeta(n+1)\big) = \frac{\pi^2}{6} - \gamma - \ln 2. \tag{6}$$

Proof. The proof comes from Abel's lemma again, where we take $p = 2$, $a_k = \frac{(-1)^{k-1}}{k}$ and $b_k = \zeta(k) - 1$. Then $A_k = H_k^- - 1$ and the lemma yields for every $n \geq 3$

$$\sum_{k=2}^{n} \frac{(-1)^{k-1}}{k}\big(\zeta(k)-1\big) = (H_n^- - 1)\big(\zeta(n)-1\big) + \sum_{k=2}^{n-1}(H_k^- - 1)\big(\zeta(k)-\zeta(k+1)\big).$$

From here with $n \to \infty$ we find

$$\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k}\,\zeta(k) - \ln 2 + 1 = \sum_{k=2}^{\infty} H_k^- \big(\zeta(k)-\zeta(k+1)\big) - \sum_{k=2}^{\infty}\big(\zeta(k)-\zeta(k+1)\big).$$

The first sum equals $-\gamma$ and for the last sum we write

$$\sum_{k=2}^{\infty}\big(\zeta(k) - \zeta(k+1)\big) = \sum_{k=2}^{\infty}\big((\zeta(k) - 1) - (\zeta(k+1) - 1)\big),$$

which is a telescoping series equal to its first term $\zeta(2) - 1$. Thus

$$-\gamma - \ln 2 + 1 = \sum_{k=2}^{\infty} H_k^- \big(\zeta(k) - \zeta(k+1)\big) - \zeta(2) + 1$$

and the proof is finished. □

It was noted by the referee that another proof follows from the fact that the series in (6) has a telescoping property. This property leads to a convenient representation of its partial sum

$$\sum_{n=2}^{m} H_n^- \left(\zeta(n) - \zeta(n+1)\right) = \frac{\zeta(2)}{2} - H_{m+1}^-(\zeta(m+1) - 1) - H_{m+1}^- + \sum_{n=2}^{m} \frac{(-1)^n \zeta(n+1)}{n+1}.$$

Setting here $m \to \infty$ and using equation (4), the desired evaluation follows.

It is also interesting to compare parts (a) and (c) of Theorem 2 to the result in the following proposition.

Proposition 7. *For any $p > 1$ it holds*

$$\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^p} = -\zeta'(p) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\zeta(p+k)}{k}.$$

Proof. We have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\log(1+n)}{n^p} &= \sum_{n=1}^{\infty} \frac{1}{n^p} \log\left[n\left(1+\frac{1}{n}\right)\right] = \sum_{n=1}^{\infty} \frac{\log n}{n^p} + \sum_{n=1}^{\infty} \frac{1}{n^p} \log\left(1+\frac{1}{n}\right) \\ &= -\zeta'(p) + \sum_{n=1}^{\infty} \frac{1}{n^p} \left\{\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{1}{n}\right)^k\right\} \\ &= -\zeta'(p) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left\{\sum_{n=1}^{\infty} \frac{1}{n^{k+p}}\right\} \\ &= -\zeta'(p) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\zeta(k+p)}{k}.\end{aligned}$$

The change of order of summation is justified by the absolute convergence of the series. □

The next proposition parallels some of the results in Theorem 2.

Proposition 8. *Let*

$$M_1 = \int_1^2 \frac{\psi(1+t)+\gamma}{t}\, \mathrm{d}t \approx 0.86062.$$

Then the following statements hold:

(k) $\sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1+\frac{1}{n+1}\right) = M_1$;

(l) $\sum_{n=1}^{\infty} H_n^{-}\big(\zeta(n+1)-1\big) = M_1$;

(m) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\big(n-\zeta(2)-\zeta(3)-\cdots-\zeta(n)\big) = M_1$

(the first term in the sum is 1).

Proof. Here (k) follows immediately from the well-known representation

$$\frac{\psi(1+t)+\gamma}{t} = \sum_{n=1}^{\infty} \frac{1}{n(n+t)}$$

by integration from 1 to 2 (integration between 0 and 1 implies (b) in Theorem 2). Independently (l) follows from Theorem 12 in Section 4 (see the remark at the end of that section). Further details are left to the reader. □

3. Variations on a problem of Ovidiu Furdui

The identity (d) of Theorem 2 can be written in the form

$$\sum_{n=2}^{\infty} H_n\big(\zeta(n+1)-1\big) = M - \zeta(2) + 1$$

by starting the summation from $n=2$. Subtracting this from (e) we find

$$\sum_{n=2}^{\infty} H_n\big(\zeta(n)-\zeta(n+1)\big) = \frac{\pi^2}{6} - \gamma. \tag{7}$$

This result was displayed in Problem W10 from [6]. We shall extend (7) to power series involving the Hurwitz zeta function

$$\zeta(s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}, \quad \zeta(s,1) = \zeta(s),$$

where $\mathrm{Re}(s) > 1$ and $a > 0$.

Theorem 9. *For $a>0$ and $|x| < a$ we have*

$$\sum_{n=2}^{\infty} H_n\big(\zeta(n,a) - x\zeta(n+1,a)\big)x^n = \zeta(2,a)x^2 + \psi(a)x + \log\Gamma(a-x) - \log\Gamma(a). \tag{8}$$

Proof. The left hand side can be written this way

$$\begin{aligned}
\sum_{n=2}^{\infty} H_n\,\zeta(n,a)x^n &= \sum_{n=2}^{\infty}\left(H_{n-1} + \frac{1}{n}\right)\zeta(n,a)x^n \\
&= \sum_{n=2}^{\infty} H_{n-1}\,\zeta(n,a)x^n + \sum_{n=2}^{\infty}\frac{1}{n}\,\zeta(n,a)x^n \\
&= \sum_{n=1}^{\infty} H_n\,\zeta(n+1,a)x^{n+1} + \log\Gamma(a-x) - \log\Gamma(a) + \psi(a)x
\end{aligned}$$

by using the well-known series from [3, p. 78] or [12, p. 159]

$$\sum_{n=2}^{\infty}\frac{1}{n}\,\zeta(n,a)x^n = \log\Gamma(a-x) - \log\Gamma(a) + \psi(a)x \tag{9}$$

for $|x| < a$. Then we separate the first term in the series

$$\sum_{n=1}^{\infty} H_n\,\zeta(n+1,a)x^{n+1} = \sum_{n=2}^{\infty} H_n\,\zeta(n+1,a)x^{n+1} + \zeta(2,a)x^2$$

and from here

$$\sum_{n=2}^{\infty} H_n\,\zeta(n,a)x^n - \sum_{n=2}^{\infty} H_n\,\zeta(n+1,a)x^{n+1} = \zeta(2,a)x^2 + \psi(a)x + \log\Gamma(a-x) - \log\Gamma(a).$$

Thus (8) is proved.

The convergence in these series is justified by the estimate given in the Appendix

$$\frac{1}{(a+1)^n} + \frac{1}{a^n} < \zeta(n,a) \leq \frac{1}{a^n} + \frac{1}{(a+1)^n}\left(\frac{n+a}{n-1}\right)$$

($n \geq 2$) and by the slow growth of the harmonic numbers at infinity, more precisely, $H_n \sim \ln n$. □

Corollary 10. *For* $|x| < 1$ *it holds*

$$\sum_{n=2}^{\infty} H_n\big(\zeta(n) - x\zeta(n+1)\big)x^n = \zeta(2)x^2 - \gamma x + \log\Gamma(1-x), \tag{10}$$

and for $|x| < 2$

$$\sum_{n=2}^{\infty} H_n\big(\zeta(n) - x\zeta(n+1) + x - 1\big)x^n = \zeta(2)x^2 - x^2 + (1-\gamma)x + \log\Gamma(2-x). \tag{11}$$

Proof. Setting $a = 1$ in (8) we have $\psi(1) = -\gamma$ and (10) follows. With $a = 2$ in (8) we have $\zeta(n,2) = \zeta(n) - 1$, $\zeta(n+1,2) = \zeta(n+1) - 1$, and $\psi(2) = 1 - \gamma$. Thus (8) turns into (11). □

Note that by putting $x = 1$, (11) becomes (7). Setting $x = -1$ in (11) we also find

$$\sum_{n=2}^{\infty} (-1)^n H_n\big(\zeta(n) + \zeta(n+1) - 2\big) = \zeta(2) + \gamma - 2 + \log 2. \tag{12}$$

The series in (8) can be regularized, modified to extend the interval of convergence.

Corollary 11. *For every* $a > 0$ *and every* $|x| < a + 1$ *we have*

$$\sum_{n=2}^{\infty} H_n\left(\zeta(n,a) - x\zeta(n+1,a) - \frac{a-x}{a^{n+1}}\right)x^n \tag{13}$$

$$= \zeta(2,a)x^2 + \psi(a)x + \frac{(a-x)x}{a^2} + \log\Gamma(a+1-x) - \log\Gamma(a+1).$$

Proof. For $|x| < a$ we write using the generating function for the harmonic numbers

$$\begin{aligned}\log(a-x) &= (a-x)\frac{\log(a-x)}{a-x} = (a-x)\left\{\left(\frac{1}{a-x}\right)\left(\log a + \log\left(1-\frac{x}{a}\right)\right)\right\}\\ &= (a-x)\left\{\frac{\log a}{a-x} + \frac{1}{a}\left(1-\frac{x}{a}\right)^{-1}\log\left(1-\frac{x}{a}\right)\right\}\\ &= \log a - (a-x)\sum_{n=1}^{\infty}\frac{H_n\, x^n}{a^{n+1}}\\ &= \log a - (a-x)\frac{x}{a^2} - \sum_{n=2}^{\infty} H_n \frac{(a-x)}{a^{n+1}}\, x^n.\end{aligned}$$

This result can be put in the form

$$-\sum_{n=2}^{\infty} H_n \frac{(a-x)}{a^{n+1}}\, x^n = \log(a-x) - \log a + (a-x)\frac{x}{a^2}.$$

Now we add this equation to equation (8). On the left hand side we combine the two sums into one. On the right hand side we use the identities $\log(a-x) + \log\Gamma(a-x) = \log\Gamma(a+1-x)$ and $\log a + \log\Gamma(a) = \log\Gamma(a+1)$. This way from (8) we obtain (13).

The right side in (13) is obviously defined for $|x| < a+1$. To see that the series on the left side converges in this interval we write it in the form

$$\sum_{n=2}^{\infty} H_n\left(\zeta(n,a) - x\zeta(n+1,a) - \frac{a-x}{a^{n+1}}\right)x^n = \sum_{n=2}^{\infty} H_n\left(\zeta(n,a) - \frac{1}{a^n}\right)x^n$$
$$-\sum_{n=2}^{\infty} H_n\left(\zeta(n+1,a) - \frac{1}{a^{n+1}}\right)x^{n+1}.$$

Here both series converge for $|x| < a+1$ in view of the estimate in the Appendix

$$\frac{1}{(a+1)^m} < \zeta(m,a) - \frac{1}{a^m} \le \frac{1}{(a+1)^m}\left(\frac{m+a}{m-1}\right),$$

which is true for every integer $m \ge 2$. The proof is complete. □

4. Two interesting generating functions

The series in (10) is a power series, the difference of the two powers series with terms $H_n\zeta(n)x^n$ and $H_n\zeta(n+1)x^{n+1}$.

Although the difference of these two series can be evaluated in closed form, evaluating each one separately leads to difficult integrals. We shall show this in the following theorem. For comparison we also include the series with skew-harmonic numbers H_n^-.

Theorem 12. *For $a > 0$ and $|x| < a$ one has*

$$\sum_{n=1}^{\infty} H_n \, \zeta(n+1,a) x^n = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \zeta(k+1, a-x)}{k} \, x^k$$
$$= \int_0^x \frac{\psi(a-x+t) - \psi(a-x)}{t} \, \mathrm{d}t, \tag{14}$$

$$\sum_{n=1}^{\infty} H_n^- \, \zeta(n+1,a) x^n = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (2^k - 1) \zeta(k+1, a-x)}{k} \, x^k$$
$$= \int_0^x \frac{\psi(a-x+2t) - \psi(a-x+t)}{t} \, \mathrm{d}t. \tag{15}$$

The proof is based on the next lemma

Lemma 13. *Let $f(t) = a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots$ be a power series. Then the following representations hold:*

$$\sum_{n=1}^{\infty} H_n a_n x^n = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k! k} \, x^k f^{(k)}(x), \tag{16}$$

$$\sum_{n=1}^{\infty} H_n^- a_n x^n = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (2^k - 1)}{k! k} \, x^k f^{(k)}(x). \tag{17}$$

Proof. The harmonic numbers have the well-known integral representation

$$H_n = \int_0^1 \frac{t^n - 1}{t - 1} \, \mathrm{d}t.$$

Multiplying both sides here by $a_n x^n$ and summing for $n = 1, 2, \ldots$, we find for x small enough

$$\sum_{n=1}^{\infty} H_n a_n x^n = \int_0^1 \frac{f(xt) - f(x)}{t - 1} \, \mathrm{d}t$$

(cf. [10, Exercise 20, p. 79]). Now Taylor's formula applied to the function $g(t) = f(xt)$ and centered at $t = 1$ gives

$$f(xt) = f(x) + \sum_{k=1}^{\infty} \frac{x^k f^{(k)}(x)}{k!} (t-1)^k.$$

Substituting this into the integral and integrating term by term we arrive at (16).

The proof of (17) is done the same way by using the representations

$$\sum_{n=1}^{\infty} H_n^- a_n x^n = \int_0^1 \frac{f(x) - f(-xt)}{t + 1} \, \mathrm{d}t$$

and

$$f(-xt) = f(x) + \sum_{k=1}^{\infty} \frac{(-1)^k x^k f^{(k)}(x)}{k!} (t+1)^k.$$

□

Proof of Theorem 12. We apply the previous lemma to the function (see [12, p. 159])

$$f(t) = \sum_{n=1}^{\infty} \zeta(n+1, a)\, t^n = -\psi(a-t) + \psi(a), \quad |t| < |a|. \tag{18}$$

Here

$$f^{(k)}(x) = (-1)^{k+1} \psi^{(k)}(a-x) = k!\zeta(k+1, a-x)$$

and from Lemma 13 we get

$$\sum_{n=1}^{\infty} H_n\, \zeta(n+1, a) x^n = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \zeta(k+1, a-x)}{k}\, x^k. \tag{19}$$

Now we write (18) in the form

$$\sum_{k=1}^{\infty} (-1)^{k-1} \zeta(k+1, a)\, t^k = \psi(a+t) - \psi(a), \quad |t| < |a|.$$

Dividing by t and integrating between 0 and x we find

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \zeta(k+1, a)}{k}\, x^k = \int_0^x \frac{\psi(a+t) - \psi(a)}{t}\, \mathrm{d}t.$$

Replacing here a by $a - x$ gives

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \zeta(k+1, a-x)}{k}\, x^k = \int_0^x \frac{\psi(a-x+t) - \psi(a-x)}{t}\, \mathrm{d}t. \tag{20}$$

Now (19) and (20) lead to (14).

The proof of (15) is left to the reader. □

Remark 14. By setting $a = 2$ and $x = 1$ in (14) we obtain

$$\sum_{n=1}^{\infty} H_n \big(\zeta(n+1) - 1\big) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \zeta(k+1)}{k} = \int_0^1 \frac{\psi(1+t) + \gamma}{t}\, \mathrm{d}t$$

that is, a new proof of equations (a) and (d) in Theorem 2, as $\zeta(n+1, 2) = \zeta(n+1) - 1$.

With $a = 2,\ x = 1$ in (15) we find

$$\sum_{n=1}^{\infty} H_n^- \big(\zeta(n+1) - 1\big) = \int_0^1 \frac{\psi(1+2t) - \psi(1+t)}{t}\,\mathrm{d}t = \int_1^2 \frac{\psi(1+t)+\gamma}{t}\,\mathrm{d}t, \tag{21}$$

which is (l) in Proposition 8. To show the equality of the above two integrals we write

$$\int_0^1 \frac{\psi(1+2t) - \psi(1+t)}{t}\,\mathrm{d}t = \int_0^1 \frac{\psi(1+2t)+\gamma}{t}\,\mathrm{d}t - \int_0^1 \frac{\psi(1+t)+\gamma}{t}\,\mathrm{d}t$$

and then in the first integral on the right hand side we make the substitution $u = 2t$.

5. Appendix

Proof of Goldbach's theorem. The theorem has a simple two-line proof:

$$\sum_{n=2}^{\infty}\big(\zeta(n) - 1\big) = \sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{k^n} = \sum_{k=2}^{\infty}\sum_{n=2}^{\infty}\left(\frac{1}{k}\right)^n = \sum_{k=2}^{\infty}\frac{1}{k^2}\cdot\frac{1}{1-\frac{1}{k}}$$
$$= \sum_{k=2}^{\infty}\frac{1}{k(k-1)} = \sum_{k=2}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k}\right) = 1,$$

where the last sum is a well-known telescoping series. □

Proof of Lemma 1 (extended version). Let $a > 0,\ s > 1$. Then

$$\frac{1}{(a+1)^s} < \zeta(s,a) - \frac{1}{a^s} \le \frac{1}{(a+1)^s}\left(\frac{s+a}{s-1}\right).$$

Furdui's proof from [7, p.51] is modified here for the Hurwitz zeta function. The left hand side inequality is obvious and for the right hand side inequality we write using the remainder estimate from the integral test for series

$$\sum_{k=2}^{\infty}\frac{1}{(k+a)^s} \le \int_1^{\infty}\frac{1}{(t+a)^s}\,\mathrm{d}t = \frac{1}{(s-1)(a+1)^{s-1}}.$$

This way

$$\zeta(s,a) - \frac{1}{a^s} = \frac{1}{(a+1)^s} + \sum_{k=2}^{\infty}\frac{1}{(k+a)^s} \le \frac{1}{(a+1)^s} + \frac{1}{(s-1)(a+1)^{s-1}}$$
$$= \frac{1}{(a+1)^s}\left(\frac{s+a}{s-1}\right).$$

□

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A note on an exam problem

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Abstract. This note is about generalizing an exam problem given at Technical University of Cluj-Napoca on 16 July 2018.

Keywords: Limits of integrals, fractional part function, continuous function.

MSC: 26A06, 26A09.

1. Introduction and the main results

Problem 4 on the entrance examination test, which was given at Technical University of Cluj-Napoca on 16 July 2018, is about calculating the

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following limit

$$\lim_{n\to\infty} n\int_0^{\pi} \{x\}^n \,\mathrm{d}x,$$

where $\{x\}$ denotes the fractional part of the real number x.

In this note we generalize this problem. Our main result is contained in Theorem 1. In what follows, we denote by $\lfloor x \rfloor$ the floor of x, also known as the integer part of x.

Theorem 1. ***An exam problem and its generalizations.***

(i) Let $0 \le a < b$. *Then*

$$\lim_{n\to\infty} n\int_a^{b} \{x\}^n \,\mathrm{d}x = \lfloor b \rfloor - \lfloor a \rfloor .$$

(ii) Let $0 \le a < b$ *and let* $f : [0,1] \to \mathbb{R}$ *be a Riemann integrable function which is continuous at* 1. *Then*

$$\lim_{n\to\infty} n\int_a^{b} \{x\}^n f(\{x\}) \,\mathrm{d}x = (\lfloor b \rfloor - \lfloor a \rfloor) f(1).$$

(iii) Let $0 \le a < b$ *and let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function. Then*

$$\begin{aligned}\lim_{n\to\infty} n\int_a^{b} \{x\}^n f(x) \,\mathrm{d}x &= f(1) + f(2) + \cdots + f(\lfloor b \rfloor + 1)\\ &\quad - (f(1) + f(2) + \cdots + f(\lfloor a \rfloor + 1)) .\end{aligned}$$

Proof. (i) We show that, if $\alpha > 0$ then

$$\lim_{n\to\infty} n\int_0^{\alpha} \{x\}^n \,\mathrm{d}x = \lfloor \alpha \rfloor ,$$

from which the first part of the theorem follows.

Let $k = \lfloor \alpha \rfloor$. We have

$$\begin{aligned}\int_0^{\alpha} \{x\}^n \,\mathrm{d}x &= \sum_{i=0}^{k-1} \int_i^{i+1} \{x\}^n \,\mathrm{d}x + \int_k^{\alpha} \{x\}^n \,\mathrm{d}x\\ &= \sum_{i=0}^{k-1} \int_i^{i+1} (x-i)^n \mathrm{d}x + \int_k^{\alpha} (x-k)^n \mathrm{d}x\\ &= \sum_{i=0}^{k-1} \int_0^{1} t^n \mathrm{d}t + \left.\frac{(x-k)^{n+1}}{n+1}\right|_k^{\alpha}\\ &= \frac{k}{n+1} + \frac{\{\alpha\}^{n+1}}{n+1},\end{aligned}$$

and it follows that

$$\lim_{n\to\infty} n\int_0^{\alpha} \{x\}^n \,\mathrm{d}x = k \lim_{n\to\infty} \frac{n}{n+1} + \lim_{n\to\infty} \frac{n}{n+1} \{\alpha\}^{n+1} = k.$$

(ii) It suffices to show that, if $\alpha > 0$ and $f : [0,1] \to \mathbb{R}$ is a Riemann integrable function which is continuous at 1, then

$$\lim_{n\to\infty} n \int_0^\alpha \{x\}^n f(\{x\})\,\mathrm{d}x = \lfloor \alpha \rfloor f(1).$$

Let $k = \lfloor \alpha \rfloor$. We have

$$\begin{aligned}\int_0^\alpha \{x\}^n f(\{x\})\,\mathrm{d}x &= \sum_{i=0}^{k-1} \int_i^{i+1} (x-i)^n f(x-i)\mathrm{d}x + \int_k^\alpha (x-k)^n f(x-k)\mathrm{d}x \\ &= \sum_{i=0}^{k-1} \int_0^1 y^n f(y)\mathrm{d}y + \int_0^{\{\alpha\}} y^n f(y)\mathrm{d}y \\ &= k \int_0^1 y^n f(y)\mathrm{d}y + \int_0^{\{\alpha\}} y^n f(y)\mathrm{d}y.\end{aligned}$$

This implies that

$$n \int_0^\alpha \{x\}^n f(\{x\})\,\mathrm{d}x = kn \int_0^1 y^n f(y)\mathrm{d}y + n \int_0^{\{\alpha\}} y^n f(y)\mathrm{d}y. \tag{1}$$

Since f is Riemann integrable we get that f is bounded by $M \geq 0$ and we obtain that

$$\left| n \int_0^{\{\alpha\}} y^n f(y)\mathrm{d}y \right| \leq Mn \int_0^{\{\alpha\}} y^n \mathrm{d}y = M\frac{n}{n+1}\{\alpha\}^{n+1}$$

and it follows that

$$\lim_{n\to\infty} n \int_0^{\{\alpha\}} y^n f(y)\mathrm{d}y = 0. \tag{2}$$

On the other hand, it is an easy exercise to show that if $f : [0,1] \to \mathbb{R}$ is Riemann integrable and continuous at 1, then

$$\lim_{n\to\infty} n \int_0^1 y^n f(y)\mathrm{d}y = f(1). \tag{3}$$

Combining (1), (2) and (3), part *(ii)* of the theorem is proved.

(iii) We prove that, if $\alpha > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then

$$\lim_{n\to\infty} n \int_0^\alpha \{x\}^n f(x)\,\mathrm{d}x = f(1) + f(2) + \cdots + f(\lfloor \alpha \rfloor + 1).$$

Let $k = \lfloor \alpha \rfloor$. We have

$$\begin{aligned}\int_0^\alpha \{x\}^n f(x)\,\mathrm{d}x &= \sum_{i=0}^{k-1} \int_i^{i+1} (x-i)^n f(x)\mathrm{d}x + \int_k^\alpha (x-k)^n f(x)\mathrm{d}x \\ &= \sum_{i=0}^{k-1} \int_0^1 y^n f(y+i)\mathrm{d}y + \int_0^{\{\alpha\}} y^n f(k+y)\mathrm{d}y \\ &= \int_0^1 y^n \left(f(y) + f(y+1) + \cdots + f(y+k)\right)\mathrm{d}y \\ &\quad + \int_0^{\{\alpha\}} y^n f(k+y)\mathrm{d}y.\end{aligned}$$

It follows that

$$\begin{aligned}\lim_{n\to\infty} n \int_0^\alpha \{x\}^n f(x)\,\mathrm{d}x &= n \int_0^1 y^n \left(f(y) + f(y+1) + \cdots + f(y+k)\right)\mathrm{d}y \\ &\quad + n \int_0^{\{\alpha\}} y^n f(k+y)\mathrm{d}y.\end{aligned}$$

Exactly as in the proof of part *(ii)* of the theorem one has, based on the continuity of f, that

$$\lim_{n\to\infty} n \int_0^{\{\alpha\}} y^n f(k+y)\mathrm{d}y = 0.$$

Let $g(x) = f(x) + f(x+1) + \cdots + f(x+k)$. We have, based on formula (3), that

$$\begin{aligned}\lim_{n\to\infty} n \int_0^1 y^n \left(f(y) + f(y+1) + \cdots + f(y+k)\right)\mathrm{d}y &= \lim_{n\to\infty} n \int_0^1 y^n g(y)\mathrm{d}y \\ &= g(1) = \sum_{j=1}^{k+1} f(j),\end{aligned}$$

and Theorem 1 is proved. □

Corollary 2. *(i) The following equality holds*

$$\lim_{n\to\infty} n \int_0^\pi \{x\}^n\,\mathrm{d}x = 3.$$

(ii) Let $0 \le a < b$. The following equality holds

$$\lim_{n\to\infty} n \left(n \int_a^b \{x\}^n\,\mathrm{d}x - \lfloor b \rfloor + \lfloor a \rfloor \right) = \lfloor a \rfloor - \lfloor b \rfloor.$$

Proof. *(i)* This follows from part *(i)* of Theorem 1 with $a = 0$ and $b = \pi$.
(ii) It suffices to show that if $\alpha > 0$, then

$$\lim_{n\to\infty} n\left(n\int_0^\alpha \{x\}^n \,\mathrm{d}x - \lfloor\alpha\rfloor\right) = -\lfloor\alpha\rfloor .$$

We have, based on the proof of part *(i)* of Theorem 1, that

$$n\left(n\int_0^\alpha \{x\}^n \,\mathrm{d}x - \lfloor\alpha\rfloor\right) = -\frac{n}{n+1}\lfloor\alpha\rfloor + \frac{n^2}{n+1}\{\alpha\}^{n+1},$$

and the result follows. □

We mention that nonstandard problems involving limits of integrals as well as exercises about the fractional part function can be found in [1] and [2].

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An extension of a convergence criterion

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Abstract. We provide an extension of a convergence criterion for integrals and series involving functions of dominated variation, which is more appropriate for certain applications.

Keywords: Convergence criterion, function of dominated variance.
MSC: 40A05, 40A10, 28A25.

1. Introduction and the main results

Let $f : X \to \overline{\mathbb{R}}_+$ be a measurable function on the measure space $(X, \mathcal{A}, \mu)$, where μ is a finite measure on the σ-algebra $\mathcal{A}$ of the set X. The following exercise in measure theory and integration (see, e.g., [3, Prop. 4.2.9, p. 46] or [1, Ex. 8.6, p. 67 and Ex. 15.3, p. 141]) states the following equivalence:

$$\int_X f(x)\,\mathrm{d}\mu(x) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} \mu\{x \in X : f(x) \geq n\}) < \infty, \tag{1}$$

thus reducing the convergence of a Lebesgue integral to that of a numerical series, and vice-versa.

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The purpose of this note is to provide an extension of formula (1), better suited for certain applications. Specifically, we are seeking classes of functions g, defined on both the range of f and the set of non-negative integers $\mathbb{N}$, such that

$$\int_X f(x)\, g(f(x))\, \mathrm{d}\mu(x) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} g(n)\, \mu\{x \in X : f(x) \geq n\}) < \infty, \tag{2}$$

where $g(f(x))$ is the composite function of g and f.

We discovered that successful candidates g are given by *non-negative, non-decreasing functions of dominated variance* on $\mathbb{N}$; the latter means:

$$g(2n) \leq c\, g(n) \text{ for some constant } c > 0 \text{ and all } n \in \mathbb{N}$$

(cf., e.g., Section 2.1 in [2]). Such functions have the following useful properties:

(i) $(n+1)g(n+1) \leq 2c\, ng(n)$

and

(ii) $\dfrac{1}{3c}\, ng(n) \leq \displaystyle\sum_{j=1}^{n} g(j) \leq ng(n)$

for all $n \geq 1$.

To prove (i), use that $g(n+1) \leq g(2n) \leq c\, g(n)$, hence $(n+1)g(n+1) \leq c\,(n+1)g(n) \leq 2c\, ng(n)$.

The right hand side of (ii) is obvious because g is non-decreasing. As for the left hand side, note that it is true for $n = 1$. Moreover, denoting by $[n]$ the integer part of n, we have that $[n/2] \geq n/3$ for $n \geq 2$ and $n-1 < [n] \leq n$, hence

$$\begin{aligned} \sum_{j=1}^{n} g(j) &\geq \sum_{j=\left[\frac{n}{2}\right]+1}^{n} g(j) \geq \left(n - \left[\frac{n}{2}\right]\right) g\left(\left[\frac{n}{2}\right] + 1\right) \geq \left[\frac{n}{2}\right] g\left(\left[\frac{n}{2}\right] + 1\right) \\ &\geq \frac{1}{3c}\left[\frac{n}{2}\right] g\left(2\left[\frac{n}{2}\right] + 2\right) \geq \frac{1}{3c}\left[\frac{n}{2}\right] g(n) \geq \frac{1}{3c}\, ng(n). \end{aligned}$$

Our result is the following.

Theorem 1. *Let $(X, \mathcal{A}, \mu)$ be as above, $f : X \to [1, \infty]$ a measurable function, and g a non-negative, non-decreasing function of dominated variance, defined on $\mathbb{N}$ and on the range of f. Then formula* (2) *holds.*

Proof. We shall use throughout the equation

$$\int_X f(x)\, g(f(x))\, \mathrm{d}\mu(x) = \sum_{n=0}^{\infty} \int_{X_n} f(x)\, g(f(x))\, \mathrm{d}\mu(x),$$

where $X_n := \{x \in X : n \leq f(x) < n+1\}$ are the levels sets of f. As $f(x) \in [n, n+1)$ for all $x \in X_n$ and using that g is non-decreasing, we obtain

$$ng(n)\mu(X_n) \leq \int_{X_n} f(x)\, g(f(x))\, \mathrm{d}\mu(x) \leq (n+1)g(n+1)\mu(X_n) \text{ for } n \geq 0.$$

Moreover, using property (i), we have that

$$\int_{X_n} f(x)\, g(f(x))\, \mathrm{d}\mu(x) \leq 2c\, ng(n)\mu(X_n) \text{ for } n \geq 1.$$

Thus

$$\begin{aligned}\sum_{n=1}^{\infty} ng(n)\mu(X_n) &= \sum_{n=0}^{\infty} ng(n)\mu(X_n)\\ &\leq \int_X f(x)\, g(f(x))\, \mathrm{d}\mu(x)\\ &\leq g(1)\mu(X_0) + 2c \sum_{n=1}^{\infty} ng(n)\mu(X_n).\end{aligned}$$

The latter equation says that the integral $\int_X f(x)\, g(f(x))\, \mathrm{d}\mu(x)$ converges simultaneously with the numerical series $\sum_{n=1}^{\infty} ng(n)\mu(X_n)$, which in turn converges simultaneously with the series $\sum_{n=1}^{\infty}\Big(\sum_{m=1}^{n} g(m)\Big)\mu(X_n)$, according to property (ii). Finally, by rearranging the (non-negative) terms, it follows that

$$\begin{aligned}\sum_{n=1}^{\infty}\Big(\sum_{m=1}^{n} g(m)\Big)\mu(X_n) &= \sum_{1\leq m\leq n<\infty} g(m)\mu(X_n)\\ &= \sum_{m=1}^{\infty}\Big(g(m)\sum_{n=m}^{\infty}\mu(X_n)\Big)\\ &= \sum_{m=1}^{\infty} g(m)\mu\Big(\bigcup_{n=m}^{\infty} X_n\Big)\\ &= \sum_{m=1}^{\infty} g(m)\mu(\{x \in X : f(x) \geq m\}),\end{aligned}$$

and the proof is complete. □

Examples. $g(x) = x^{p-1}$ with $p > 1$ or $g(x) = \log x$, both defined on $[1, \infty)$ and completed with $g(\infty) = \infty$, satisfy the hypotheses of the above theorem. In the first case, formula (1) gives

$$\int_X f^p(x)\,\mathrm{d}\mu(x) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} \mu\{x \in X : f(x) \geq n^{1/p}\}) < \infty,$$

whereas formula (2) gives

$$\int_X f^p(x)\,\mathrm{d}\mu(x) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} n^{p-1}\,\mu\{x \in X : f(x) \geq n\}) < \infty.$$

Note that the superlevel sets in the latter formula are much easier to handle than the corresponding ones in the former formula.

In the second case, formula (2) yields

$$\int_X f(x)\,\log f(x)\,\mathrm{d}\mu(x) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} \log n\,\mu\{x \in X : f(x) \geq n\}) < \infty,$$

whereas formula (1) involves more complicated superlevel sets:

$$\int_X f(x)\,\log f(x)\,\mathrm{d}\mu(x) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} \mu\{x \in X : f(x)\,\log f(x) \geq n\}) < \infty.$$

Another example: if f is as in the theorem and the function $g := f^{-1}$ satisfies the hypotheses therein, then formula (2) says that

$$\int_X |x| f(x)\,\mathrm{d}\mu(x) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} |f^{-1}(n)|\,\mu\{x \in X : f(x) \geq n\}) < \infty,$$

whereas formula (1) reads

$$\int_X |x| f(x)\,\mathrm{d}\mu(x) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} \mu\{x \in X : |x| f(x) \geq n\}) < \infty.$$

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Traian Lalescu national mathematics contest for university students, 2018 edition

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Abstract. This note deals with the problems proposed at the 2018 edition of the Traian Lalescu mathematics contest for university students, hosted by the University Politehnica of Bucharest between May 10^{th} and May 12^{th}, 2018.

Keywords: Ellipse, sheaf of planes, tangent plane, symmetric matrix, trace, determinant, rank of a matrix, orthonormal basis, eigenvector, eigenvalue, Jordan canonical form, Riemann integral, improper integral, uniformly bounded sequence.

MSC: 15A03, 15A18, 15A21, 26D15, 51N10.

The University Politehnica of Bucharest organized between May 10^{th} and May 12^{th} the 2018 edition of the Traian Lalescu national mathematics contest for university students.

The contest saw a participation of 83 students representing 12 universities from Bucharest, Cluj, Constanţa, Iaşi and Timişoara, and was organized in five sections:

- Section A for students of faculties of Mathematics,
- Section B for first-year students of technical faculties, electric specializations, and Computer Science faculties,
- Section C for first-year students of technical faculties, non-electric specializations,
- Section D for second-year students of technical faculties, electric specializations, and
- Section E for second-year students of technical faculties, non-electric specializations.

We present in the sequel the problems proposed in Sections A and B of the contest and their solutions.

Section A

Problem 1. We inscribe in the ellipse $\mathcal{E}_1$ the rectangle $\mathcal{D}_1$ of maximum area. We inscribe in $\mathcal{D}_1$ the ellipse $\mathcal{E}_2$ with the axes parallel to the sides of

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$\mathcal{D}_1$. We continue by inscribing in $\mathcal{E}_2$ the rectangle $\mathcal{D}_2$ of maximum area, and so on. Show that

$$\sum_{n=1}^{\infty} \operatorname{area}(\mathcal{E}_n) = 2\operatorname{area}(\mathcal{E}_1) \text{ and } \sum_{n=1}^{\infty} \operatorname{area}(\mathcal{D}_n) = 2\operatorname{area}(\mathcal{D}_1).$$

Gheorghe Costovici

The jury considered this problem to be easy. The contestants confirmed this opinion, more than half of them managing to fully solve the problem. The solutions they gave went along the lines of the solution below.

Solution. We start by proving that if a rectangle is inscribed in an ellipse, then its sides are parallel to the symmetry axes of the ellipse (see also [1, Problem 3, Section A]).

Since this claim is obvious if the ellipse is a circle, we will suppose it is not. So, let $\mathcal{E}$ be an ellipse of parametric equations $\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$, $a \neq b$, and let $\mathcal{D} = A_1A_2A_3A_4$ be a rectangle inscribed in it, the coordinates of A_i being $(a\cos t_i, b\sin t_i)$, $1 \le i \le 4$, with

$$0 \le t_1 < t_2 < t_3 < t_4 < 2\pi. \tag{1}$$

Since $\overrightarrow{A_1A_2} = \overrightarrow{A_4A_3}$, we get

$$a(\cos t_2 - \cos t_1)\overrightarrow{i} + b(\sin t_2 - \sin t_1)\overrightarrow{j} = a(\cos t_3 - \cos t_4)\overrightarrow{i} + b(\sin t_3 - \sin t_4)\overrightarrow{j},$$

so

$$\begin{cases} \cos t_2 - \cos t_1 = \cos t_3 - \cos t_4, \\ \sin t_2 - \sin t_1 = \sin t_3 - \sin t_4. \end{cases} \tag{2}$$

Squaring these equations and adding the results yields $\cos(t_1 - t_2) = \cos(t_3 - t_4)$, whence

$$\exists k \in \mathbb{Z}, \quad t_1 - t_2 - t_3 + t_4 = 2k\pi, \tag{3}$$

or

$$\exists k' \in \mathbb{Z}, \quad t_1 - t_2 + t_3 - t_4 = 2k'\pi. \tag{4}$$

But, since $0 < t_4 - t_1 < 2\pi$ and $-2\pi < -t_3 + t_2 < 0$, relation (4) would imply $t_4 - t_3 + t_2 - t_1 = 0$, and thus the contradiction $t_2 + t_4 = t_1 + t_3$. Consequently, relation (3) must hold.

Similarly, from $\overrightarrow{A_2A_3} = \overrightarrow{A_1A_4}$ we derive the existence of $l \in \mathbb{Z}$ such that

$$-t_1 - t_2 + t_3 + t_4 = 2l\pi. \tag{5}$$

From relations (3) and (5) we get

$$t_3 - t_1 = (l - k)\pi \text{ and } t_4 - t_2 = (l + k)\pi,$$

and thus, since $0 < t_3 - t_1 < 2\pi$ and $0 < t_4 - t_2 < 2\pi$, we obtain

$$t_3 = t_1 + \pi \text{ and } t_4 = t_2 + \pi. \tag{6}$$

Now, from $\overrightarrow{A_1A_2} \perp \overrightarrow{A_2A_3}$ we get

$$a^2(\cos t_2 - \cos t_1)(\cos t_3 - \cos t_2) + b^2(\sin t_2 - \sin t_1)(\sin t_3 - \sin t_2) = 0;$$

in view of relations (6), this becomes

$$a^2(\cos^2 t_1 - \cos^2 t_2) + b^2(\sin^2 t_1 - \sin^2 t_2) = 0,$$

whence $(a^2 - b^2)(\sin^2 t_2 - \sin^2 t_1) = 0$. Since $a \neq b$, we obtain $\sin^2 t_2 = \sin^2 t_1$. But $0 < t_1 < t_2 = t_4 - \pi < \pi$, and thus $\sin t_2 = \sin t_1$, so $A_3A_4 \parallel A_1A_2 \parallel Ox$, and $t_2 = \pi - t_1$, so, since $t_3 = \pi + t_1$ from relation (6), $A_1A_4 \parallel A_2A_3 \parallel Oy$.

Next, if $\mathcal{D}$ is the rectangle of maximum area inscribed in $\mathcal{E}$, since its area equals $2a \cos t_1 \cdot 2b \sin t_1 = 2ab \sin 2t_1$ and $t_1 \in \left(0, \frac{\pi}{2}\right)$, we get $t_1 = \frac{\pi}{4}$, and $\operatorname{area}(\mathcal{D}) = 2ab$. Since the area of $\mathcal{E}$ is πab, we get that $\operatorname{area}(\mathcal{D}) = \frac{2}{\pi} \cdot \operatorname{area}(\mathcal{E})$.

If, on the other hand, we consider an ellipse (H) inscribed in a rectangle Δ, the axes of (H) being parallel to the sides of the rectangle, then denoting by L and l the length and width of the rectangle, respectively, we get $\operatorname{area}(H) = \pi \cdot \frac{L}{2} \cdot \frac{l}{2} = \frac{\pi}{4} \cdot \operatorname{area}(\Delta)$.

In view of all of the above, in the specific conditions of Problem 1, we have for all $n \in \mathbb{N}^*$

$$\operatorname{area}(\mathcal{E}_{n+1}) = \frac{\pi}{4} \cdot \operatorname{area}(\mathcal{D}_n) = \frac{\pi}{4} \cdot \frac{2}{\pi} \cdot \operatorname{area}(\mathcal{E}_n) = \frac{1}{2} \cdot \operatorname{area}(\mathcal{E}_n)$$

and

$$\operatorname{area}(\mathcal{D}_{n+1}) = \frac{2}{\pi} \cdot \operatorname{area}(\mathcal{E}_{n+1}) = \frac{2}{\pi} \cdot \frac{\pi}{4} \cdot \operatorname{area}(\mathcal{D}_n) = \frac{1}{2} \cdot \operatorname{area}(\mathcal{D}_n),$$

so

$$\sum_{n=1}^{\infty} \operatorname{area}(\mathcal{E}_n) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \operatorname{area}(\mathcal{E}_1) = \operatorname{area}(\mathcal{E}_1) \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \operatorname{area}(\mathcal{E}_1)$$

and

$$\sum_{n=1}^{\infty} \operatorname{area}(\mathcal{D}_n) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \operatorname{area}(\mathcal{D}_1) = \operatorname{area}(\mathcal{D}_1) \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \operatorname{area}(\mathcal{D}_1).$$

Problem 2. Let a be a real number, let d be the line described by the equations $x + a = y + a = z - 2a$, and let S^2 be the unit sphere centered at the origin. Find the values of a for which there exist planes that contain d and are tangent to S^2. For these values of a also write down the equations of the tangent planes.

Gabriel Mincu

The jury considered that, although the problem is not difficult, failing to find a suitable expression of the geometric conditions via equations could result in heavy calculations. The contestants confirmed this to a certain extent, only a few of them managing to identify strategies that avoided spending *vast amounts of time on calculations. We will present two solutions in the sequel, the first one being closer to the solutions given by the students who fully solved the problem.*

Solution 1. Let us first notice that a plane π contains d and is tangent to S^2 at the point $M(\alpha,\beta,\gamma)$ iff $OM \perp \pi$, π contains a point of d ($N(0,0,3a)$ for instance), the direction of d is perpendicular to the normal vector of π, and $M \in \pi$. An equivalent manner of expressing this system of conditions is:

$$\begin{cases} \pi \perp \alpha\vec{i} + \beta\vec{j} + \gamma\vec{k}, \\ \pi \ni (0,0,3a), \\ \alpha\vec{i} + \beta\vec{j} + \gamma\vec{k} \perp \vec{i} + \vec{j} + \vec{k}, \\ \pi \ni (\alpha,\beta,\gamma). \end{cases}$$

These conditions are met simultaneously iff

$$\begin{cases} \text{the equation of } \pi \text{ is } \alpha x + \beta y + \gamma(z-3a) = 0, \\ \alpha+\beta+\gamma = 0, \text{ and} \\ \alpha^2+\beta^2+\gamma(\gamma-3a) = 0. \end{cases}$$

Since $\alpha^2+\beta^2+\gamma^2 = 1$, the last equation is equivalent to $3a\gamma = 1$, and thus the condition for planes tangent to S^2 and containing d to exist is that the system

$$\begin{cases} \alpha^2+\beta^2+\gamma^2 = 1, \\ \alpha+\beta+\gamma = 0, \\ 3a\gamma = 1 \end{cases}$$

with the unknowns α, β and γ have solutions.

Now, this system obviously has no solutions for $a = 0$, whilst for $a \neq 0$ it is equivalent to

$$\begin{cases} \gamma = \frac{1}{3a}, \\ \alpha+\beta = -\frac{1}{3a}, \\ \alpha^2+\beta^2+\gamma^2 = 1, \end{cases} \Leftrightarrow \begin{cases} \gamma = \frac{1}{3a}, \\ \alpha+\beta = -\frac{1}{3a}, \\ \frac{1}{9a^2} - 2\alpha\beta + \frac{1}{9a^2} = 1, \end{cases} \Leftrightarrow \begin{cases} \gamma = \frac{1}{3a}, \\ \alpha+\beta = -\frac{1}{3a}, \\ \alpha\beta = \frac{1}{9a^2} - \frac{1}{2}. \end{cases}$$

Now, two values α and β satisfy the last two equations of this system iff they are the solutions of the equation $u^2 + \frac{1}{3a}u + \frac{1}{9a^2} - \frac{1}{2} = 0$. This equation (and the system for that matter) has real solutions iff $2 - \frac{1}{3a^2} \geq 0$, i.e., iff $a \in \left(-\infty, -\frac{1}{\sqrt{6}}\right] \cup \left[\frac{1}{\sqrt{6}}, +\infty\right)$.

By plugging the solutions (α,β,γ) of the system into the equation $\alpha x + \beta y + \gamma(z-3a) = 0$ of π, we draw the following conclusions:

If $|a| > \frac{1}{\sqrt{6}}$, there are two planes containing d and tangent to S^2. Their equations are

$$\left(-\frac{1}{6a}+\sqrt{\frac{1}{2}-\frac{1}{12a^2}}\right)x+\left(-\frac{1}{6a}-\sqrt{\frac{1}{2}-\frac{1}{12a^2}}\right)y+\frac{1}{3a}z=1$$

and

$$\left(-\frac{1}{6a}-\sqrt{\frac{1}{2}-\frac{1}{12a^2}}\right)x+\left(-\frac{1}{6a}+\sqrt{\frac{1}{2}-\frac{1}{12a^2}}\right)y+\frac{1}{3a}z=1.$$

If $|a| = \frac{1}{\sqrt{6}}$, there is only one plane containing d and tangent to S^2 (meaning that d itself is tangent to S^2); its equation is

$$-\frac{1}{6a}x-\frac{1}{6a}y+\frac{1}{3a}z=1.$$

Solution 2. We first notice that for $a = 0$ the line d contains the origin, so no plane containing d is tangent to S^2. Thus, we will consider in the sequel $a \neq 0$ (so d does not pass through the origin).

Since the points $M(0,0,3a)$ and $N(-a,-a,2a)$ belong to d, the origin does not, whilst $P(1,-1,0)$ does not lie in the plane ρ spanned by d and the origin, the equation of the sheaf of planes of axis d (leaving ρ out) is

$$\begin{vmatrix} 1 & x & y & z \\ 1 & 0 & 0 & 3a \\ 1 & -a & -a & 2a \\ 1 & 1 & -1 & 0 \end{vmatrix} + \lambda \begin{vmatrix} 1 & x & y & z \\ 1 & 0 & 0 & 3a \\ 1 & -a & -a & 2a \\ 1 & 0 & 0 & 0 \end{vmatrix} = 0,$$

which is equivalent to

$$\begin{vmatrix} 1 & x & y & z \\ 1 & 0 & 0 & 3a \\ 1 & -a & -a & 2a \\ 1+\lambda & 1 & -1 & 0 \end{vmatrix} = 0.$$

After computing the determinant, we conclude that each plane in the sheaf has an equation of the form

$$(1-3a(1+\lambda))x+(1+3a(1+\lambda))y-2z+6a=0 \tag{7}$$

for a specific value of the parameter λ.

The plane given by equation (7) is tangent to S^2 iff the distance from it to the origin equals 1, i.e., iff

$$\frac{|6a|}{\sqrt{(1-3a(1+\lambda))^2+(1+3a(1+\lambda))^2+4}}=1.$$

This condition is equivalent to $|6a| = \sqrt{6 + 18a^2(1+\lambda)^2}$, which is rewritten as

$$(\lambda + 1)^2 = 2 - \frac{1}{3a^2}. \tag{8}$$

Consequently, there exist planes containing d and tangent to S^2 iff there are values λ satisfying $(\lambda + 1)^2 = 2 - \frac{1}{3a^2}$, i.e., iff $2 - \frac{1}{3a^2} \geq 0$, that is, iff $a \in \left(-\infty, -\frac{1}{\sqrt{6}}\right] \cup \left[\frac{1}{\sqrt{6}}, +\infty\right)$.

If a meets this condition, the equations of the tangent planes, obtained by plugging the solutions of equation (8) into equation (7), are

$$(1 - \sqrt{18a^2 - 3})x + (1 + \sqrt{18a^2 - 3})y - 2z + 6a = 0$$

and

$$(1 + \sqrt{18a^2 - 3})x + (1 - \sqrt{18a^2 - 3})y - 2z + 6a = 0$$

(we notice that the two equations coincide for $|a| = \frac{1}{\sqrt{6}}$).

Remark. Had it not been for the requirement to actually write down the equations of the tangent planes, we would have found the values of a for which a plane meeting the conditions in the statement exists with considerably lesser effort by merely requiring d either to not cut S^2 at all or to be tangent to it, i.e., by requiring the system

$$\begin{cases} x^2 + y^2 + z^2 = 1, \\ x + a = y + a = z - 2a, \end{cases} \Leftrightarrow \begin{cases} y = x, \\ z = x + 3a, \\ 3x^2 + 6ax + 9a^2 - 1 = 0, \end{cases}$$

to have at most one solution; this happens iff

$$9a^2 - 3(9a^2 - 1) \leq 0 \Leftrightarrow 18a^2 \geq 3 \Leftrightarrow a \in \left(-\infty, -\frac{1}{\sqrt{6}}\right] \cup \left[\frac{1}{\sqrt{6}}, +\infty\right).$$

Problem 3. Let A be an $n \times n$ real matrix. Show that A is symmetric if and only if there exist an orthonormal basis $u_1, u_2, \ldots, u_n$ in $\mathbb{R}^n$ and real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$A = \lambda_1 u_1 u_1^t + \lambda_2 u_2 u_2^t + \cdots + \lambda_n u_n u_n^t. \tag{9}$$

How many such bases are there if $\lambda_i \neq \lambda_j$ for all $i \neq j$?

Vasile Pop

The jury considered the only difficulty of the problem was making the connection between the theoretic notions and the form relation (9) was presented in. This was indeed the case, and thus the participants either solved the problem completely or did not manage to start a solution at all.

Solution. Since A is symmetric, all the eigenvalues of A are real, and there exists an orthonormal basis formed of eigenvectors of A. Thus, there exist an orthonormal basis $u_1, \ldots, u_n$ of $\mathbb{R}^n$ and n real numbers $\lambda_1, \ldots, \lambda_n$

such that, denoting by S the matrix having for each $i \in \{1, \ldots, n\}$ the components of u_i on the i^{th} column,

$$A = S \operatorname{diag}(\lambda_1, \ldots, \lambda_n) S^{-1},$$

i.e., since $S^{-1} = S^t$, such that

$$A = \lambda_1 u_1 u_1^t + \cdots + \lambda_n u_n u_n^t.$$

The converse is obvious.

If $\lambda_i \neq \lambda_j$ for all $i \neq j$, then all the eigenspaces of A are one-dimensional, so they are $\mathbb{R}u_1, \ldots, \mathbb{R}u_n$. But then the choice of an orthonormal basis of $\mathbb{R}^n$ such that relation (9) holds consists in fact of n independent choices between u_i and $-u_i$, these being the only unit vectors in $\mathbb{R}u_i$, $i \in \{1, \ldots, n\}$. Thus, the required number of bases is 2^n.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that there exists a positive integer p such that $f^{(p+1)} = f'$, where $f^{(i)}$ denotes the ith derivative of f. For every $x \in \mathbb{R}^*$ compute the following limits:

a) $\displaystyle\lim_{n\to\infty} \left(\frac{f(x) - f(0)}{x}\right)^{(n)}$.

b) $\displaystyle\lim_{n\to\infty} n \left(\frac{e^x - 1}{x}\right)^{(n)}$.

Mircea Ivan

The jury expected the participants to encounter difficulties with this problem. This was indeed the case, the highest mark scored on the problem being 4 out of 10 (managed by the winner of the section).

Solution. Since this problem is a part of Problem 4 of Section B of the contest, we refer the reader to the solution thereof.

Section B

The jury of Section B selected the following four problems, ordered by difficulty from easy to hard. The solutions presented here are (with minor changes) the ones given by the authors or other members of the jury.

Problem 1. Let $n \in \mathbb{N}$, $n \geq 3$.

(a) If n is odd, find all symmetric matrices $A \in \mathcal{M}_n(\mathbb{R})$ which satisfy $\operatorname{Tr} A^{n-1} = \det A < n^n$.

(b) If n is odd and $A \in \mathcal{M}_n(\mathbb{R})$ is a symmetric matrix which satisfies $\operatorname{Tr} A^{n-1} = \det A = n^n$, prove that $A^2 = n^2 I_n$.

(c) If n is even, prove that there exists a symmetric matrix $A \in \mathcal{M}_n(\mathbb{R})$ which satisfies $\operatorname{Tr} A^{n-1} = \det A = n^n$ and $A^2 \neq n^2 I_n$.

Cristian Ghiu

Solution. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a symmetric matrix such that $\operatorname{Tr} A^{n-1} = \det A$. If $\lambda_1, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of A, then they are real numbers (since A is symmetric) and

$$\lambda_1^{n-1} + \cdots + \lambda_n^{n-1} = \operatorname{Tr} A^{n-1} = \det A = \lambda_1 \cdots \lambda_n. \tag{10}$$

Also, A is diagonalizable, i.e., $A = PDP^{-1}$ where $P \in \mathcal{M}_n(\mathbb{R})$ is invertible and D is a diagonal matrix with the eigenvalues $\{\lambda_i\}$ on the main diagonal.

(a) & (b) If n is odd, then $n-1$ is even and by (10) it follows that $\det A \geq 0$.

If $\det A = 0$, then $\lambda_1 = \cdots = \lambda_n = 0$ by (10), since $n-1$ is even, hence $D = O_n$, which leads to $A = O_n$.

If $\det A > 0$, then by (10) and the AM-GM inequality it follows that

$$\det A = \lambda_1^{n-1} + \cdots + \lambda_n^{n-1} \geq n \sqrt[n]{\lambda_1^{n-1} \cdots \lambda_n^{n-1}} = n\,(\det A)^{\frac{n-1}{n}}, \tag{11}$$

which leads to $\det A \geq n^n$.

Consequently, if $\det A < n^n$, then $\det A = 0$, hence $A = O_n$ (proving (a)). If $\det A = n^n$, then we have equality in (11), which leads to $|\lambda_1| = \cdots = |\lambda_n| = n$. It follows that $D^2 = n^2 I_n$, hence $A^2 = n^2 I_n$ (proving (b)).

(c) If n is even, $n \geq 4$, then we construct A to be a diagonal matrix with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ on the main diagonal satisfying

$$\begin{aligned} \operatorname{Tr} A^{n-1} = \lambda_1^{n-1} + \cdots + \lambda_n^{n-1} &= n^n, \\ \det A = \lambda_1 \cdots \lambda_n &= n^n, \end{aligned}$$

and $\lambda_i \notin \{-n, n\}$ for some i (so that $A^2 \neq n^2 I_n$) by first taking $\lambda_3 = -1 \notin \{-n, n\}$ and $\lambda_4 = \cdots = \lambda_n = 1$. Then the remaining λ_1, λ_2 must satisfy

$$\begin{aligned} \lambda_1^{n-1} + \lambda_2^{n-1} &= n^n - n + 4, \\ \lambda_1 \lambda_2 &= -n^n. \end{aligned}$$

Since $n-1$ is odd, we may choose λ_i to be $\sqrt[n-1]{x_i}$ $(i = 1, 2)$, x_1 and x_2 being the (real) solutions of the equation $x^2 - (n^n - n + 4)\,x - n^{n(n-1)} = 0$.

Problem 2. Let $f : [1, \infty) \to (0, \infty)$ be a continuous function such that $\displaystyle\int_1^\infty f(x)\,\mathrm{d}x$ is convergent.

(a) Prove that for every $n \in \mathbb{N}^*$, there exists a unique $a_n \in [1, \infty)$ such that

$$\int_1^{a_n} x^n f(x)\,\mathrm{d}x = \int_1^\infty f(x)\,\mathrm{d}x.$$

(b) Find $\lim\limits_{n\to\infty} a_n$ and study the convergence of the series $\sum\limits_{n=1}^{\infty}(a_n-1)$.

Bogdan Sebacher

Solution. (a) Let $I=\displaystyle\int_1^{\infty} f(x)\,\mathrm{d}x\in(0,\infty)$. Fix $n\in\mathbb{N}^*$ and let

$$F:[1,\infty)\to\mathbb{R},\quad F(t)=\int_1^t x^n f(x)\,\mathrm{d}x-I.$$

Since $F'(t)=t^nf(t)>0$ for all $t\geq 1$, it follows that F is strictly increasing. Moreover, $F(1)=-I<0$ and $\lim\limits_{t\to\infty}F(t)=\lim\limits_{t\to\infty}\displaystyle\int_1^t(x^n-1)\,f(x)\,\mathrm{d}x>0$, so we can conclude that the equation $F(t)=0$ has a unique solution $t=a_n\in[1,\infty)$.

(b) For every $n\in\mathbb{N}^*$,

$$\int_1^{a_n}x^nf(x)\,\mathrm{d}x=\int_1^{a_{n+1}}x^{n+1}f(x)\,\mathrm{d}x>\int_1^{a_{n+1}}x^nf(x)\,\mathrm{d}x,$$

which leads to $a_n>a_{n+1}$, hence $(a_n)_{n\geq 1}$ is convergent (being decreasing and bounded from below by 1). Let $L=\lim\limits_{n\to\infty}a_n\geq 1$ and $m=\min\limits_{x\in[1,L]}f(x)>0$. Then

$$I=\int_1^{a_n}x^nf(x)\,\mathrm{d}x\geq\int_1^{L}x^nf(x)\,\mathrm{d}x\geq m\int_1^L x^n\,\mathrm{d}x=m\cdot\frac{L^{n+1}-1}{n+1}\tag{12}$$

for all $n\in\mathbb{N}^*$. If $L>1$, then $\lim\limits_{n\to\infty}\dfrac{L^{n+1}-1}{n+1}=\infty$ which contradicts (12). Therefore, $\lim\limits_{n\to\infty}a_n=1$.

Next, let $M=\max\limits_{x\in[1,a_1]}f(x)\in(0,\infty)$, $a=\frac{I}{M}$ and $b=\frac{I}{M}+1$. Then

$$I=\int_1^{a_n}x^nf(x)\,\mathrm{d}x\leq M\int_1^{a_n}x^n\,\mathrm{d}x=M\cdot\frac{a_n^{n+1}-1}{n+1},$$

which leads to

$$a_n\geq\sqrt[n+1]{\frac{(n+1)I}{M}+1}=\sqrt[n+1]{an+b}\ \text{for all } n\in\mathbb{N}^*.\tag{13}$$

Since $\lim\limits_{n\to\infty}\dfrac{\mathrm{e}^{\frac{\ln(an+b)}{n+1}}-1}{\frac{\ln(an+b)}{n+1}}=1$, the numerical series

$$\sum_{n=1}^{\infty}\left(\sqrt[n+1]{an+b}-1\right)=\sum_{n=1}^{\infty}\left(\mathrm{e}^{\frac{\ln(an+b)}{n+1}}-1\right)$$

has the same nature as $\sum_{n=1}^{\infty} \frac{\ln(an+b)}{n+1}$. Also, since $\frac{\ln(an+b)}{n+1} \geq \frac{1}{n+1}$ for all $n \geq \frac{e-b}{a}$, the numerical series $\sum_{n=1}^{\infty} \frac{\ln(an+b)}{n+1}$ is divergent. Consequently, $\sum_{n=1}^{\infty} \left(\sqrt[n+1]{an+b} - 1 \right)$ is divergent, hence, by (13), $\sum_{n=1}^{\infty} (a_n - 1)$ is also divergent.

Problem 3. Let $n \geq 2$ be an integer. Prove that:
(a) $\operatorname{rank} A^n = \operatorname{rank} A^{n+1}$ for every $A \in \mathcal{M}_n(\mathbb{C})$;
(b) there exists $B \in \mathcal{M}_n(\mathbb{C})$ such that $\operatorname{rank} B^{n-1} \neq \operatorname{rank} B^n$;
(c) all matrices $B \in \mathcal{M}_n(\mathbb{C})$ which satisfy $\operatorname{rank} B^{n-1} \neq \operatorname{rank} B^n$ are similar.

Vasile Pop

Solution 1. For every $M \in \mathcal{M}_n(\mathbb{C})$, let

$$\operatorname{Ker} M = \{X \in \mathcal{M}_{n,1}(\mathbb{C}) : M \cdot X = O_{n,1}\}$$

be the kernel of M, which is a linear subspace in $\mathcal{M}_{n,1}(\mathbb{C}) = \mathbb{C}^n$ of dimension $n - \operatorname{rank} M$.

(a) We prove $\operatorname{Ker} A^n = \operatorname{Ker} A^{n+1}$. The inclusion $\operatorname{Ker} A^n \subseteq \operatorname{Ker} A^{n+1}$ is immediate. Assume that the inclusion is strict, i.e., there exists $X \in \mathcal{M}_{n,1}(\mathbb{C})$ such that

$$A^n \cdot X \neq 0 \quad \text{and} \quad A^{n+1} \cdot X = 0. \tag{14}$$

By the Hamilton-Cayley formula,

$$a_0 \cdot I_n + a_1 \cdot A + \cdots + a_{n-1} \cdot A^{n-1} + A^n = 0$$

for some real numbers a_0, a_1, …, a_{n-1}; multiplying this by X to the right it follows that

$$a_0 \cdot X + a_1 \cdot A \cdot X + \cdots + a_{n-1} \cdot A^{n-1} \cdot X + A^n \cdot X = 0. \tag{15}$$

Next, multiplying (15) by A^n to the left and using (14), it follows that $a_0 \cdot A^n \cdot X = 0$, hence $a_0 = 0$. By replacing $a_0 = 0$ in (15) and multiplying to the left by A^{n-1} it will follow that $a_1 = 0$. Repeating this process, we obtain $a_0 = a_1 = \cdots = a_{n-1} = 0$, hence $A^n = 0$, so $A^n \cdot X = 0$, which contradicts (14).

Consequently, $\operatorname{Ker} A^n = \operatorname{Ker} A^{n+1}$, hence $\operatorname{rank} A^n = n - \dim(\operatorname{Ker} A^n) = n - \dim\left(\operatorname{Ker} A^{n+1}\right) = \operatorname{rank} A^{n+1}$.

(b) Let $B = J_0 = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}$. Then $B^{n-1} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 1 & \dots & 0 \end{pmatrix}$ and $B^n = O_n$, hence $\operatorname{rank} B^{n-1} = 1$ and $\operatorname{rank} B^n = 0$.

(c) We denote by $\operatorname{col}(v_1, \dots, v_n)$ the matrix that has, for each $i \in \{1, \dots, n\}$, the components of v_i on the i^{th} column.
Let $B \in \mathcal{M}_n(\mathbb{C})$ such that $\operatorname{rank} B^{n-1} \neq \operatorname{rank} B^n$. We claim that B is similar to J_0: Since $\operatorname{rank} B^{n-1} \neq \operatorname{rank} B^n$, there exists $X_0 \in \mathcal{M}_{n,1}(\mathbb{C})$ such that $B^{n-1} \cdot X_0 \neq 0$ and $B^n \cdot X_0 = 0$. The vectors $X_0, BX_0, \dots, B^{n-1}X_0$ are linearly independent (since from $a_0X_0 + a_1BX_0 + \dots + a_{n-1}B^{n-1}X_0 = 0$ it follows, using the argument from (a), that $a_0 = a_1 = \dots = a_{n-1} = 0$). Consequently, the matrix

$$P = \operatorname{col}(X_0, BX_0, \dots, B^{n-1}X_0)$$

is invertible. Moreover,

$$B \cdot P = \operatorname{col}(BX_0, B^2X_0, \dots, B^{n-1}X_0, O_{n,1}) = P \cdot J_0,$$

which concludes the proof.

Solution 2. (a) Let $J_k(\lambda)$ be the $k \times k$ Jordan block corresponding to $\lambda \in \mathbb{C}$. We notice that $\operatorname{rank}(J_k(\lambda)^m) = k = \operatorname{rank}(J_k(\lambda))$ for all $\lambda \neq 0$ and $k, m \in \mathbb{N}^*$, whilst

$$\operatorname{rank}(J_k(0)^m) = \begin{cases} k - m, & \text{if} \quad m < k, \\ 0, & \text{if} \quad m \geq k. \end{cases} \tag{16}$$

Suppose that the Jordan canonical form of A is

$$\operatorname{diag}(J_{k_1}(\lambda_1), \dots, J_{k_r}(\lambda_r), J_{k_{r+1}}(\lambda_{r+1}), \dots, J_{k_s}(\lambda_s)),$$

with $\lambda_1 = \dots = \lambda_r = 0$ and $\lambda_{r+1} \cdots \lambda_s \neq 0$. Then, in view of relation (16), we have

$$\begin{aligned} \operatorname{rank}(A^n) &= \sum_{i=1}^{s} \operatorname{rank}((J_{k_i}(\lambda_i))^n) = \sum_{i=r+1}^{s} \operatorname{rank}((J_{k_i}(\lambda_i))^n) \\ &= \sum_{i=r+1}^{s} \operatorname{rank}(J_{k_i}(\lambda_i)) = \sum_{i=r+1}^{s} \operatorname{rank}((J_{k_i}(\lambda_i))^{n+1}) \\ &= \sum_{i=1}^{s} \operatorname{rank}((J_{k_i}(\lambda_i))^{n+1}) = \operatorname{rank}(A^{n+1}). \end{aligned}$$

(b) In view of relation (16), $B = J_n(0)$ satisfies the requirement.

(c) If B has the eigenvalues $\lambda_1 = \dots = \lambda_r = 0$ and $\lambda_{r+1}, \dots, \lambda_s$ such that $\lambda_{r+1} \cdots \lambda_s \neq 0$, $\operatorname{rank}(B^{n-1}) = \operatorname{rank}(B^n)$ is equivalent to $\operatorname{rank}(J_B^{n-1}) =$

$\operatorname{rank}(J_B^n)$, thus to

$$\sum_{i=1}^{s} \operatorname{rank}((J_{k_i}(\lambda_i))^{n-1}) \neq \sum_{i=1}^{s} \operatorname{rank}((J_{k_i}(\lambda_i))^{n})$$

and, further, to

$$\sum_{i=1}^{r} \operatorname{rank}((J_{k_i}(\lambda_i))^{n-1}) \neq \sum_{i=1}^{s} \operatorname{rank}((J_{k_i}(\lambda_i))^{n}),$$

whence there exists $i \in \{1, 2, \ldots, r\}$ such that $k_i = n$, and therefore $r = s = 1$, $\lambda_1 = 0$ and $k_1 = n$. Consequently, $J_B = J_n(0)$, so B is similar to $J_n(0)$.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that $f^{(p+1)} = f'$ for some positive integer p, where $f^{(i)}$ denotes the ith derivative of f. For every $x \in \mathbb{R}^*$, compute the following limits:

(a) $\displaystyle\lim_{n\to\infty} \left(\frac{f(x) - f(0)}{x}\right)^{(n)}$;

(b) $\displaystyle\lim_{n\to\infty} n \left(\frac{f(x) - f(0)}{x}\right)^{(np)}$;

(c) $\displaystyle\lim_{n\to\infty} n \left(\frac{\mathrm{e}^x - 1}{x}\right)^{(n)}$.

Mircea Ivan

Solution 1. (a) Fix $x \in \mathbb{R}^*$. Since $\dfrac{f(x) - f(0)}{x} = \int_0^1 f'(tx)\,\mathrm{d}t$, it follows that

$$\left(\frac{f(x) - f(0)}{x}\right)^{(n)} = \int_0^1 t^n f^{(n+1)}(tx)\mathrm{d}t, \quad n \in \mathbb{N}. \tag{17}$$

For every $n \in \mathbb{N}$, let

$$M_n = \sup\left\{\left|f^{(n+1)}(y)\right| : y \in [\min\{0, x\},\ \max\{0, x\}]\right\} \in \mathbb{R}.$$

Because $f^{(p+1)} = f'$, it follows that

$$\{M_n : n \in \mathbb{N}\} = \{M_0, M_1, \ldots, M_p\},$$

and therefore $M = \sup\{M_n : n \in \mathbb{N}\} \in \mathbb{R}_+$. Then, for every $n \in \mathbb{N}$,

$$0 \leq \left|\left(\frac{f(x) - f(0)}{x}\right)^{(n)}\right| \leq \int_0^1 t^n \left|f^{(n+1)}(tx)\right| \mathrm{d}t \leq M \int_0^1 t^n \mathrm{d}t = \frac{M}{n+1}, \tag{18}$$

hence the limit is 0.

Remark. The result holds in a more general setting, namely if we drop the condition $f^{(p+1)} = f'$ and assume instead that the sequence of derivatives $\left(f^{(n)}\right)_{n\geq 0}$ is uniformly bounded on every compact interval.

(b) By (17), it follows that for every $n \in \mathbb{N}$,

$$\begin{aligned}
n\left(\frac{f(x)-f(0)}{x}\right)^{(np)} &= n\int_0^1 t^{np} f^{(np+1)}(tx)\mathrm{d}t = n\int_0^1 t^{np} f'(tx)\mathrm{d}t \\
&= \frac{n}{np+1}\int_0^1 \left(t^{np+1}\right)' f'(tx)\mathrm{d}t \\
&= \frac{n}{np+1}\left(t^{np+1}f'(tx)\Big|_0^1 - x\int_0^1 t^{np+1} f''(tx)\mathrm{d}t\right) \\
&= \frac{n}{np+1}\left(f'(x) - x\int_0^1 t^{np+1} f''(tx)\mathrm{d}t\right).
\end{aligned}$$

Since the function $t \in [0,1] \longmapsto f''(tx)$ is bounded, it follows that

$$\lim_{n\to\infty}\int_0^1 t^{np+1} f''(tx)\mathrm{d}t = 0$$

by an argument similar to (18). Consequently, the required limit is $\dfrac{f'(x)}{p}$.

(c) By plugging $f(x) = \mathrm{e}^x$ and $p = 1$ in (b), we obtain that the required limit is e^x.

Solution 2. (*Mircea Rus*) (a) Since $f^{(p+1)} = f'$, it follows that the sequence of derivatives $\left(f^{(n)}\right)_{n\geq 0}$ is uniformly bounded on every compact interval (by the argument given in Solution 1), hence f is analytic over $\mathbb{R}$, i.e.,

$$f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k, \quad x \in \mathbb{R}.$$

Also, $M = \sup\left\{|f^{(k)}(0)|\right\} < \infty$. It follows that

$$\begin{aligned}
\left(\frac{f(x)-f(0)}{x}\right)^{(n)} &= \left(\sum_{k=1}^{\infty}\frac{f^{(k)}(0)}{k!}x^{k-1}\right)^{(n)} \\
&= \sum_{k=n+1}^{\infty}\frac{(k-1)(k-2)\cdots(k-n)\cdot f^{(k)}(0)}{k!}x^{k-n-1} \\
&= \sum_{k=n+1}^{\infty}\frac{f^{(k)}(0)}{k\cdot(k-n-1)!}x^{k-n-1} = \sum_{k=0}^{\infty}\frac{f^{(k+n+1)}(0)}{(k+n+1)\cdot k!}x^k,
\end{aligned}$$

hence

$$0 \le \left| \left(\frac{f(x) - f(0)}{x} \right)^{(n)} \right| \le \sum_{k=0}^{\infty} \frac{|f^{(k+n+1)}(0)|}{(k+n+1) \cdot k!} |x|^k$$
$$\le \frac{M}{n+1} \sum_{k=0}^{\infty} \frac{|x|^k}{k!} = \frac{M}{n+1} \mathrm{e}^{|x|}$$

for every $x \in \mathbb{R}^*$ and $n \in \mathbb{N}$. This proves that the required limit is 0.

(b) For every $x \in \mathbb{R}^*$, let $g(x) = \dfrac{f(x) - f(0)}{x}$, hence $f(x) = f(0) + xg(x)$. Next, by the Leibniz rule, it follows that

$$f'(x) = f^{(np+1)}(x) = xg^{(np+1)}(x) + (np+1)g^{(np)}(x), \ n \in \mathbb{N},$$

hence

$$n \left(\frac{f(x) - f(0)}{x} \right)^{(np)} = n \cdot g^{(np)}(x) = \frac{n}{np+1} \left(f'(x) - xg^{(np+1)}(x) \right), \ n \in \mathbb{N}.$$

Since, by (a),

$$\lim_{m \to \infty} g^{(m)}(x) = 0$$

we conclude that the required limit is $\dfrac{f'(x)}{p}$.

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NOTE MATEMATICE

A remark on an identity for derangement numbers

Ulrich Abel[1)]

Abstract. We give a short proof of an identity involving derangement numbers. The method is an application of the binomial convolution for formal power series.

Keywords: Combinatorial identities, derangement numbers.
MSC: 05A19

1. Introduction

Let $d(n)$ denote the number of permutations of $\{1,\ldots,n\}$ with no fixed points, the so-called derangements. With the convention $d(0)=1$ they possess the closed expression

$$d(n) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \tag{1}$$

and the exponential generating function

$$\sum_{n=0}^{\infty} d(n) \frac{z^n}{n!} = \frac{e^{-z}}{1-z} \qquad (|z|<1). \tag{2}$$

The recent papers [1, 4, 5] demonstrate the vivid interest in these numbers. Vinh [7, Theorem 1] proved, for integers $n \geq m \geq 0$, the identity

$$\sum_{k=0}^{n} k^m \binom{n}{k} d(n-k) = n! B_m, \tag{3}$$

where B_m denotes the Bell number given by

$$B_m = \sum_{k=0}^{m} S(m,k). \tag{4}$$

The Stirling numbers of the second kind $S(m,k)$ are the coefficients when expanding the power z^m in a linear combination of falling factorials $z^{\underline{k}} = \prod_{i=0}^{k-1}(z-i)$, i.e.,

$$z^m = \sum_{k=0}^{m} S(m,k)\, z^{\underline{k}}. \tag{5}$$

The quantities in Eq. (3) have a very concrete interpretation which is nicely described in [3]. Let X_n be the random variable representing the number of fixed points of a permutation of n objects (n-permutation) chosen

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by random. Since the number of n-permutations with exactly k fixed points is equal to $\binom{n}{k}d(n-k)$, the probability that the random variable X_n takes the value k is given by

$$P(X_n = k) = \frac{1}{n!}\binom{n}{k}d(n-k).$$

Hence, for the expectation of the random variable X_n we obtain

$$E(X_n) = \sum_{k=0}^{n} k \cdot P(X_n = k) = \frac{1}{n!}\sum_{k=0}^{n} k\binom{n}{k}d(n-k).$$

For $m \in \mathbb{N}$, the moments of the random variable X_n possess the representation

$$E(X_n^m) = \sum_{k=0}^{n} k^m P(X_n = k) = \frac{1}{n!}\sum_{k=0}^{n} k^m\binom{n}{k}d(n-k).$$

Vinh's identity (3) tells us that $E(X_n^m) = B_m$ $(m = 0, 1, \ldots, n)$. Using well-known values of the Bell numbers we find the table (cf. [3, pg. 5251])

m	0	1	2	3	4	5	6	7	8	9	10
$E(X_n^m)$	1	1	2	5	15	52	203	877	4140	21147	115975

provided that in each case $n \geq m$.

The purpose of this note is a short proof of Vinh's identity (3) by an application of the binomial convolution for formal power series

$$\left(\sum_{n=0}^{\infty} a_n \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} b_n \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!} \quad \text{with} \quad c_n = \sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k}.$$

2. Proof of Identity (3)

First note that

$$\sum_{n=0}^{\infty} n^m \frac{z^n}{n!} = \sum_{k=0}^{m} S(m,k)\sum_{n=k}^{\infty} n^{\underline{k}}\frac{z^n}{n!} = \sum_{k=0}^{m} S(m,k)\sum_{n=k}^{\infty}\frac{z^n}{(n-k)!}$$
$$= \sum_{k=0}^{m} S(m,k)\, z^k e^z.$$

Put

$$c_n = \sum_{k=0}^{n}\binom{n}{k}k^m d(n-k).$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty} c_n \frac{z^n}{n!} &= \left(\sum_{n=0}^{\infty} n^m \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} d(n)\frac{z^n}{n!}\right) = \sum_{k=0}^{m} S(m,k)\, z^k e^z \cdot \frac{e^{-z}}{1-z}\\ &= \sum_{k=0}^{m} S(m,k) \sum_{i=0}^{\infty} z^{k+i} = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} S(m,k)\,.\end{aligned}$$

Comparing the coefficients we conclude that

$$c_n = n! \sum_{k=0}^{n} S(m,k) \qquad (n \geq 0)\,.$$

Noting that $S(m,k) = 0$, for $k > m$, completes the proof of (3), for $0 \leq m \leq n$.

3. A generalization and Dobiński's formula

In the special case $m = 0$, Vinh's identity (3) reduces to the well-known formula

$$\sum_{k=0}^{n} \binom{n}{k} d(n-k) = n!$$

possessing the following combinatorial proof: There are exactly $\binom{n}{k}$ ways to choose k objects from n distinguable objects, so that $\binom{n}{k} d(n-k)$ is the number of n-permutations with exactly k fixed points. Summing up we get the total number $n!$ of n-permutations. It follows that

$$\sum_{k=0}^{n} k^{\underline{m}} \binom{n}{k} d(n-k) = n^{\underline{m}} \sum_{k=0}^{n-m} \binom{n-m}{k} d(n-m-k) = n^{\underline{m}} (n-m)! = n!$$

if $m \leq n$, while it is obvious that the sum is equal to zero, for $m > n$. Put

$$\phi(x) = \sum_{m=0}^{\infty} c_m x^{\underline{m}} \qquad \text{with} \qquad \sum_{m=0}^{\infty} |c_m| < \infty.$$

Then

$$\sum_{k=0}^{n} \phi(k) \binom{n}{k} d(n-k) = n! \sum_{m=0}^{n} c_m$$

which is equivalent to a result found by Rousseau [6, Eq. (2)].

Finally, Eq. (3) can be rewritten in the form

$$B_m = \sum_{k=0}^{n} \frac{k^m}{k!} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} = \sum_{i=0}^{n} \frac{(-1)^i}{i!} \sum_{k=0}^{n-i} \frac{k^m}{k!}.$$

Letting $n \to \infty$ we obtain Dobiński's formula

$$B_m = e^{-1} \sum_{k=0}^{\infty} \frac{k^m}{k!}$$

(cf. [6, pg. 5253]).

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Ordering convex functions according to their lengths

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Abstract. Let $f, g : [0, 1] \to \mathbb{R}$ be two convex functions. We are interested by the relation between the fact that $f(x) \geq g(x)$ for all $x \in [0, 1]$ and the graph lengths of them. We prove that the order between functions implies an order between lengths and with a supplementary condition we obtain the converse.

Keywords: Real function, length, Lorenz curve.
MSC: 26A06, 26A48, 26A51, 26A24

Let us consider $f, g : [0, 1] \to [0, 1]$ with the properties:
(i) f, g are increasing, convex, continuous functions;
(ii) $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$.

Any such function is called a *Lorenz curve* (see, for instance, [2]). Obviously if $f : [0, 1] \to [0, 1]$ is a Lorenz curve then $f(x) \leq x$ for all $x \in [0, 1]$.

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If $f \geq g$ (in the sense that $f(x) \geq g(x)$ for all $x \in [0,1]$) we say that f is *Lorenz dominated* by g.

Let us denote by $l_f\big([a,b]\big)$ and $l_g\big([a,b]\big)$ the graph lengths of functions f and g on $[a,b] \subset [0,1]$.

We this notation we can state our first result.

Proposition 1. *If $f \geq g$ then $l_f\left([0,1]\right) \leq l_g\left([0,1]\right)$.*

Proof. Let us consider

$$A = \{(x,y) \in [0,1]^2 : f(x) \leq y \leq x\}$$

and

$$B = \{(x,y) \in [0,1]^2 : g(x) \leq y \leq x\}.$$

Then both sets A, B are convex and $A \subseteq B$. Therefore the perimeter of A is less than or equal to the perimeter of B (see for example [1]). $\square$

The converse of Proposition 1 is not true. See, for instance, the following **Counterexample.** Consider $f,\ g : [0,1] \to [0,1]$ with $f(x) = x^2$ for all $x \in [0,1]$ and

$$g(x) = \begin{cases} 0 & \text{if} \quad x \in \left[0, \frac{1}{4}\right], \\ \frac{4}{3}x - \frac{1}{3} & \text{if} \quad x \in \left(\frac{1}{4}, 1\right]. \end{cases}$$

It is easy to check that $l_f\left([0,1]\right) \leq l_g\left([0,1]\right)$, but is not true that $f \geq g$.

Although $f \geq g$ implies $l_f\left([0,1]\right) \leq l_g\left([0,1]\right)$, it is possible that the inequality between lengths is reversed on $[0,r]$ for small r. Our next result points out a specific case where this happens.

Proposition 2. *Consider $f,\ g : [0,1] \to [0,1]$ defined by $f(x) = x$ and $g(x) = x^2$. Then there exists $u \in (0,1)$ such that $l_f([0,r]) > l_g([0,r])$ for all $r \leq u$.*

Proof. It is obvious that $f \geq g$.

The lengths are

$$l_f([0,x]) = \int_0^x \sqrt{1 + \left(f'(t)\right)^2}\mathrm{d}t = \int_0^x \sqrt{2}\mathrm{d}t = \sqrt{2}x$$

and

$$\begin{aligned} l_g([0,x]) &= \int_0^x \sqrt{1 + \left(g'(t)\right)^2}\mathrm{d}t = \int_0^x \sqrt{1 + (2t)^2}\mathrm{d}t = \int_0^x \sqrt{1 + 4t^2}\mathrm{d}t \\ &= \frac{1}{2}\left(\frac{1}{2}\ln\left(2x + \sqrt{4x^2+1}\right) + x\sqrt{4x^2+1}\right). \end{aligned}$$

It follows that $\lim\limits_{n\to\infty} n\big(l_f([0,\frac{1}{n}]) - l_g([0,\frac{1}{n}])\big)$ equals

$$\lim_{n\to\infty}\left(\sqrt{2} - \frac{1}{4}\cdot\frac{\ln\left(\frac{2}{n} + \sqrt{\frac{4}{n^2}+1}\right)}{\frac{1}{n}} - \frac{1}{2}\cdot\sqrt{\frac{4}{n^2}+1}\right) = \sqrt{2} - 1 > 0,$$

because $\lim\limits_{n\to\infty} \frac{\ln\left(\frac{2}{n}+\sqrt{\frac{4}{n^2}+1}\right)}{\frac{1}{n}} = \lim\limits_{\substack{y\to 0\\ y>0}} \frac{\ln\left(2y+\sqrt{4y^2+1}\right)}{y} = 2.$

Therefore there exists $u \in (0,1)$ such that $l_f([0,r]) > l_g([0,r])$, for all $r \leq u$. $\square$

Our last result gives a sufficient condition for the converse of Proposition 1 to hold.

Proposition 3. *Let be $f,\ g \in C^1\big([0,1]\big)$. If there exists a unique point $u \in (0,1)$ such that $l_f([a,b]) \geq l_g([a,b])$ for all $[a,b] \subseteq [0,u]$ and $l_f([a,b]) \leq l_g([a,b])$ for all $[a,b] \subset [u,1]$, then $f \geq g$.*

Proof. Take $c \in (0,u)$. For all h with $0 < h < c$ we have

$$\frac{1}{2h}\int_{c-h}^{c+h}\sqrt{1+(f'(t))^2}\mathrm{d}t \geq \frac{1}{2h}\int_{c-h}^{c+h}\sqrt{1+(g'(t))^2}\mathrm{d}t.$$

Taking $h \to 0$, it follows that $f'(c) \geq g'(c)$.

But $f,\, g \in C^1\ ([0,1])$, so $f'(x) \geq g'\,(x)$ for all $x \in [0,u]$. It follows that $x \mapsto f(x) - g(x)$ is increasing on $[0,u]$, thus $f(x) - g(x) \geq f(0) - g(0) = 0$, in other words $f(x) \geq g(x)$ for all $x \in [0,u]$.

Similarly, for $c \in (u,1)$ we obtain that $f'(c) \leq g'\,(c)$. Therefore the function $x \mapsto f(x) - g(x)$ is decreasing on $[u,1]$, thus $f(x) - g(x) \geq f(1) - g(1)$, in other words $f(x) \geq g(x)$ for all $x \in [u,1]$. $\square$

References

[1] https://math.stackexchange.com/questions/950838/is-the-perimeter-of-a-nested-convex-set-smaller-than-the-containing-sets.

[2] https://en.wikipedia.org/wiki/Lorenz_curve

PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of November 2019**.

PROPOSED PROBLEMS

482. Let $x \in \mathbb{R}$. Prove the series

$$\sum_{n=0}^{\infty} 3^n \left(\sin x - x + \frac{1}{3!}x^3 - \cdots - \frac{\sin \frac{n\pi}{2}}{n!} x^n \right)$$

converges absolutely and calculate its sum.

Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

483. Suppose that $0 < a_1 \leq a_2 \leq \ldots$ and $\sum_{n=1}^{\infty} 1/a_n < \infty$. Let $A := \{n \in \mathbb{N}^* : a_n < n \log n\}$. Prove that A has logarithmic density 0, that is, $\lim_{x\to\infty} \frac{1}{\log x} \sum_{k \leq x, k \in A} \frac{1}{k} = 0$.

Proposed by George Stoica, New Brunswick, Canada.

484. Prove that for a continuous nonconstant function $f : \mathbb{R} \to \mathbb{R}$, the following conditions are equivalent:

(1) $f(x) - f(y) \in \mathbb{Q}$ for all $x, y \in \mathbb{R}$ such that $x - y \in \mathbb{Q}$;
(2) $f(x) - f(y) \in \mathbb{R} \setminus \mathbb{Q}$ for all $x, y \in \mathbb{R}$ such that $x - y \in \mathbb{R} \setminus \mathbb{Q}$;
(3) there exist $a \in \mathbb{Q}^*$ and $b \in \mathbb{R}$ such that $f(x) = ax + b$ for all $x \in \mathbb{R}$.

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

485. Assume that ABC is a triangle with $a \geq b \geq c$, where the angle A has a fixed value. We denote by Σ the sum

$$\sqrt{\frac{b+c-a}{a}} + \sqrt{\frac{c+a-b}{b}} + \sqrt{\frac{a+b-c}{c}}.$$

Then the only possible values of A are $\pi/3 \leq A < \pi$ and we have:

(i) The smallest possible value Σ is

$$\frac{4\sin\frac{A}{2}+\sqrt{2\left(1-\sin\frac{A}{2}\right)}}{\sqrt{2\sin\frac{A}{2}}}.$$

(ii) If $\pi/3 \le A < \pi/2$ then the largest possible value of Σ is

$$\frac{4\cos A+\sqrt{2\left(1-\cos A\right)}}{\sqrt{2\cos A}}.$$

(iii) If $\pi/2 \le A < \pi$ then there is no finite upper bound for Σ.

Proposed by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania.

486. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ which satisfy the equation

$$f(-x)=x+\int_0^x \sin t\, f(x-t)\,\mathrm{d}t, \quad \forall x\in\mathbb{R}.$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

487. We consider the complex matrices A and B of dimensions $m\times n$ and $n\times m$, where $m>n\ge 1$. Let $C=AB$. If 0 is an eigenvalue of C of order $m-n$ and $C^{k+1}=\lambda C^k$ for some $k\ge 1$ and $\lambda\in\mathbb{C}\setminus\{0\}$, then determine BA.

Proposed by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, and Costel Bălcău, University of Piteşti, Romania.

488. Let $0\le a<b$ and $f:[a,b]\to\mathbb{R}$ be a continuous function with $f(a)=f(b)$. Then there exist $\alpha,\beta\in[a,b]$ such that $f(\alpha)=f(\beta)$ and $\dfrac{\alpha}{\beta}\notin\mathbb{Q}$.

Proposed by George Stoica, New Brunswick, Canada.

SOLUTIONS

464. Prove that the only twice differentiable functions $f:(0,\infty)^2\to\mathbb{R}$ satisfying:

$$x^2\frac{\partial^2 f}{\partial x^2}(x,y)+2xy\frac{\partial^2 f}{\partial x\partial y}(x,y)+y^2\frac{\partial^2 f}{\partial y^2}(x,y)+\frac{1}{4}f(x,y)=0$$

are

$$f(x,y)=C(x,y)+D(x,y)\log x,$$

where C and D are twice differentiable, homogeneous functions of degree $1/2$.

We say that a function C is homogeneous of degree r if

$$C(tx,ty)=t^rC(x,y) \text{ for all } t>0.$$

Proposed by George Stoica, Saint John, New Brunswick, Canada.

Solution by the author. Fix $x, y > 0$ and define $g(t) = g_{x,y}(t) = f(tx, ty)$ for $t > 0$. Using the hypothesis, we obtain that

$$\frac{\partial^2 g}{\partial t^2}(t) = -\frac{1}{4}t^2 g(t).$$

Furthermore, define $h(t) := g(t)t^{-1/2}$, so $g(t) = t^{1/2}h(t)$. Then the equation $g''(t) = -\frac{1}{4t^2}g(t)$ is equivalent to $th''(t) + h'(t) = 0$, i.e., $(th'(t))' = 0$. Hence $th'(t) = E$, so $h'(t) = \frac{1}{t}E$ for some constant E. It follows that $h(t) = E + D\log t$, for some constants E and D. Note that E and D are constants in the sense that they don't depend on t. But they depend on x, y so we have

$$f(tx, ty) = E(x, y)t^{1/2} + D(x, y)t^{1/2}\log t \text{ for all } t, x, y > 0.$$

Note that the relation $g''(t) = -\frac{1}{4t^2}g(t)$ for $g(t) = f(tx, ty)$ is equivalent to our differential equation at (tx, ty). It follows that $f(tx, ty) = E(x, y)t^{1/2} + D(x, y)t^{1/2}\log t\ \forall x, y, t$ is also sufficient, i.e., every twice differentiable function with this property is a solution of the differential equation.

If we put $t = 1$ in the relation above we get $f(x, y) = E(x, y)$ so in fact we have

$$f(tx, ty) = f(x, y)t^{1/2} + D(x, y)t^{1/2}\log t.$$

If $t \neq 1$, so $\log t \neq 0$, we have $D(x, y) = \frac{1}{\log t}(f(tx, ty)t^{-1/2} - f(x, y))$. Since f is twice differentiable, so is D.

For $s, t > 0$ we have

$$\begin{aligned} f(stx, sty) &= f(x, y)(st)^{1/2} + D(x, y)(st)^{1/2}\log(st) \\ &= f(tx, ty)s^{1/2} + D(tx, ty)s^{1/2}\log s \\ &= (f(x, y)t^{1/2} + D(x, y)t^{1/2}\log t)s^{1/2} + D(tx, ty)s^{1/2}\log s \\ &= f(x, y)(st)^{1/2} + D(x, y)(st)^{1/2}\log t + D(tx, ty)s^{1/2}\log s. \end{aligned}$$

After simplifying $s^{1/2}$ we get

$$D(x, y)t^{1/2}\log(st) = D(x, y)t^{1/2}\log t + D(tx, ty)\log s,$$

i.e., $D(tx, ty)\log s = D(x, y)t^{1/2}\log(st) - D(tx, ty)\log t = D(x, y)t^{1/2}\log s$. If we take $s \neq 1$, so that $\log s \neq 0$ can be simplified, we get $D(tx, ty) = D(x, y)t^{1/2}$, which shows that D is homogeneous of degree $1/2$.

Let now $C(x, y) = f(x, y) - D(x, y)\log x$ so that $f(x, y) = C(x, y) + D(x, y)\log x$. Since both f and D are twice differentiable, so is C. Then the relation $f(tx, ty) = f(x, y)t^{1/2} + D(x, y)t^{1/2}\log t$ writes as

$$\begin{aligned} C(tx, ty) + D(x, y)t^{1/2}\log(tx) &= C(tx, ty) + D(tx, ty)\log(tx) \\ &= (C(x, y) + D(x, y)\log x)t^{1/2} + D(x, y)t^{1/2}\log t \\ &= C(x, y)t^{1/2} + D(x, y)t^{1/2}\log(tx), \end{aligned}$$

so $C(tx, ty) = C(x, y)t^{1/2}$. Hence C is also homogeneous of degree $1/2$.

Conversely, if $f(x,y) = C(x,y) + D(x,y)\log x$, with C, D twice differentiable and homogeneous of degree $1/2$ then f is twice differentiable and we have

$$\begin{aligned} f(tx,ty) &= C(tx,ty) + D(tx,ty)\log(tx) = C(x,y)t^{1/2} + D(x,y)t^{1/2}\log(tx) \\ &= C(x,y)t^{1/2} + D(x,y)t^{1/2}\log x + D(x,y)t^{1/2}\log t \\ &= f(x,y) + D(x,y)t^{1/2}\log t. \end{aligned}$$

Hence f is a solution of the differential equation. $\square$

A note from the editor. This problem is a particular case of a more general family of differential equations. Namely, if $m \geq 1$, $D \subseteq \mathbb{R}^m \setminus \{(0,\ldots,0)\}$ is an open set such that $tx = (tx_1,\ldots,tx_m) \in D$ $\forall x = (x_1,\ldots,x_m) \in D$, $t > 0$ and $a_0, a_1, \ldots, a_n \in \mathbb{C}$, with $a_n \neq 0$, we want to determine all n times differentiable functions $f : D \to \mathbb{C}$ such that

$$\sum_{k=0}^{n} a_k \sum_{j_1+\cdots+j_m=k} \binom{k}{j_1,\ldots,j_m} x_1^{j_1}\cdots x_m^{j_m} \frac{\partial^k f}{\partial x_1^{j_1}\cdots\partial x_m^{j_m}} = 0. \qquad (1)$$

The answer is given by the following result.

Theorem. *Put $P(X) = a_0 + a_1X + a_2X(X-1) + \cdots + a_nX(X-1)\cdots(X-n+1)$ and let $r_1,\ldots,r_h$ be the roots of P, with the multiplicities $n_1,\ldots,n_h$, respectively. If z is an n times differentiable, homogeneous of degree 1 function on D, which takes only positive values, then the solutions of the equation (1) are*

$$f = \sum_{j=1}^{h}\sum_{k=0}^{n_j-1} f_{j,k}(\log z)^k,$$

where each $f_{j,k}$ is an n times differentiable, homogeneous of degree r_j function.

A function z with the properties from the Theorem is, e.g., $z(x) = |x| = \sqrt{x_1^2+\cdots+x_m^2}$.

If we assume that $a_0,\ldots,a_n$ are real and restrict ourselves to functions with real values, then let $r_1,\ldots,r_h$ be the real roots of P, with the multiplicities $n_1,\ldots,n_h$, and let $r_1' \pm ir_1'',\ldots,r_l' + ir_l''$ be the non-real roots of P, with the multiplicities $n_1',\ldots,n_l'$.

If $r \in \mathbb{R}$ then every homogeneous of degree r function g writes as $g = g' + ig''$, where g', g'' are homogeneous of degree r, but also with real values. Since z is homogeneous of degree 1, $z^{ir''} = \cos r''\log z + i\sin r''\log z$ is homogeneous of degree ir''. Hence a homogeneous function of degree $r' + ir''$ has the form $g(\cos r''\log z + i\sin r''\log z)$, where $g = g' + ig''$ is homogeneous of degree r'.

Then, by identifying the real part in the Theorem, we get that the real solutions of the differential equation (1) are

$$f=\sum_{j=1}^{h}\sum_{k=0}^{n_j-1}f_{j,k}(\log z)^k+\sum_{j=1}^{l}\sum_{k=0}^{n'_j-1}(f'_{j,k}\cos r''_j\log z+f''_{j,k}\cos r''_j\log z)(\log z)^k,$$

where all $f_{j,k}$, $f'_{j,k}$ and $f''_{j,k}$ are n times differentiable and homogeneous, $f_{j,k}$ of degree r_j, and $f'_{j,k}, f''_{j,k}$ of degree $r'_{j,k}$.

Remarks. (i) If $n=1$, $a_1=1$ and $a_0=-r$ then equation (1) becomes

$$\sum_{l=1}^{m}x_l\frac{\partial f}{\partial x_l}=rf.$$

By a theorem of Euler, its solutions are the differentiable functions that are homogeneous of degree r, which is what the Theorem states.

(ii) If $m=1$ we obtain the well known Cauchy-Euler equation

$$a_nx^nf^{(n)}(x)+a_{n-1}x^{n-1}f^{(n-1)}(x)+\cdots+a_1xf'(x)+a_0f(x)=0.$$

(iii) If $m=2$ then the equation (1) writes as

$$\sum_{k=0}^{n}a_n\sum_{j=0}^{k}\binom{k}{j}x^{k-j}y^j\frac{\partial^k f}{\partial x^{k-j}\partial y^j}=0.$$

In the case of problem 464, $m=2$, $n=2$, $D=(0,\infty)^2$, $a_2=1$, $a_1=0$ and $a_0=\frac14$. Hence $P=\frac14+X(X-1)+X^2-X+\frac14$, which has $r_1=\frac12$ as a double solution. If we take the function z to be $z(x,y)=x$, then the solutions are $f(x,y)=f_{1,0}(x,y)+f_{1,1}(x,y)\log x$, where $f_{1,0}$ and $f_{1,1}$ are twice differentiable and homogeneous of degree 1/2.

465. Let $A\in M_{m,n}(\mathbb{C})$, $B\in M_{n,m}(\mathbb{C})$. Prove that

$$\operatorname{rank}(I_m-AB)-\operatorname{rank}(I_n-BA)=m-n.$$

`Proposed by Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

Solution by the author. We consider the matrices $M,N\in M_{m+n}(\mathbb{C})$ defined with blocks: $M=\begin{pmatrix}I_m-AB&0\\0&I_n\end{pmatrix}$ and $N=\begin{pmatrix}I_m&0\\0&I_n-BA\end{pmatrix}$. Then we have $M=UPV$ and $N=VPU$, where $P=\begin{pmatrix}I_m&A\\B&I_n\end{pmatrix}$, $U=\begin{pmatrix}I_m&-A\\0&I_n\end{pmatrix}$ and $V=\begin{pmatrix}I_m&0\\-B&I_n\end{pmatrix}$. Since U and V are invertible, the matrices M,N and P are equivalent, so they have the same rank. Hence

$$\operatorname{rank}(I_m-AB)+n=\operatorname{rank}M=\operatorname{rank}N=m+\operatorname{rank}(I_n-BA),$$

which proves our claim. □

Solution by Moubinool Omarjee, Lycée Henry IV, Paris, France. By the rank theorem we have $\dim\ker(I_m-AB)=m-\operatorname{rank}(I_m-AB)$ and

$\dim\ker(I_n - BA) = n - \operatorname{rank}(I_n - BA)$. Therefore the relation we want to prove is equivalent to $\dim\ker(I_m - AB) = \dim\ker(I_n - BA)$. We will prove that there is an isomorphism between $F = \ker(I_m - AB)$ and $G = \ker(I_n - BA)$.

If $x \in F$ then $0 = (I_m - AB)x = x - ABx$. It follows that $0 = B(x - ABx) = Bx - BABx = (I_n - BA)Bx$ so $Bx \in G$. Hence we have a linear map $f : F \to G$ given by $f(x) = Bx\ \forall x \in F$. Similarly, if $y \in G$ then $Ay \in F$ so we have a linear map $g : G \to F$ given by $g(y) = Ay$.

For any $x \in F$ we have $0 = (I_m - AB)x = x - ABx$ so $ABx = x$. It follows that $g(f(x)) = g(Bx) = ABx = x$. Hence $g \circ f = 1_F$. Similarly, for any $y \in G$ we have $f(g(y)) = y$ so $f \circ g = 1_G$. Hence f and g are inverse to each other isomorphisms between F and G. This proves our claim. □

We also received two solutions from Victor Makanin, Sankt Petersburg, Russia. His first solution is the same as the author's, while his second solution is essentially the same as the solution by Moubinool Omarjee.

Victor Makanin also pointed out that the author's solution appears in Fuzhen Zhang's book, *Linear Algebra. Challenging Problems for Students*, Problem 2.65. Although in the book only the case when B is the conjugate transpose of A is considered, the same proof applies to the general case.

A note from the editor. This result can be slightly generalized to $\operatorname{rank}(\lambda I_m - AB) - \operatorname{rank}(\lambda I_n - BA) = m - n\ \forall \lambda \in \mathbb{C},\ \lambda \neq 0$. This follows from $\operatorname{rank}(I_m - AB') - \operatorname{rank}(I_n - B'A) = m - n$, where $B' = \lambda^{-1}B$.

An equivalent statement is $\dim\ker(\lambda I_m - AB) = \dim\ker(\lambda I_n - BA)$.

This can be used to determine a relation between the similarity classes of XY and YX when $X \in M_{m,n}(\mathbb{C})$, $Y \in M_{n,m}(\mathbb{C})$. Namely, one proves that $XY \sim Z \oplus M$ and $YX \sim Z \oplus N$ where $Z \in M_k(\mathbb{C})$, $M \in M_{m-k}(\mathbb{C})$ and $N \in M_{n-k}(\mathbb{C})$ for some $k \leq \min\{m, n\}$, such that Z is invertible and M, N are nilpotent.

For any $\lambda \in \mathbb{C}$ and any $i \geq 1$ we denote by $J_i(\lambda)$ the Jordan block of dimension $i \times i$ corresponding to the eigenvalue λ. Then $XY \sim Z \oplus M$, where Z is the sum of all components of the form $J_i(\lambda)$ with $\lambda \neq 0$ and M is the sum of all components of the for $J_i(0)$ from the Jordan decomposition of XY. We have that Z is invertible and M is nilpotent. For YX we have a similar decomposition, $YX = Z' \oplus N$, with Z' invertible and N nilpotent. We want to prove that $Z \sim Z'$. This is equivalent to the fact that, for every $\lambda \neq 0$ and every $i \geq 1$, $J_i(\lambda)$ appears the same number of times in the Jordan decompositions of XY and YX. But the number of occurrences of $J_i(\lambda)$ in XY is $m_i(\lambda) = 2\dim\ker(XY - \lambda I_m)^i - \dim\ker(XY - \lambda I_m)^{i-1} - \dim\ker(XY - \lambda I_m)^{i+1}$ and in YX it is $n_i(\lambda) = 2\dim\ker(YX - \lambda I_n)^i - \dim\ker(YX - \lambda I_n)^{i-1} - \dim\ker(YX - \lambda I_n)^{i+1}$. To prove that $m_i(\lambda) = n_i(\lambda)$ $\forall \lambda \neq 0$, $i \geq 1$, one must show that $\dim\ker(XY - \lambda I_m)^j = \dim\ker(YX - \lambda I_n)^j$ $\forall \lambda \neq 0$, $i \geq 1$.

We have

$$(XY - \lambda I_m)^j = (-\lambda)^j I_m + \binom{j}{1}(-\lambda)^{j-1}XY + \binom{j}{2}(-\lambda)^{j-2}(XY)^2 + \cdots + (XY)^j = (-\lambda)^j I_m + XT$$

and similarly

$$(YX - \lambda I_n)^j = (-\lambda)^j I_n + TX,$$

where

$$T = \binom{j}{1}(-\lambda)^{j-1}Y + \binom{j}{2}(-\lambda)^{j-2}YXY + \cdots + \binom{j}{j}(YX)^{j-1}Y \in M_{n,m}(\mathbb{C}).$$

But, as seen above, $\dim\ker((-\lambda)^j I_m + XT) = \dim\ker((-\lambda)^j I_n + TX)$, i.e., $\dim\ker(XY - \lambda I_m)^j = \dim\ker(YX - \lambda I_n)^j$, as claimed.

This result leads to the following open problem. Given $Z \in M_k(\mathbb{C})$, $M \in M_{m-k}(\mathbb{C})$ and $N \in M_{n-k}(\mathbb{C})$, such that Z is invertible and M, N are nilpotent, determine necessary and sufficient conditions such that there are $X \in M_{m,n}(\mathbb{C})$ and $Y \in M_{n,m}(\mathbb{C})$ such that $XY \sim Z \oplus M$ and $YX \sim Z \oplus N$.

For convenience, we put $m_i = m_i(0)$ and $n_i = n_i(0)$. Since $M \in M_{m-k}(\mathbb{C})$ is the sum of all Jordan blocks of XY of the form $J_i(0)$ we have $M = m_1 J_1(0) \oplus m_2 J_2(0) \oplus \cdots$ and so $m_1 + 2m_2 + 3m_3 + \cdots = m - k$. Similarly, $N = n_1 J_1(0) \oplus n_2 J_2(0) \oplus \cdots$ and $n_1 + 2n_2 + 3n_3 + \cdots = n - k$. Thus the similarity classes of M and N are determined by $m_1, m_2, m_3, \ldots$ and $n_1, n_2, n_3, \ldots$.

Some necessary conditions on the numbers m_i and n_i, i.e., on the similarity classes of M and N, can be easily obtained from the fact that for every $i \geq 1$ we have $(XY)^i = X(YX)^{i-1}Y$ so $\operatorname{rank}(XY)^i \leq \operatorname{rank}(YX)^{i-1}$. Similarly, $\operatorname{rank}(YX)^i \leq \operatorname{rank}(XY)^{i-1}$.

We have $\dim\ker(XY)^i = m_1 + 2m_2 + \cdots + im_i + im_{i+1} + \cdots$. Together with $m_1 + 2m_2 + 3m_3 + \cdots = m - k$, this implies, by subtracting, that

$$\operatorname{rank}(XY)^i = m - \dim\ker(XY)^i = k + m_{i+1} + 2m_{i+2} + 3m_{i+3} + \cdots .$$

Similarly,

$$\operatorname{rank}(YX)^i = k + n_{i+1} + 2n_{i+2} + 3n_{i+3} + \cdots .$$

Therefore the relations

$$\operatorname{rank}(XY)^i \leq \operatorname{rank}(YX)^{i-1} \quad \text{and} \quad \operatorname{rank}(YX)^i \leq \operatorname{rank}(XY)^{i-1}$$

imply

$$m_{i+1} + 2m_{i+2} + 3m_{i+3} + \cdots \leq n_i + 2n_{i+1} + 3n_{i+2} + \cdots$$

and

$$n_{i+1} + 2n_{i+2} + 3n_{i+3} + \cdots \leq m_i + 2m_{i+1} + 3m_{i+2} + \cdots .$$

However, it is not clear whether these conditions are also sufficient.

466. Let $f : [1, \infty) \to \mathbb{R}$ be differentiable with $\lim\limits_{x\to\infty} f(x) = 0$. Suppose that f' is strictly monotone, does not vanish anywhere and

$$\lim_{n\to\infty} \frac{f'(n+1)}{f'(n)} = 1.$$

Prove that the limit

$$\gamma_f := \lim_{n\to\infty} \left(\sum_{k=1}^{n} f(k) - \int_1^n f(x)\mathrm{d}x \right),$$

which we call the generalized Euler constant, exists and is finite. Calculate

$$\lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} f(k) - \int_1^n f(x)\mathrm{d}x - \gamma_f}{f(n)}.$$

`Proposed by Mircea Ivan, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

Solution by the author. The derivative f' has the Darboux property. Since it is monotone and it doesn't vanish anywhere, it has a constant sign. It follows that f is strictly monotone. Since also $\lim\limits_{x\to\infty} f(x) = 0$, f has a constant sign. We will assume that f is positive, so it decreases to 0 as $x \to \infty$. Hence $f'(x) < 0\ \forall x$. Since f' is monotone, it has a (possibly infinite) limit L at ∞. Since f is convergent at ∞, the limit L can only be 0. (We have $f(n+1) - f(n) = f'(x_n)$ for some $x_n \in (n, n+1)$. Since $x_n \to \infty$ when $n \to \infty$, we have $\lim\limits_{n\to\infty} f'(x_n) = L$. But $\lim\limits_{n\to\infty} (f(n+1) - f(n)) = 0 - 0 = 0$, so $L = 0$.) Since f' is strictly monotone, everywhere negative and $\lim\limits_{x\to\infty} f'(x) = 0$, f' must be strictly increasing.

We now start our proof. Let $S_n = \sum\limits_{k=1}^{n} f(k) - \int_1^n f(x)\mathrm{d}x$. Since f is decreasing, for every $x \in [k-1, k]$ we have $f(k-1) \geq f(x) \geq f(k)$ and therefore

$$f(k-1) \geq \int_{k-1}^{k} f(x)\mathrm{d}x \geq f(k).$$

We have

$$S_n = f(1) + \sum_{k=2}^{n} \left(f(k) - \int_{k-1}^{k} f(x)\mathrm{d}x \right).$$

But for every k we have $f(k) - \int_{k-1}^{k} f(x)\mathrm{d}x \leq 0$, so that the sequence S_n is decreasing. On the other hand, we also have

$$S_n = \sum_{k=2}^{n} \left(f(k-1) - \int_{k-1}^{k} f(x)\mathrm{d}x \right) + f(n).$$

As $f(n) \geq 0$ and for for every k we have $f(k-1) - \int_{k-1}^{k} f(x)\mathrm{d}x \geq 0$, it results that $S_n \geq 0$. Since S_n is decreasing and bounded, it is convergent to a real number γ_f, as claimed. (In fact, as seen from the proof, we have $0 \le \gamma_f \le f(1)$.)

Using Taylor's formula for $F(t) = \int_1^t f(x)\mathrm{d}x$ we get

$$F(n+1) = F(n) + F'(n) + \frac{1}{2}F''(x_n), \quad \text{for some} \quad x_n \in (n, n+1),$$

i.e.,

$$\int_n^{n+1} f(x)\mathrm{d}x - f(n) = \frac{1}{2}f'(x_n), \quad \text{with} \quad x_n \in (n, n+1).$$

On the other hand, we also have

$$f(n+1) - f(n) = f'(y_n), \quad \text{for some} \quad y_n \in (n, n+1).$$

Since f' is increasing and everywhere negative, from $n < x_n, y_n < n+1$ we get $f(n) \le f(x_n), f(y_n) \le f(n+1) < 0$, which implies

$$\frac{f'(n+1)}{f'(n)} \le \frac{f'(x_n)}{f'(y_n)} \le \frac{f'(n)}{f'(n+1)}.$$

By hypothesis, this implies

$$\frac{f'(x_n)}{f'(y_n)} \to 1 \quad \text{as} \quad n \to \infty.$$

Since both $\sum\limits_{k=1}^{n} f(k) - \int_1^n f(x)\mathrm{d}x - \gamma_f$ and $f(n)$ have limit 0 as $n \to \infty$, we can apply the Stolz-Cesàro theorem and we get

$$\begin{aligned}\frac{\sum\limits_{k=1}^{n} f(k) - \int_1^n f(x)\mathrm{d}x - \gamma_f}{f(n)} &\sim \frac{f(n+1) - \int_n^{n+1} f(x)\mathrm{d}x}{f(n+1) - f(n)} \\ &= 1 - \frac{\int_n^{n+1} f(x)\mathrm{d}x - f(n)}{f(n+1) - f(n)} = 1 - \frac{\frac{1}{2}f'(x_n)}{f'(y_n)} \sim \frac{1}{2}.\end{aligned}$$

Hence our limit is $\frac{1}{2}$. □

We received essentially the same solution from Victor Makanin, Sankt Petersburg, Russia.

We also received a slightly different solution from Leonard Giugiuc, Drobeta Turnu Severin, Romania. He also assumed that f' is negative and decreasing, so that f is decreasing. Then he proved the existence of $\gamma_f = \lim\limits_{n\to\infty} \left(\sum\limits_{k=1}^{n} f(k) - \int_1^n f(x)\mathrm{d}x \right)$ by the same method as the author, and used Stolz-Cesàro to prove that

$$\begin{aligned}\lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} f(k) - \int_1^n f(x)\mathrm{d}x - \gamma_f}{f(n)} &= \lim_{n\to\infty} \frac{f(n+1) - \int_n^{n+1} f(x)\mathrm{d}x}{f(n+1) - f(n)} \\ &= \lim_{n\to\infty} \frac{\int_n^{n+1} f(x)\mathrm{d}x - f(n+1)}{f(n) - f(n+1)},\end{aligned}$$

provided that the last limit exists. For the purpose of computing this limit he used the following result.

Lemma. *If $f : [a, b] \to \mathbb{R}$ is a convex function then*

$$f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

Proof. Let $x \in [0, \frac{b-a}{2}]$. Since f is convex we have $f(\frac{a+b}{2} - x) + f(\frac{a+b}{2} + x) \geq 2f(\frac{a+b}{2})$. On the other hand, by using the property that $f(c) \leq \frac{(b-c)f(a)+(c-a)f(b)}{b-a}$ $\forall c \in [a, b]$ one gets $f(\frac{a+b}{2} - x) + f(\frac{a+b}{2} + x) \leq \frac{(\frac{b-a}{2}+x)f(a)+(\frac{b-a}{2}-x)f(b)}{b-a} + \frac{(\frac{b-a}{2}-x)f(a)+(\frac{b-a}{2}+x)f(b)}{b-a} = f(a) + f(b)$.

Hence $2f(\frac{a+b}{2}) \leq f(\frac{a+b}{2} - x) + f(\frac{a+b}{2} + x) \leq f(a) + f(b)$ $\forall x \in [0, \frac{b-a}{2}]$. By integrating we get

$$\begin{aligned}(b-a)f\left(\frac{a+b}{2}\right) &\leq \int_0^{\frac{b-a}{2}} f\left(\frac{a+b}{2} - x\right)\mathrm{d}x + \int_0^{\frac{b-a}{2}} f\left(\frac{a+b}{2} + x\right)\mathrm{d}x \\ &\leq \frac{b-a}{2}(f(a)+f(b)).\end{aligned}$$

But the middle term of the inequalities above is equal to $\int_a^{\frac{a+b}{2}} f(x)\mathrm{d}x + \int_{\frac{a+b}{2}}^b f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$. Hence the conclusion. □

Back to our problem, since f' is increasing, f is convex. By taking $a = n$, $b = n+1$ in the Lemma, we get $f(n + \frac{1}{2}) \leq \int_n^{n+1} f(x)\mathrm{d}x \leq \frac{1}{2}(f(n) + f(n+1))$. It follows that

$$\begin{aligned}\frac{f(n+\frac{1}{2}) - f(n+1)}{f(n) - f(n+1)} &\leq \frac{\int_n^{n+1} f(x)\mathrm{d}x - f(n+1)}{f(n) - f(n+1)} \\ &\leq \frac{\frac{1}{2}(f(n)+f(n+1)) - f(n+1)}{f(n) - f(n+1)} = \frac{1}{2}.\end{aligned}$$

By Lagrange's mean value theorem we have $f(n+\frac{1}{2}) - f(n+1) = -\frac{1}{2}f'(u_n)$ and $f(n) - f(n+1) = -f'(v_n)$, with $n + \frac{1}{2} < u_n < n+1$, $n < v_n < n+1$. Since $-f'$ is positive and decreasing, we have $\frac{-f'(n+1)}{-f'(n)} \leq \frac{-f'(u_n)}{-f'(v_n)} \leq \frac{-f'(n)}{-f'(n+1)}$. But we have $\lim\limits_{n\to\infty} \frac{f'(n+1)}{f'(n)} = \lim\limits_{n\to\infty} \frac{f'(n)}{f'(n+1)} = 1$ so, by the squeeze theorem, $\lim\limits_{n\to\infty} \frac{f'(u_n)}{f'(v_n)} = 1$. It follows that

$$\lim_{n\to\infty} \frac{f(n+\frac{1}{2}) - f(n+1)}{f(n) - f(n+1)} = \lim_{n\to\infty} \frac{-\frac{1}{2}f'(u_n)}{-f'(v_n)} = \frac{1}{2}.$$

Then, again by the squeeze theorem,

$$\lim_{n\to\infty} \frac{\int_n^{n+1} f(x)\mathrm{d}x - f(n+1)}{f(n) - f(n+1)} = \frac{1}{2},$$

so the desired limit is $\frac{1}{2}$.

467. Let $a, b, c, d, n \geq 0$ be some integers with $a+c, b+d \geq n$. Then
$$\sum_{k+l=n} k!l!\binom{a}{l}\binom{b}{l}\binom{d}{k}\binom{a+c-l}{k} = \sum_{k+l=n} k!l!\binom{a}{l}\binom{c}{k}\binom{d}{k}\binom{b+d-k}{l}.$$
(Here we make the convention that $\binom{n}{k} = 0$ if $k < 0$ or $k > n$.)

`Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.`

Solution by the author. Let A_n and B_n be the sides of the equality we want to prove. We actually prove that $A_n = C_n$ and $B_n = C_n$, where
$$C_n = \sum_{i+j+h=n} \frac{(i+j)!(j+h)!}{j!}\binom{a}{j+h}\binom{b}{h}\binom{c}{i}\binom{d}{i+j}.$$
We use the well-known formula $\binom{m+n}{k} = \sum\limits_{i+j=k}\binom{m}{i}\binom{n}{j}$, which follows by identifying the coefficient of X^k in the identity $(1+X)^{m+n} = (1+X)^m(1+X)^n$, i.e., $\sum\limits_k \binom{m+n}{k}X^k = \sum\limits_i \binom{m}{i}X^i \sum\limits_j \binom{n}{j}X^j$.

We have $\binom{a+c-l}{k} = \sum\limits_{i+j=k}\binom{c}{i}\binom{a-l}{j}$ so
$$\begin{aligned} A_n &= \sum_{k+l=n}\sum_{i+j=k} k!l!\binom{a}{l}\binom{b}{l}\binom{c}{i}\binom{d}{k}\binom{a-l}{j} \\ &= \sum_{k+l=n}\sum_{i+j=k} (i+j)!l!\binom{a}{l}\binom{b}{l}\binom{c}{i}\binom{d}{i+j}\binom{a-l}{j}. \end{aligned}$$
But the sum $\sum\limits_{k+l=n}\sum\limits_{i+j=k} x_{i,j,l}$ also writes as $\sum\limits_{i+j+l=n} x_{i,j,l}$. Also, after simplifying factorials, one gets
$$(i+j)!l!\binom{a}{l}\binom{a-l}{j} = \frac{(i+j)!a!}{(a-j-l)!j!} = \frac{(i+j)!(j+l)!}{j!}\binom{a}{j+l}.$$
It follows that
$$A_n = \sum_{i+j+l=n} \frac{(i+j)!(j+l)!}{j!}\binom{a}{j+l}\binom{b}{l}\binom{c}{i}\binom{d}{i+j} = C_n.$$
The equality $B_n = C_n$ follows from $A_n = C_n$ by replacing $a, b, c, d, k, l, i, j, h$ with $d, c, b, a, l, k, h, j, i$, respectively. □

Note. This result is related to the properties of the Weyl algebras. The Weyl algebra A_1 over an arbitrary commutative ring with unity R is generated by X and Y, with the relation $[Y, X] := YX - XY = 1$. It is known that X^aY^b, with $a, b \geq 0$, are a basis for A_1. Also we have the formula
$$Y^bX^a = \sum_{n \leq a,b} n!\binom{a}{n}\binom{b}{n}X^{a-n}Y^{b-n},$$

which allows us to write the multiplication table on A_1 in terms of the basis. Namely, we have

$$(X^cY^b)(X^aY^d) = X^c(Y^bX^a)Y^d = \sum_{n\le a,b} n!\binom{a}{n}\binom{b}{n}X^{a+c-n}Y^{b+d-n}.$$

Then $A_n = B_n$ is equivalent to the associativity property for Y^d, X^cY^b and X^a. Indeed, we have

$$\begin{aligned}
Y^d((X^cY^b)X^a) &= Y^d\sum_l l!\binom{a}{l}\binom{b}{l}X^{a+c-l}Y^{b-l}\\
&= \sum_l l!\binom{a}{l}\binom{b}{l}Y^d(X^{a+c-l}Y^{b-l})\\
&= \sum_l l!\binom{a}{l}\binom{b}{l}\sum_k k!\binom{a+c-l}{k}\binom{d}{k}X^{a+c-l-k}Y^{d+b-l-k}\\
&= \sum_n \sum_{k+l=n} k!l!\binom{a}{l}\binom{b}{l}\binom{d}{k}\binom{a+c-l}{k}X^{a+c-n}Y^{b+d-n}\\
&= \sum_n A_nX^{a+c-n}Y^{b+d-n}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(Y^d(X^cY^b))X^a &= \left(\sum_k k!\binom{c}{k}\binom{d}{k}X^{c-k}Y^{b+d-k}\right)X^a\\
&= \sum_k k!\binom{c}{k}\binom{d}{k}(X^{c-k}Y^{b+d-k})X^a\\
&= \sum_k k!\binom{c}{k}\binom{d}{k}\sum_l l!\binom{a}{l}\binom{b+d-k}{l}X^{a+c-k-l}Y^{b+d-k-l}\\
&= \sum_n \sum_{k+l=n} k!l!\binom{a}{l}\binom{c}{k}\binom{d}{k}\binom{b+d-k}{l}X^{a+c-n}Y^{b+d-n}\\
&= \sum_n B_nX^{a+c-n}Y^{b+d-n}.
\end{aligned}$$

Then $A_n = B_n$ $\forall n$ and $\sum_n A_nX^{a+c-n}Y^{b+d-n} = \sum_n B_nX^{a+c-n}Y^{b+d-n}$ are equivalent.

Recall that $A_n = B_n$ was proved by showing that $A_n = C_n$ and $B_n = C_n$. This proof may look very unnatural, as there is no apparent reason to define the sums C_n. We prove that in fact $Y^dX^cY^bX^a$ also writes as $\sum_n C_nX^{a+c-n}Y^{b+d-n}$, which explains the relations $A_n = C_n$ and $B_n = C_n$. Although this is true in general, we will prove this result in the case when the base ring R is $\mathbb{Q}$ (or it contains $\mathbb{Q}$), so that we can define the exponential formal series.

First note that the relations $Y^bX^a = \sum_n n!\binom{a}{n}\binom{b}{n}X^{a-n}Y^{b-n}$ are equivalent to an equality of formal series from $A_1[[s,t]]$, viz.,

$$\exp(Yt)\exp(Xs) = \exp(Xs)\exp(Yt)\exp(st).$$

Indeed, for every $a, b \geq 0$ the coefficient of s^at^b in $\exp(Yt)\exp(Xs) = \left(\sum_b \frac{1}{b!}Y^bt^b\right)\left(\sum_a \frac{1}{a!}X^as^a\right)$ is $\frac{1}{a!b!}Y^bX^a$. We also have

$$\begin{aligned}\exp(Xs)\exp(Yt)\exp(st) &= \left(\sum_c \frac{1}{c!}X^cs^c\right)\left(\sum_d \frac{1}{d!}Y^dt^d\right)\left(\sum_n \frac{1}{n!}s^nt^n\right)\\ &= \sum_{c,d,n} \frac{1}{c!d!n!}X^cY^ds^{c+n}t^{d+n}.\end{aligned}$$

The coefficient of s^at^b in $\exp(Xs)\exp(Yt)\exp(st)$ is obtained by considering indices with $c+n=a$ and $d+n=b$, so it is $\sum_n \frac{1}{(a-n)!(b-n)!n!}X^{a-n}Y^{b-n}$. By identifying the coefficients of s^at^b $\forall a,b$, the equality $\exp(Yt)\exp(Xs) = \exp(Xs)\exp(Yt)\exp(st)$ is equivalent to

$$\frac{1}{a!b!}Y^bX^a = \sum_n \frac{1}{(a-n)!(b-n)!n!}X^{a-n}Y^{b-n} \quad \forall a,b.$$

But this also writes as

$$Y^bX^a = \sum_n \frac{a!b!}{(a-n)!(b-n)!n!}X^{a-n}Y^{b-n} = \sum_n n!\binom{a}{n}\binom{b}{n}X^{a-n}Y^{b-n}.$$

As a consequence, in $A_1[[s,t,u,v]]$ we have

$$\begin{aligned}&\exp(Yv)\exp(Xu)\exp(Yt)\exp(Xs)\\ &\quad= \exp(Yv)\exp(Xu)\exp(Xs)\exp(Yt)\exp(st)\\ &\quad= \exp(Yv)\exp(X(s+u))\exp(Yt)\exp(st)\\ &\quad= \exp(X(s+u))\exp(Yv)\exp((s+u)v)\exp(Yt)\exp(st)\\ &\quad= \exp(X(s+u))\exp(Y(t+v))\exp(st+sv+uv).\end{aligned}$$

The coefficient of $s^at^bu^cv^d$ in $\exp(Yv)\exp(Xu)\exp(Yt)\exp(Xs)$, i.e., in $\left(\sum_d \frac{1}{d!}Y^dv^d\right)\left(\sum_c \frac{1}{c!}X^cu^c\right)\left(\sum_b \frac{1}{b!}Y^bt^b\right)\left(\sum_a \frac{1}{a!}X^as^a\right)$, is $\frac{1}{a!b!c!d!}Y^dX^cY^bX^a$. On the other hand, the equality

$$\begin{aligned}&\exp(X(s+u))\exp(Y(t+v))\exp(st+sv+uv)\\ &\quad= \exp(Xs)\exp(Xu)\exp(Yt)\exp(Yv)\exp(st)\exp(sv)\exp(uv)\end{aligned}$$

writes as

$$\sum_{a',b',c',d',i,j,k} \frac{1}{a'!}X^{a'}s^{a'}\frac{1}{c'!}X^{c'}u^{c'}\frac{1}{b'!}Y^{b'}t^{b'}\frac{1}{d'!}Y^{d'}v^{d'}\frac{1}{h!}(st)^h\frac{1}{j!}(sv)^j\frac{1}{i!}(uv)^i$$
$$= \sum_{a',b',c',d',i,j,k} \frac{1}{a'!b'!c'!d'!i!j!h!}X^{a'+c'}Y^{b'+d'}s^{a'+j+h}t^{b'+h}u^{c'+i}v^{d'+i+j}.$$

To determine the coefficient of $s^a t^b u^c v^d$ we consider only indices with $a' + j + h = a$, $b' + h = b$, $c' + i = c$, $d' + i + j = d$, i.e., $a' = a - j - h$, $b' = b - h$, $c' = c - i$, $d' = d - i - j$. It follows that $\frac{1}{a!b!c!d!}Y^d X^c Y^b X^a$ is equal to

$$\sum_{i,j,k} \frac{1}{(a-j-h)!(b-h)!(c-i)!(d-i-j)!i!j!h!}X^{a+c-i-j-h}Y^{b+d-i-j-h}$$
$$= \sum_n \sum_{i+j+k=n} \frac{1}{(a-j-h)!(b-h)!(c-i)!(d-i-j)!i!j!h!}X^{a+c-n}Y^{b+d-n}.$$

Hence, $Y^d X^c Y^b X^a$ is equal to

$$\sum_n \sum_{i+j+k=n} \frac{a!, b!c!d!}{(a-j-h)!(b-h)!(c-i)!(d-i-j)!i!j!h!}X^{a+c-n}Y^{b+d-n}$$
$$= \sum_n \sum_{i+j+k=n} \frac{(i+j)!(j+h)!}{j!}\binom{a}{j+h}\binom{b}{h}\binom{c}{i}\binom{d}{i+j}X^{a+c-n}Y^{b+d-n}$$
$$= \sum_n C_n X^{a+c-n}Y^{b+d-n}.$$

468. Find all functions $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $f(xu + yv, -xv + yu) = f(x, y)f(u, v)$ for all $x, y, u, v \in \mathbb{R}$. Which ones are continuous?

`Proposed by George Stoica, Saint John, New Brunswick, Canada.`

Solution by the author. Interchanging x with u and y with v, we obtain

$$f(xu + yv, -yu + xv) = f(x, y)f(u, v) \text{ for all } x, y, u, v \in \mathbb{R}.$$

Comparing the latest formula with the one in the hypothesis, gives

$$f(xu + yv, -xv + yu) = f(xu + yv, -yu + xv) \text{ for all } x, y, u, v \in \mathbb{R}.$$

For $v = 0$ and $u = 1$, the previous formula gives

$$f(x, y) = f(x, -y) \text{ for all } x, y \in \mathbb{R}. \tag{1}$$

Letting $v = y = 0$ in the hypothesis gives

$$f(xu, 0) = f(x, 0)f(u, 0) \text{ for all } x, u \in \mathbb{R}. \tag{2}$$

Define $m : \mathbb{R} \to \mathbb{R}$ by

$$m(x) = f(x, 0) \text{ for all } x \in \mathbb{R}, \tag{3}$$

and from (2) we see that m satisfies $m(xu) = m(x)m(u)$ for all $x, u \in \mathbb{R}$. Thus m is a multiplicative function on $\mathbb{R}$.

From the hypothesis and (1) it also follows that

$$\begin{aligned} f(xu+yv, -xv+yu) &= f(x,y)f(u,v) \\ &= f(x,y)f(u,-v) \\ &= f(xu-yv, xv+yu) \text{ for all } x,y,u,v \in \mathbb{R}. \end{aligned}$$

Let $x_1 = xu+yv, x_2 = -xv+yu, y_1 = xu-yv$ and $y_2 = xv+yu$. Then it is easy to see that $x_1^2+x_2^2 = (u^2+v^2)(x^2+y^2) = y_1^2+y_2^2$. Moreover, the mapping

$$(x,y,u,v) \longmapsto (xu+yv, yu-xv, xu-yv, yu+xv)$$

maps $\mathbb{R}^4$ onto the set $\{(x_1,x_2,y_1,y_2) \in \mathbb{R}^4 \mid x_1^2+x_2^2 = y_1^2+y_2^2\}$. Hence, the last displayed equation gives

$$f(x_1,x_2) = f(y_1,y_2) \text{ for all } x_1,x_2,y_1,y_2 \in \mathbb{R} \text{ such that } x_1^2+x_2^2 = y_1^2+y_2^2.$$

Letting $y_2 = 0$ here and using (3), we obtain

$$f(x,y) = f(\sqrt{x^2+y^2}, 0) = m(\sqrt{x^2+y^2}), \tag{4}$$

where $m : \mathbb{R} \to \mathbb{R}$ is a multiplicative function.

Note that for $x \geq 0$ we have $m(x) = m(\sqrt{x})^2 \geq 0$. Therefore, we have a multiplicative function $M : [0,\infty) \to [0,\infty)$, given by $M(x) = m(\sqrt{x}) = \sqrt{m(x)}$. Then the equation (4) writes as $f(x,y) = M(x^2+y^2)$, where $M\colon [0,\infty) \to [0,\infty)$ is a multiplicative function. Conversely, these functions satisfy the functional equation.

In order that f is continuous, M must be continuous. If $M(x) = 0$ for some $x > 0$ then $M(y) = M(x)M(y/x) = 0\ \forall y$ so $M \equiv 0$. Otherwise M sends $(0,\infty)$ to $(0,\infty)$. But it is known that the only continuous multiplicative functions with positive values on $(0,\infty)$ are $x \mapsto x^\alpha$, for some $\alpha \in \mathbb{R}$. Such a function extends to a continuous function M on $[0,\infty)$ iff $\alpha \geq 0$. If $\alpha = 0$ we get $M \equiv 1$, if $\alpha > 0$ then $M(x) = x^\alpha$.

In conclusion, the only continuous solutions are $f \equiv 0$, $f \equiv 1$ and $f(x,y) = (x^2+y^2)^\alpha$, where $\alpha > 0$ is a real constant.

A note from the editor. In fact the author only proved that the image of the function $(x,y,u,v) \mapsto (xu+yv, yu-xv, xu-yv, yu+xv)$ is contained in $\{(x_1,x_2,y_1,y_2) \mid x_1^2+x_2^2 = y_1^2+y_2^2\}$. For the reverse inclusion, note that $(x+iy)(u+iv) = (xu-yv)+i(yu+xv)$ and $(x+iy)(u-iv) = (xu+yv)+i(yu-xv)$, so if we denote $\alpha = x+iy$, $\beta = u+iv$, $z = x_1+ix_2$ and $z' = y_1+iy_2$ then $(xu+yv, yu-xv, xu-yv, yu+xv) = (x_1,x_2,y_1,y_2)$ is equivalent to $z' = \alpha\beta$ and $z = \alpha\overline{\beta}$. The relation $x_1^2+x_2^2 = y_1^2+y_2^2$ also writes as $|z| = |z'|$. It is an obvious consequence of $z' = \alpha\beta$ and $z = \alpha\overline{\beta}$.

Conversely, we must prove that if $|z| = |z'|$ then there are $\alpha, \beta \in \mathbb{C}$ such that $z' = \alpha\beta$ and $z = \alpha\overline{\beta}$. If $|z| = |z'| = 0$, so $z = z' = 0$, then we may take

$\alpha = \beta = 0$. Otherwise we prove that there is $\beta \in \mathbb{C}^*$ such that $z'/z = \beta/\overline{\beta}$ and we take $\alpha = z'/\beta = z/\overline{\beta}$, so that $z' = \alpha\beta$ and $z = \alpha\overline{\beta}$.

If $w = z'/z$ then $|w| = 1$ and the existence of $\beta \in \mathbb{C}^*$ with $w = \beta/\overline{\beta}$ follows from Hilbert's Theorem 90. Explicitly, we have $1 = |w|^2 = w\overline{w}$, so if $w \neq -1$ and we take $\beta = 1 + w \neq 0$ then $\beta/\overline{\beta} = \frac{1+w}{1+\overline{w}} = \frac{1+w}{1+w^{-1}} = w$, as claimed. If $w = -1$ then we take $\beta = i$ and we have $\beta/\overline{\beta} = \frac{i}{-i} = -1 = w$.

469. Let $A, B \in M_3(\mathbb{R})$ such that $A^2 + BA + B^2 = AB$ and $S(A) \leq 0$, where $S(A)$ is the sum of minors corresponding to the entries on the diagonal of A. Prove that $(AB)^2 = A^2B^2$.

Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

Solution by the author. Let a_1, a_2, a_3 be the eigenvalues of A and let b_1, b_2, b_3 be the eigenvalues of B. We have $\mathrm{Tr}(A) = a_1 + a_2 + a_3$, $S(A) = a_1a_2 + a_1a_3 + a_2a_3$ and $\det A = a_1a_2a_3$ and, by the Hamilton-Cayley theorem, $A^3 - \mathrm{Tr}(A)A^2 + S(A)A - (\det A)I_3 = 0$. Similarly for B.

The relation from the hypothesis also writes as $AB - BA = A^2 + B^2$. Consequently, $(A+B)(A-B) = A^2 - AB + BA - B^2 = -2B^2$ and $(A-B)(A+B) = A^2 + AB - BA - B^2 = 2A^2$. It follows that $-2B^2$ and $2A^2$ have the same characteristic polynomial, i.e., $\det(2A^2) = \det(-2B^2)$, $\mathrm{Tr}(2A^2) = \mathrm{Tr}(-2B^2)$ and $S(2A^2) = S(-2B^2)$. The first relation writes as $8(\det A)^2 = -8(\det B)^2$. Since $\det A, \det B \in \mathbb{R}$, this implies that $\det A = \det B = 0$, i.e., $a_1a_2a_3 = b_1b_2b_3 = 0$. We will assume that $a_1 = b_1 = 0$. Since the eigenvalues of $2A^2$ are $2a_1^2, 2a_2^2, 2a_3^2$, we have $\mathrm{Tr}(2A^2) = 2a_1^2 + 2a_2^2 + 2a_3^2 = 2(a_2^2 + a_3^2)$ and $S(2A^2) = 4a_1^2a_2^2 + 4a_1^2a_3^2 + 4a_2^2a_3^2 = 4a_2^2a_3^2$. Similarly, the eigenvalues of $-2B^2$ are $-2b_1^2, -2b_2^2, -2b_3^2$, so $Tr(-2B^2) = -2b_1^2 - 2b_2^2 - 2b_3^2 = -2(b_2^2 + b_3^2)$ and $S(-2B^2) = 4b_1^2b_2^2 + 4b_1^2b_3^2 + 4b_2^2b_3^2 = 4b_2^2b_3^2$. Then the relations $\mathrm{Tr}(2A^2) = \mathrm{Tr}(-2B^2)$ and $S(2A^2) = S(-2B^2)$ write as

$$a_2^2 + a_3^2 + b_2^2 + b_3^2 = 0 \quad \text{and} \quad a_2^2a_3^2 = b_2^2b_3^2, \quad \text{i.e.,} \quad a_2a_3 = \pm b_2b_3.$$

We have three cases:

Case (i): $a_2, a_3, b_2, b_3 \in \mathbb{R}$. Then $a_2 = a_3 = b_2 = b_3 = 0$. Hence the eigenvalues of both A and B are all 0, so their characteristic polynomials are X^3. By Hamilton-Cayley theorem we get $A^3 = B^3 = 0$. Then $A^2B^2 = A(AB)B = A(A^2 + BA + B^2)B = A^3B + (AB)^2 + AB^3 = (AB)^2$.

Case (ii): $a_2 = \overline{a_3} \in \mathbb{C} \setminus \mathbb{R}$. In this case $S(A) = a_1a_2 + a_1a_3 + a_2a_3 = a_2a_3 = |a_2|^2 > 0$, which contradicts the hypothesis.

Case (iii): $a_2, a_3 \in \mathbb{R}$ and $b_2 = \overline{b_3} \in \mathbb{C} \setminus \mathbb{R}$. We have $0 \geq S(A) = a_1a_2 + a_1a_3 + a_2a_3 = a_2a_3$ and $S(B) = b_1b_2 + b_1b_3 + b_2b_3 = b_2b_3 = |b_2|^2 > 0$. Then $a_2a_3 = \pm b_2b_3$ implies $a_2a_3 = -b_2b_3$, i.e., $S(A) = -S(B)$. We also have $(a_2+a_3)^2 + (b_2+b_3)^2 = a_2^2 + a_3^2 + b_2^2 + b_3^2 + 2(a_2a_3 + b_2b_3) = a_2^2 + a_3^2 + b_2^2 + b_3^2 = 0$. Since $a_2 + a_3 = \mathrm{Tr}(A)$ and $b_2 + b_3 = \mathrm{Tr}(B)$ are real, this implies $a_2 + a_3 = b_2 +$

$b_3 = 0$, i.e., $\mathrm{Tr}(A) = \mathrm{Tr}(B) = 0$. Since $\mathrm{Tr}(A) = \det A = \mathrm{Tr}(B) = \det B = 0$, the Hamilton-Cayley theorem for A and B writes as $A^3 + S(A)A = 0$ and $B^3 + S(B)B = 0$. Then as in case (i), we have $A^2B^2 = A^3B + (AB)^2 + AB^3$. But $A^3B + AB^3 = (-S(A)A)B + A(-S(B)B) = -(S(A) + S(B))AB = 0$, so $A^2B^2 = (AB)^2$. $\square$

470. Let $a \in \mathbb{R}$ and let $b, c \in \mathbb{R}$ with $bc > 0$. Calculate $\lim\limits_{n\to\infty} \begin{pmatrix} 1 & \frac{b}{n} \\ \frac{c}{n} & 1 + \frac{a}{n^2} \end{pmatrix}^n$.

`Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

Solution by the author. We prove that the limit is

$$\begin{pmatrix} \cosh\sqrt{bc} & \frac{b}{\sqrt{bc}}\sinh\sqrt{bc} \\ \frac{c}{\sqrt{bc}}\sinh\sqrt{bc} & \cosh\sqrt{bc} \end{pmatrix}.$$

Let $A = \begin{pmatrix} 1 & \frac{b}{n} \\ \frac{c}{n} & 1 + \frac{a}{n^2} \end{pmatrix}$. A calculation shows that the eigenvalues of A are given by $\lambda_1 = 1 + \frac{a}{2n^2} + \frac{1}{2n}\sqrt{4bc + \frac{a^2}{n^2}}$ and $\lambda_2 = 1 + \frac{a}{2n^2} - \frac{1}{2n}\sqrt{4bc + \frac{a^2}{n^2}}$. Now we need Theorem 4.7 on page 194, see also Remark 3.1 on page 109, in [1], which states that if $n \in \mathbb{N}$, $A \in \mathcal{M}_2(\mathbb{C})$ and $\lambda_1 \neq \lambda_2$ are the eigenvalues of A, then $A^n = \dfrac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}A + \dfrac{\lambda_1\lambda_2^n - \lambda_2\lambda_1^n}{\lambda_1 - \lambda_2}I_2$.

Let $k \in \mathbb{N}$. An easy calculation, based on the previous formula, shows that

$$A^k = \frac{\lambda_1^k(1-\lambda_2) + \lambda_2^k(\lambda_1 - 1)}{\lambda_1 - \lambda_2}I_2 + \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}\begin{pmatrix} 0 & \frac{b}{n} \\ \frac{c}{n} & \frac{a}{n^2} \end{pmatrix}.$$

When $k = n$, one has that

$$\begin{aligned} A^n &= \frac{\lambda_1^n(1-\lambda_2) + \lambda_2^n(\lambda_1 - 1)}{\lambda_1 - \lambda_2}I_2 + \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}\begin{pmatrix} 0 & \frac{b}{n} \\ \frac{c}{n} & \frac{a}{n^2} \end{pmatrix} \\ &= \frac{\lambda_1^n\left(\frac{1}{2n}\sqrt{4bc + \frac{a^2}{n^2}} - \frac{a}{2n^2}\right) + \lambda_2^n\left(\frac{1}{2n}\sqrt{4bc + \frac{a^2}{n^2}} + \frac{a}{2n^2}\right)}{\frac{1}{n}\sqrt{4bc + \frac{a^2}{n^2}}}I_2 \qquad (1) \\ &\qquad + \frac{\lambda_1^n - \lambda_2^n}{\frac{1}{n}\sqrt{4bc + \frac{a^2}{n^2}}}\begin{pmatrix} 0 & \frac{b}{n} \\ \frac{c}{n} & \frac{a}{n^2} \end{pmatrix}. \end{aligned}$$

We have $\lim\limits_{n\to\infty} n(\lambda_1 - 1) = \sqrt{bc}$ and $\lim\limits_{n\to\infty} n(\lambda_2 - 1) = -\sqrt{bc}$, so $\lim\limits_{n\to\infty} \lambda_1^n = \mathrm{e}^{\sqrt{bc}}$ and $\lim\limits_{n\to\infty} \lambda_2^n = \mathrm{e}^{-\sqrt{bc}}$.

Passing to the limit as $n \to \infty$ in (1) we get

$$\lim_{n\to\infty} A^n = \frac{\mathrm{e}^{\sqrt{bc}} + \mathrm{e}^{-\sqrt{bc}}}{2} I_2 + \left(\mathrm{e}^{\sqrt{bc}} - \mathrm{e}^{-\sqrt{bc}}\right) \begin{pmatrix} 0 & \frac{b}{2\sqrt{bc}} \\ \frac{c}{2\sqrt{bc}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\sqrt{bc} & \frac{b}{\sqrt{bc}} \sinh\sqrt{bc} \\ \frac{c}{\sqrt{bc}} \sinh\sqrt{bc} & \cosh\sqrt{bc} \end{pmatrix}.$$

It is interesting to note that the limit does not depend on a. The problem is solved. □

Solution by Julio Cesar Mohnsam and Oscar Schmitt, Instituto Federal Sul-rio-grandense, Câmpus Pelotas, RS, Brasil.

Let $n \in \mathbb{N}$, let f be the polynomial function $f(x) = \left(1 + \frac{x}{n}\right)^n$, and let $A = \begin{pmatrix} 0 & b \\ c & \frac{a}{n} \end{pmatrix}$. The eigenvalues of A are $\lambda_1 = \frac{1}{2}(\frac{a}{n} + \sqrt{\frac{a^2}{n^2} + 4bc})$ and $\lambda_2 = \frac{1}{2}(\frac{a}{n} - \sqrt{\frac{a^2}{n^2} + 4bc})$. By Theorem 4.7 in [1] we have

$$\begin{pmatrix} 1 & \frac{b}{n} \\ \frac{c}{n} & 1 + \frac{a}{n^2} \end{pmatrix}^n = \left(I_2 + \frac{A}{n}\right)^n$$

$$= \frac{\left(1 + \frac{\lambda_1}{n}\right)^n - \left(1 + \frac{\lambda_2}{n}\right)^n}{\lambda_1 - \lambda_2} A + \frac{\lambda_1 \left(1 + \frac{\lambda_2}{n}\right)^n - \lambda_2 \left(1 + \frac{\lambda_1}{n}\right)^n}{\lambda_1 - \lambda_2} I_2$$

But for $i = 1, 2$ we have $\lim\limits_{n\to\infty} (1 + \frac{\lambda_i}{n})^n = e^{\lim\limits_{n\to\infty} \lambda_i}$. It is easy to get $\lim\limits_{n\to\infty} \lambda_1 = \sqrt{bc}$ and $\lim\limits_{n\to\infty} \lambda_2 = -\sqrt{bc}$. Therefore

$$\lim_{n\to\infty} \begin{pmatrix} 1 & \frac{b}{n} \\ \frac{c}{n} & 1 + \frac{a}{n^2} \end{pmatrix}^n = \frac{e^{\sqrt{bc}} - e^{-\sqrt{bc}}}{2\sqrt{bc}} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + \frac{e^{\sqrt{bc}} + e^{-\sqrt{bc}}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sqrt{\frac{b}{c}} \sinh\sqrt{bc} \\ \sqrt{\frac{c}{b}} \sinh\sqrt{bc} & 0 \end{pmatrix} + \begin{pmatrix} \cosh\sqrt{bc} & 0 \\ 0 & \cosh\sqrt{bc} \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\sqrt{bc} & \sqrt{\frac{b}{c}} \sinh\sqrt{bc} \\ \sqrt{\frac{c}{b}} \sinh\sqrt{bc} & \cosh\sqrt{bc} \end{pmatrix}.$$

References

[1] Vasile Pop, Ovidiu Furdui, *Square Matrices of Order 2. Theory, Applications, and Problems*, Berlin, Springer, 2017.

We also received a solution from Leonard Giugiuc, who, in order to find an exact formula for $\begin{pmatrix} 1 & \frac{b}{n} \\ \frac{c}{n} & 1 + \frac{a}{n^2} \end{pmatrix}^n$, wrote the matrix $\begin{pmatrix} 1 & \frac{b}{n} \\ \frac{c}{n} & 1 + \frac{a}{n^2} \end{pmatrix}$ in the form $U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^{-1}$.

Note from the editor. It is well known that if $a_1, a_2, \ldots$ is a sequence of complex numbers with $\lim\limits_{n\to\infty} n(a_n - 1) = \ell \in \mathbb{C}$ then $\lim\limits_{n\to\infty} a_n^n = e^\ell$. The same results holds for sequences of matrices. Namely, if $A_1, A_2, \ldots \in M_k(\mathbb{C})$ satisfy $\lim\limits_{n\to\infty} n(A_n - I_k) = L \in M_k(\mathbb{C})$ then $\lim\limits_{n\to\infty} A_n^n = e^L$. The proof is similar, with some complications, to that for sequences of complex numbers.

In our case, if we denote $A_n = \begin{pmatrix} 1 & b/n \\ c/n & 1 + a/n^2 \end{pmatrix}$ then

$$\lim_{n\to\infty} n(A_n - I_2) = L := \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

It follows that $\lim\limits_{n\to\infty} A_n^n = e^L = \sum\limits_{n\geq 0} \frac{1}{n!} L^n$. Note that $L^2 = bcI_2$. Hence if n is even then $L^n = (L^2)^{n/2} = (bc)^{n/2} I_2 = \sqrt{bc}^n I_2$. If n is odd then $L^{n-1} = \sqrt{bc}^{n-1} I_2 = \frac{1}{\sqrt{bc}} \sqrt{bc}^n I_2$ so $L^n = L^{n-1}L = \frac{1}{\sqrt{bc}} \sqrt{bc}^n L$. It follows that

$$e^L = \sum_{n \text{ even}} \frac{1}{n!} L^n + \sum_{n \text{ odd}} \frac{1}{n!} L^n = \sum_{n \text{ even}} \frac{1}{n!} \sqrt{bc}^n I_2 + \sum_{n \text{ odd}} \frac{1}{n!} \frac{1}{\sqrt{bc}} \sqrt{bc}^n L.$$

But we have the formulas $\sum\limits_{n \text{ even}} \frac{1}{n!} x^n = \cosh x$ and $\sum\limits_{n \text{ odd}} \frac{1}{n!} x^n = \sinh x$. Therefore

$$\begin{aligned} e^L &= \cosh \sqrt{bc}\, I_2 + \frac{1}{\sqrt{bc}} \sinh \sqrt{bc}\, L \\ &= \cosh \sqrt{bc} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{bc}} \sinh \sqrt{bc} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cosh \sqrt{bc} & \frac{b}{\sqrt{bc}} \sinh \sqrt{bc} \\ \frac{c}{\sqrt{bc}} \sinh \sqrt{bc} & \cosh \sqrt{bc} \end{pmatrix}. \end{aligned}$$

We received solutions on these lines from Francisco Perdomo and Ángel Plaza, Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Spain and Moubinool Omarjee, Lycée Henri IV, Paris, France.

Also, our reader Stanley Rabinowitz, Chelmsford, MA, USA, pointed out that the *Mathematica* command

```
Limit[MatrixPower[{{1,b/n},{c/n,1+a/n^2}},n],n->Infinity] FullSimplify
```

gives the answer to this problem:

$$\begin{pmatrix} \cosh(\sqrt{bc}) & \dfrac{b \sinh(\sqrt{bc})}{\sqrt{bc}} \\ \dfrac{c \sinh(\sqrt{bc})}{\sqrt{bc}} & \cosh(\sqrt{bc}) \end{pmatrix}.$$

471. Let R be a commutative ring of characteristic 2, i.e., where $2 = 0$. In the ring $R[[X]]$ we consider the formal power series $f(X) = \sum\limits_{n\geq 0} X^{2^n - 1}$. Prove that for any $\alpha, \beta \in R$ the formal power series $f(\alpha^2 X)f(\beta^2 X)f((\alpha+\beta)^2 X)$ is a square in $R[[X]]$.

Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.

Solution by the author. Since we are in characteristic 2, we have $(a+b)^2 = a^2 + 2ab + b^2 = a^2 + b^2$. More generally, by induction on n one obtains $(a+b)^{2^n} = a^{2^n} + b^{2^n}$.

Solution 1. We first prove that if Y, Z are two more variables then $f(Y^2X)f(Z^2X)f((Y^2+Z^2)X)$ is a square in $R[Y,Z][[X]]$.

Let $F(X) = Xf(X) = \sum_{n\geq 1} X^{2^n}$. Note that one has

$F(X_1+X_2) = \sum_{n\geq 0}(X_1+X_2)^{2^n} = \sum_{n\geq 0}(X_1^{2^n} + X_2^{2^n}) = F(X_1) + F(X_2)$ and

$$F(X)^2 = \left(\sum_{n\geq 0} X^{2^n}\right)^2 = \sum_{n\geq 0}(X^{2^n})^2 = \sum_{n\geq 0} X^{2^{n+1}} = F(X) - X.$$

By using these properties of $F(X) = Xf(X)$, we get

$$\begin{aligned}
X^3Y^2Z^2(Y^2+Z^2)\, & f(Y^2X)f(Z^2X)f((Y^2+Z^2)X) \\
&= F(Y^2X)F(Z^2X)F((Y^2+Z^2)X) \\
&= F(Y^2X)F(Z^2X)(F(Y^2X)+F(Z^2X)) \\
&= F(Y^2X)^2F(Z^2X) + F(Y^2X)F(Z^2X)^2 \\
&= (F(Y^2X)-Y^2X)F(Z^2X) + F(Y^2X)(F(Z^2X)-Z^2X) \\
&= Y^2XF(Z^2X) + F(Y^2X)Z^2X \\
&= Y^2X\left(\sum_{n\geq 0}(Z^2X)^{2^n}\right) + \left(\sum_{n\geq 0}(Y^2X)^{2^n}\right)Z^2X \\
&= \sum_{n\geq 0}(Y^2Z^{2^{n+1}} + Y^{2^{n+1}}Z^2)X^{2^n+1}.
\end{aligned}$$

When $n=0$ the coefficient of $X^2 = X^{2^0+1}$ above is $Y^2Z^2 + Y^2Z^2 = 0$. If $n \geq 1$ then $Y^2Z^{2^{n+1}} + Y^{2^{n+1}}Z^2 = Y^2Z^2(Y^{2^{n+1}-2} + Z^{2^{n+1}-2}) = Y^2Z^2(Y^2 + Z^2)(\sum_{i=0}^{2^n-2} Y^{2^{n+1}-4-2i}Z^{2i})$ because $Y^{2^{n+1}-2} + Z^{2^{n+1}-2} = (Y^2)^{2^n-1} - (Z^2)^{2^n-1}$ $= (Y^2 - Z^2)(\sum_{i=0}^{2^n-2}(Y^2)^{2^n-2-i}(Z^2)^i) = (Y^2+Z^2)(\sum_{i=0}^{2^n-2} Y^{2^{n+1}-4-2i}Z^{2i})$.

In conclusion,

$$\begin{aligned}
X^3Y^2Z^2(Y^2+Z^2)\quad & f(Y^2X)f(Z^2X)f((Y^2+Z^2)X) \\
&= \sum_{n\geq 1} Y^2Z^2(Y^2+Z^2)\left(\sum_{i=0}^{2^n-2} Y^{2^{n+1}-4-2i}Z^{2i}\right)X^{2^n+1} \\
&= X^3Y^2Z^2(Y^2+Z^2)\left(\sum_{n\geq 1}\sum_{i=0}^{2^n-2} Y^{2^{n+1}-4-2i}Z^{2i}X^{2^n-2}\right).
\end{aligned}$$

By simplifying $X^3Y^2Z^2(Y^2+Z^2)$ one gets

$$f(Y^2X)f(Z^2X)f((Y^2+Z^2)X) = \sum_{n\ge 1}\sum_{i=0}^{2^n-2} Y^{2^{n+1}-4-2i}Z^{2i}X^{2^n-2}$$
$$= \sum_{n\ge 1}\sum_{i=0}^{2^n-2} (Y^{2^n-2-i}Z^iX^{2^{n-1}-1})^2 = g(Y,Z,X)^2,$$

where $g(Y,Z,X) = \sum_{n\ge 1}\sum_{i=0}^{2^n-2} Y^{2^n-2-i}Z^iX^{2^{n-1}-1}$.

It follows that $f(\alpha^2X)f(\beta^2X)f((\alpha^2+\beta^2)X) = g(\alpha,\beta,X)^2$.

Remark. We first proved that $f(Y^2X)f(Z^2X)f((Y^2+Z^2)X)$ is a square because we needed to simplify $X^3Y^2Z^2(Y^2+Z^2)$. If we tried to do the same proof directly then we had to simplify $X^3\alpha^2\beta^2(\alpha^2+\beta^2)$, which is not always possible, as $\alpha^2\beta^2(\alpha^2+\beta^2)$ might not be invertible.

Same as for $f(X)$, the only non-zero coefficients of $g(\alpha,\beta,X)$ correspond to exponents of X of the form 2^n-1, with $n\ge 0$.

Solution 2. We prove that $f(\alpha^2X)f(\beta^2X)f((\alpha+\beta)^2X)$ is a square by proving that its inverse is a square.

We use the formulas for F from the first solution: $F(X) = Xf(X)$, $F(X_1+X_2) = F(X_1)+F(X_2)$ and $F(X)^2 = F(X)-X$. The last relation also writes as $F(X)+F(X)^2 = X$ and as $F(X) = F(X)^2+X$. Then $X = F(X)+F(X)^2 = F(X)(1+F(X)) = Xf(X)(1+F(X))$, which implies $1 = f(X)(1+F(X))$, so $f(X)^{-1} = 1+F(X)$. Hence

$$\begin{aligned}
&f(\alpha^2X)^{-1}f(\beta^2X)^{-1}f((\alpha^2+\beta^2)X)^{-1}\\
&\quad= (1+F(\alpha^2X))(1+F(\beta^2X))(1+F((\alpha^2+\beta^2)X))\\
&\quad= (1+F(\alpha^2X))(1+F(\beta^2X))(1+F(\alpha^2X)+F(\beta^2X))\\
&\quad= 1+F(\alpha^2X)^2+F(\beta^2X)^2\\
&\qquad\quad+F(\alpha^2X)F(\beta^2X)+F(\alpha^2X)^2F(\beta^2X)+F(\alpha^2X)F(\beta^2X)^2\\
&\quad= 1+F(\alpha^2X)^2+F(\beta^2X)^2+(F(\alpha^2X)^2+\alpha^2X)(F(\beta^2X)^2+\beta^2X)\\
&\qquad\quad+F(\alpha^2X)^2(F(\beta^2X)^2+\beta^2X)+(F(\alpha^2X)+\alpha^2X)F(\beta^2X)^2\\
&\quad= 1+\alpha^2\beta^2X^2+F(\alpha^2X)^2+F(\beta^2X)^2+F(\alpha^2X)^2F(\beta^2X)^2\\
&\quad= h(\alpha,\beta,X)^2,
\end{aligned}$$

where $h(\alpha,\beta,X) = 1+\alpha\beta X+F(\alpha^2X)+F(\beta^2X)+F(\alpha^2X)F(\beta^2X)$. □